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Openness and Continuity in Locally Convex Cones

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Abstract. In this paper we define nearly continuity and nearly openness of linear operators between locally convex cones. Also we give some conditions on locally convex cones and linear operators which ensure that every nearly continuous (nearly open) mapping is continuous (open). We show by an example that a nearly continuous operator is not necessarily continuous.

1. Introduction

A cone is defined to be a commutative monoid $\mathcal{P}$ together with a scalar multiplication by nonnegative real numbers satisfying the same axioms as for vector spaces; that is, $\mathcal{P}$ is endowed with an addition $(a, b) \mapsto a + b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ which is associative, commutative and admits a neutral element $0 \in \mathcal{P}$, and with a scalar multiplication $(r, a) \mapsto r.a : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ satisfying the usual associative and distributive properties, where $\mathbb{R}_+$ is the set of nonnegative real numbers. We have $1a = a$ and $0a = 0$, for all $a \in \mathcal{P}$.

The theory of locally convex cones uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use mostly the latter. These cones developed in [4] and [8]. For recent researches see [2, 3, 5, 9].

Let $\mathcal{P}$ be a cone. A convex quasiuniform structure on $\mathcal{P}$ is a collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ such that the following properties hold:
(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a, a) : a \in \mathcal{P}\}$);
(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.
Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone if
(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\lambda > 0$ such that $(0, a) \in \lambda U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in \mathcal{P} : (a, b) \in U\}, \quad U \in \mathfrak{U}.$$

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The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

The extended real numbers system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha.(+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. We set $\widetilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a, b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}.$$

Then $\widetilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \widetilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a)+T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$. Uniform continuity is not just continuity. It is immediate from the definition that it implies and combines continuity for the operator $T : \mathcal{P} \to \mathcal{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We note that $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is continuous if and only if there is $U \in \mathfrak{U}$ such that $(\mu(a), \mu(b)) \in \tilde{1}$, i.e. $\mu(a) \leq \mu(b) + 1$ for $(a, b) \in U$. We denote the set of all linear functional on $\mathcal{P}$ by $\mathcal{L}(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset F of $\mathcal{P}^2$, we define polar F° as follows:

$$F^\circ = \{\mu \in \mathcal{L}(\mathcal{P}) : \mu(a) \leq \mu(b) + 1, \ \forall (a, b) \in F\}.$$

The dual cone $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars U° of neighborhoods $U \in \mathfrak{U}$.

2. Nearly Continuity and Nearly Openness

In this section we define and investigate nearly continuous and nearly open linear operators in locally convex cones.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. We say that the subset $F \subseteq \mathcal{P}^2$ has the closedness property (CP) if the following holds (see [2]):

(CP) if $(a, b) \notin F$, then there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in F$.

Indeed, the property (CP) is an alternative to closedness in quasiuniform structures.

Definition 2.1. *The uniformly closure of $F \subseteq \mathcal{P}^2$, un-cl$(F)$, is the smallest set containing F which has (CP). The set F is uniformly closed if $F =$ un-cl(F).*

Lemma 2.1. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. For every subset B of $\mathcal{P}^2$, (un-cl$(B))^\circ = B^\circ$ (the polars being taken in $\mathcal{P}^*$).*

Proof. Since $B \subseteq$ un-cl(B), then (un-cl$(B))^\circ \subseteq B^\circ$. Set

$$\tilde{B} = \{(a, b) \in \mathcal{P}^2 : \mu(a) \leq \mu(b) + 1, \ \forall \mu \in B^\circ\}.$$

We have $B \subseteq \tilde{B}$. It is easy to see that $\tilde{B}$ has (CP). Then un-cl$(B) \subseteq \tilde{B}$ and so $(\tilde{B})^\circ \subseteq$ (un-cl$(B))^\circ \subseteq B^\circ$. It is sufficient to prove that $B^\circ \subseteq (\tilde{B})^\circ$. Let $\mu \in B^\circ$ and $(a, b) \in \tilde{B}$. By the definition of $\tilde{B}$, we have $\mu(a) \leq \mu(b) + 1$. Therefore $\mu \in (\tilde{B})^\circ$. $\square$

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and $T : \mathcal{P} \to \mathcal{Q}$ be a linear operator. The *adjoint operator* $T^* : \mathcal{Q}^* \to \mathcal{L}(\mathcal{P})$ is defined as follows: For any $\mu \in \mathcal{Q}^*$ define the linear functional $T^*(\mu)$ on $\mathcal{P}$ by $T^*(\mu)(a) = \mu(T(a))$ for all $a \in \mathcal{P}$. If T is continuous, then T^* is a linear operator from $\mathcal{Q}^*$ into $\mathcal{P}^*$ (see [4], II.2.15).

Let T^* be the adjoint operator of T. We characterize the subcone

$$T^{*-1}(\mathcal{P}^*) = \{\mu \in \mathcal{Q}^* : T^*(\mu) \in \mathcal{P}^*\}$$

of $\mathcal{Q}^*$ as follows:

Proposition 2.2. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones and let $T : \mathcal{P} \to \mathcal{Q}$ be a linear operator. Then $T^{*-1}(\mathcal{P}^*) = \cup_{U \in \mathfrak{U}}((T \times T)(U))^\circ$ (the polars being taken in $\mathcal{Q}^*$).*

Proof. Let $\mu \in T^{*-1}(\mathcal{P}^*)$. Then $T^*(\mu) \in \mathcal{P}^*$. Hence there exists $U \in \mathfrak{U}$ such that $T^*(\mu) \in U^\circ$, i.e. $T^*(\mu)(a) \le T^*(\mu)(b) + 1$ for all $(a, b) \in U$. Thus $\mu(T(a)) \le \mu(T(b)) + 1$ for all $(a, b) \in U$, i.e. $\mu \in ((T \times T)(U))^\circ$. Conversely, if $\mu \in ((T \times T)(U))^\circ$, then $\mu(T(a)) \le \mu(T(b)) + 1$ for all $(a, b) \in U$ and so $T^*(\mu) \in U^\circ \subseteq \mathcal{P}^*$. Hence $\mu \in T^{*-1}(\mathcal{P}^*)$. □

Lemma 2.3. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones with duals $\mathcal{P}^*$ and $\mathcal{Q}^*$ respectively and let T be a linear operator of $\mathcal{P}$ into $\mathcal{Q}$ with adjoint T^*. Then*

(*a*) *for each subset F of $\mathcal{P}^2$, $T^{*-1}(F^\circ) = ((T \times T)(F))^\circ$,*
(*b*) *if the polar E° being taken in $\mathcal{Q}^*$, for each subset E of $\mathcal{Q}^2$, $T^*(E^\circ) \subseteq ((T \times T)^{-1}(E))^\circ$ and if T is invertible, then we have the inverse inclusion, i.e. $T^*(E^\circ) = ((T \times T)^{-1}(E))^\circ$.*

Proof. (*a*) We have $\mu \in T^{*-1}(F^\circ)$ if and only if $T^*(\mu) \in F^\circ$ if and only if $T^*(\mu)(a) \le T^*(\mu)(b) + 1$ for all $(a, b) \in F$, if and only if $\mu(T(a)) \le \mu(T(b)) + 1$ for all $(a, b) \in F$, if and only if $\mu \in ((T \times T)(F))^\circ$.
(*b*) Let $\mu \in T^*(E^\circ)$. There is $\mu_1 \in E^\circ$ such that $\mu = T^*(\mu_1) = \mu_1 \circ T$. Now, if $(a, b) \in (T \times T)^{-1}(E)$, then $\mu_1(T(a)) \le \mu_1(T(b)) + 1$, i.e. $\mu(a) \le \mu(b) + 1$. Now, let T be invertible. If $\mu \notin T^*(E^\circ)$, then there exists $(a, b) \in E$ such that $T^{*-1}(\mu)(a) > T^{*-1}(\mu)(b) + 1$. Since $(T^*)^{-1} = (T^{-1})^*$, we have $\mu(T^{-1}(a)) > \mu(T^{-1}(b)) + 1$. This yields that $\mu \notin ((T \times T)^{-1}(E))^\circ$. □

We endow $\mathcal{P}^*$ with the canonical algebraic operations and the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. As in locally convex topological vector spaces, the polar U° of a neighborhood $U \in \mathfrak{U}$ is $w(\mathcal{P}^*, \mathcal{P})$-compact and convex (see [4], Proposition II.2.4). A typical neighborhood for $\mu_0 \in \mathcal{P}^*$ in $w(\mathcal{P}^*, \mathcal{P})$ is given via a finite subset $A = \{a_1, a_2, \ldots, a_n\}$ of $\mathcal{P}$ by

$$\omega_A(\mu_0) = \left\{ \mu \in \mathcal{P}^* : \begin{array}{lll} |\mu(a_i) - \mu_0(a_i)| \le 1 & \text{if} & \mu_0(a_i) < +\infty \\ \mu(a_i) \ge 1 & \text{if} & \mu_0(a_i) = +\infty \end{array} \right\}.$$

Similar to topological vector spaces we have:

Proposition 2.4. *The topology $w(\mathcal{P}^*, \mathcal{P})$ is Hausdorff.*

Proof. Let $\mu_1, \mu_2 \in \mathcal{P}^*$ and $\mu_1 \ne \mu_2$. Then there is $a \in \mathcal{P}$ such that $\mu_1(a) \ne \mu_2(a)$. We consider two cases:

(*a*) Suppose $\mu_1(a), \mu_2(a) \ne +\infty$. We set $\varepsilon = \dfrac{3}{| \mu_1(a) - \mu_2(a) |}$. Then

$$\omega_{\{\varepsilon a\}}(\mu_1) \cap \omega_{\{\varepsilon a\}}(\mu_2) = \emptyset.$$

For, if $\mu \in \omega_{\{\varepsilon a\}}(\mu_1) \cap \omega_{\{\varepsilon a\}}(\mu_2)$, then $| \mu(\varepsilon a) - \mu_1(\varepsilon a) | \le 1$ and $| \mu(\varepsilon a) - \mu_2(\varepsilon a) | \le 1$, and so

$$| \mu_1(a) - \mu_2(a) | \le \frac{2}{\varepsilon} = \frac{2}{3} | \mu_1(a) - \mu_2(a) |,$$

which is a contradiction.

(*b*) Suppose $\mu_1(a) = +\infty$ (the case $\mu_2(a) = +\infty$ is similar). We set $\varepsilon = 1 + \mid \mu_2(a) \mid$. Then

$$\omega_{\left\{\frac{a}{2\varepsilon}\right\}}(\mu_1) \cap \omega_{\{a\}}(\mu_2) = \emptyset.$$

For, if $\mu \in \omega_{\left\{\frac{a}{2\varepsilon}\right\}}(\mu_1) \cap \omega_{\{a\}}(\mu_2)$, then $\mu_2(a) - 1 \leq \mu(a) \leq 1 + \mu_2(a)$ and $\mu(\frac{a}{2\varepsilon}) \geq 1$. Thus

$$2(1 + \mid \mu_2(a) \mid) = 2\varepsilon \leq \mu(a) \leq 1 + \mu_2(a),$$

which is a contradiction. □

Proposition 2.5. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones. If $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is a linear operator, then its adjoint $T^* : (\mathcal{Q}^*, w(\mathcal{Q}^*, \mathcal{Q})) \to (\mathcal{L}(\mathcal{P}), w(\mathcal{L}(\mathcal{P}), \mathcal{P}))$ is continuous.*

Proof. Let $\mu_0 \in \mathcal{Q}^*$ and $\omega_A(\nu_0)$ be a neighborhood of $\nu_0 = T^*(\mu_0)$ in the topology $w(\mathcal{L}(\mathcal{P}), \mathcal{P})$, where $A = \{a_1, a_2, .., a_n\}$ is a finite subset of $\mathcal{P}$. The neighborhood $\omega_A(\nu_0)$ is given by

$$\omega_A(\nu_0) = \left\{ \nu \in \mathcal{L}(\mathcal{P}) : \begin{array}{lll} |\nu(a_i) - \nu_0(a_i)| \leq 1 & \text{if} & \nu_0(a_i) < +\infty \\ \nu(a_i) \geq 1 & \text{if} & \nu_0(a_i) = +\infty \end{array} \right\}.$$

If we set $B = \{T(a_1), T(a_2), ..., T(a_n)\}$, then it is easy to check that $T^*(\omega_B(\mu_0)) \subseteq \omega_A(\nu_0)$. □

Definition 2.2. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with dual $\mathcal{P}^*$. We shall say that the subset C^* of $\mathcal{P}^*$ is nearly closed if $C^* \cap U^\circ$ is $w(\mathcal{P}^*, \mathcal{P})$-closed for every $U \in \mathfrak{U}$. Then $(\mathcal{P}, \mathfrak{U})$ is called fully complete, if every nearly closed subcone of $\mathcal{P}^*$ is $w(\mathcal{P}^*, \mathcal{P})$-closed.*

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ whenever for every $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

Proposition 2.6. *The fully complete cone $(\mathcal{P}, \mathfrak{U})$ remains fully complete under any convex quasiuniform structure finer than $\mathfrak{U}$.*

Proof. It follows from the definition. □

Example 2.3. The extended real numbers system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ endowed with the convex quasiuniform structure $\widetilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$ is a fully complete locally convex cone. Indeed, $\overline{\mathbb{R}}^* = [0, +\infty) \cup \{\bar{0}\}$, where $\bar{0}(x) = 0$ for all $x \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$. All subcones of $\overline{\mathbb{R}}^*$ are $\{0\}$, $\{0, \bar{0}\}$, $\overline{\mathbb{R}}^*$ and $[0, \infty)$. On the other hand, the subsets $(\tilde{\varepsilon})^\circ$ are $w(\overline{\mathbb{R}}^*, \overline{\mathbb{R}})$-compact for all $\varepsilon > 0$ (see [4], Proposition II.2.4). Then the intersection of $(\tilde{\varepsilon})^\circ$ with each of the first three subcones is $w(\overline{\mathbb{R}}^*, \overline{\mathbb{R}})$-compact and then $w(\overline{\mathbb{R}}^*, \overline{\mathbb{R}})$-closed (since $w(\overline{\mathbb{R}}^*, \overline{\mathbb{R}})$ is Hausdorff). This means that these subcones are nearly closed. We note that the subcone $[0, +\infty)$ is not nearly closed. In fact for each $\varepsilon > 0$, $[0, +\infty) \cap (\tilde{\varepsilon})^\circ = [0, \frac{1}{\varepsilon}]$ which is not $w(\overline{\mathbb{R}}^*, \overline{\mathbb{R}})$-closed, since $\bar{0}$ is in the closure of $[0, \frac{1}{\varepsilon}]$. We conclude that all nearly closed subcones of $\overline{\mathbb{R}}^*$ are $w(\overline{\mathbb{R}}^*, \overline{\mathbb{R}})$-closed, so $\overline{\mathbb{R}}$ is fully complete.

Definition 2.4. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. We say that the linear operator $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is nearly open if for every $U \in \mathfrak{U}$ there exists $W \in \mathcal{W}$ such that $W \subseteq$ un-cl$((T \times T)(U))$.*

Definition 2.5. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. The linear operator $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is called nearly continuous if for every $W \in \mathcal{W}$ there exists $U \in \mathfrak{U}$ such that $U \subseteq$ un-cl$((T \times T)^{-1}(W))$.*

Lemma 2.7. *If T is a nearly open linear operator of $(\mathcal{P}, \mathfrak{U})$ into $(\mathcal{Q}, \mathcal{W})$, with adjoint T^* (mapping $\mathcal{Q}^*$ into $\mathcal{L}(\mathcal{P})$), and D^* is a nearly closed subcone of $\mathcal{Q}^*$ and the polars of the subsets of $\mathcal{Q}^2$ are in $\mathcal{Q}^*$, then $T^*(D^*) \cap \mathcal{P}^*$ is nearly closed.*

Proof. We show that $T^*(D^*) \cap \mathcal{P}^* \cap U^\circ$ is $w(\mathcal{P}^*, \mathcal{P})$-closed for all $U \in \mathfrak{U}$. If $U \in \mathfrak{U}$, then

$$T^*(D^*) \cap \mathcal{P}^* \cap U^\circ = T^*(D^*) \cap U^\circ = T^*(D^* \cap T^{*-1}(U^\circ)).$$

Since T is nearly open, there is $W \in \mathcal{W}$ such that $W \subseteq$ un-cl$((T \times T)(U))$ and so un-cl$((T \times T)(U)) \in \mathcal{W}$. Thus $((T \times T)(U))^\circ = (\text{un-cl}((T \times T)(U)))^\circ$ is $w(Q^*, Q)$-compact. Since D^* is nearly closed, $D^* \cap ((T \times T)(U))^\circ$ is $w(Q^*, Q)$-closed and so it is $w(Q^*, Q)$-compact. Now by Lemma 2.3(a), $T^{*-1}(U^\circ) = ((T \times T)(U))^\circ$. By Proposition 2.5, T^* is continuous under the topologies $w(Q^*, Q)$ and $w(\mathcal{L}(\mathcal{P}), \mathcal{P})$, hence $T^*(D^* \cap T^{*-1}(U^\circ))$ is $w(\mathcal{L}(\mathcal{P}), \mathcal{P})$-compact and so $w(\mathcal{P}^*, \mathcal{P})$-compact. Since $w(\mathcal{P}^*, \mathcal{P})$ is Hausdorff (Proposition 2.4), therefore $T^*(D^*) \cap \mathcal{P}^* \cap U^\circ = T^*(D^* \cap T^{*-1}(U^\circ))$ is $w(\mathcal{P}^*, \mathcal{P})$-closed. □

Lemma 2.8. *For every locally convex cone $(\mathcal{P}, \mathfrak{U})$, $(\mathcal{P} \times \mathcal{P})^\circ = \{0\}$.*

Proof. It is straightforward. □

Proposition 2.9. *If there is a continuous nearly open linear operator of a fully complete cone onto a locally convex cone $(Q, \mathcal{W})$, then $(Q, \mathcal{W})$ is fully complete.*

Proof. Let T be a continuous nearly open linear operator of the fully complete cone $(\mathcal{P}, \mathfrak{U})$ onto $(Q, \mathcal{W})$, and let D^* be a nearly closed subcone of the dual Q^* of Q. Since T is continuous, T^* maps Q^* into the dual $\mathcal{P}^*$ of $\mathcal{P}$ and by Lemma 2.7, $T^*(D^*)$ is nearly closed. Thus $T^*(D^*)$ is $w(\mathcal{P}^*, \mathcal{P})$-closed, since $\mathcal{P}$ is fully complete. By Lemmas 2.8 and 2.3(a),

$$T^{*-1}(\{0\}) = T^{*-1}((\mathcal{P} \times \mathcal{P})^\circ) = ((T \times T)(\mathcal{P} \times \mathcal{P}))^\circ = (Q \times Q)^\circ = \{0\}.$$

Thus $D^* = T^{*-1}(T^*(D^*))$. By Proposition 2.5, D^* is $w(Q^*, Q)$-closed. Thus Q is fully complete. □

Proposition 2.10. *If T is a nearly continuous linear operator of $(\mathcal{P}, \mathfrak{U})$ into $(Q, \mathcal{W})$, with adjoint T^* (mapping Q^* into $\mathcal{L}(\mathcal{P})$), and if C^* is a nearly closed subcone of $\mathcal{P}^*$ and the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$, then $T^{*-1}(C^*)$ is nearly closed.*

Proof. Let $W \in \mathcal{W}$. We set $K^* = T^{*-1}(C^*)$ and $V = (T \times T)^{-1}(W)$. There is $U \in \mathfrak{U}$ such that $U \subseteq$ un-cl$((T \times T)^{-1}(W)) =$ un-cl(V), since T is nearly continuous. Therefore, since C^* is nearly closed, $C^* \cap (\text{un-cl}(V))^\circ$ is $w(\mathcal{P}^*, \mathcal{P})$-compact and so $w(\mathcal{L}(\mathcal{P}), \mathcal{P})$-compact, and then $w(\mathcal{L}(\mathcal{P}), \mathcal{P})$-closed. Hence by Proposition 2.5, $T^{*-1}(C^* \cap (\text{un-cl}(V))^\circ)$ is $w(Q^*, Q)$-closed. We have

$$T^{*-1}(C^* \cap (\text{un-cl}(V))^\circ) = T^{*-1}(C^* \cap V^\circ) = K^* \cap T^{*-1}(V^\circ) = K^* \cap ((T \times T)(V))^\circ.$$

Thus

$$K^* \cap W^\circ = K^* \cap W^\circ \cap ((T \times T)(V))^\circ = T^{*-1}(C^* \cap (\text{un-cl}(V))^\circ) \cap W^\circ,$$

which is an intersection of $w(Q^*, Q)$-closed sets. Thus K^* is nearly closed. □

A *barrel* is a convex subset B of $\mathcal{P}^2$ with the following properties:

($B1$) For every $b \in \mathcal{P}$ there is $U \in \mathfrak{U}$ such that for every $a \in U(b)U$ there is a $\lambda > 0$ such that $(a, b) \in \lambda B$.

($B2$) For all a, b such that $(a, b) \notin B$ there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in B$ and $\mu(a) > \mu(b) + 1$ (see [10]).

It is clear that for every barrel B, we have un-cl$(B) = B$.

An upper-barreled cone is defined in [6] as follows:
Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. Then $\mathcal{P}$ is called *upper-barreled* if for every barrel $B \subseteq \mathcal{P}^2$, there is $U \in \mathfrak{U}$ such that $U \subseteq B$.

Proposition 2.11. *Let $(\mathcal{P}, \mathfrak{U})$ and $(Q, \mathcal{W})$ be two locally convex cones and let T be a continuous and nearly open linear operator of $\mathcal{P}$ into Q. If $(\mathcal{P}, \mathfrak{U})$ is upper-barreled, so is $(Q, \mathcal{W})$.*

Proof. Let B be a barrel in $\mathcal{Q}$. Since T is continuous, $(T \times T)^{-1}(B)$ is a barrel in $\mathcal{P}$ by Lemma 4.4 of [6] and therefore there exists $U \in \mathfrak{U}$ such that $U \subseteq (T \times T)^{-1}(B)$, because $\mathcal{P}$ is upper-barrelled. Since T is nearly open, there exists $W \in \mathcal{W}$ such that $W \subseteq$ un-cl$((T \times T)(U))$. Thus

$$W \subseteq \text{un-cl}((T \times T)((T \times T)^{-1}(B))) \subseteq \text{un-cl}(B) = B.$$

□

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. The subset $A \subseteq \mathcal{P}^*$ is called *(uniformly) equicontinuous*, if there is some $U \in \mathfrak{U}$ such that for all $a, b \in \mathcal{P}$ and $\mu \in A$, $(a, b) \in U$ implies $\mu(a) \leq \mu(b) + 1$. In other words, the subset $A \subseteq \mathcal{P}^*$ is equicontinuous if and only if there is some $U \in \mathfrak{U}$ such that $A \subseteq U^\circ$. Thus for every $U \in \mathfrak{U}$, U° is an equicontinuous subset of $\mathcal{P}^*$ ([4], IV.1.1).

Proposition 2.12. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones, T be a linear operator of $\mathcal{P}$ into $\mathcal{Q}$ and the polars of the subsets of $\mathcal{Q}^2$ be in $\mathcal{Q}^*$. Then T is nearly open if and only if for each equicontinuous subset B of $\mathcal{P}^*$, $T^{*-1}(B)$ is an equicontinuous subset of $\mathcal{Q}^*$.*

Proof. Let B be an equicontinuous subset of $\mathcal{P}^*$. Then $B \subseteq U^\circ$ for some $U \in \mathfrak{U}$. Since T is nearly open, there is $W \in \mathcal{W}$ such that $W \subseteq$ un-cl$((T \times T)(U))$. Therefore by Lemmas 2.1 and 2.3(a),

$$T^{*-1}(B) \subseteq T^{*-1}(U^\circ) = ((T \times T)(U))^\circ = (\text{un-cl}((T \times T)(U)))^\circ \subseteq W^\circ.$$

To prove the converse, let $U \in \mathfrak{U}$. The set U° is an equicontinuous subset of $\mathcal{P}^*$. So by the assumption, $T^{*-1}(U^\circ)$ is an equicontinuous subset of $\mathcal{Q}^*$. Thus there is $W \in \mathcal{W}$ such that $((T \times T)(U))^\circ = T^{*-1}(U^\circ) \subseteq W^\circ$. We claim that $W \subseteq$ un-cl$((T \times T)(U))$. For, if $(a, b) \notin$ un-cl$((T \times T)(U))$, then by (CP), there is $\mu \in \mathcal{Q}^*$ such that $\mu(a) > \mu(b)+1$ and $\mu(c) \leq \mu(d)+1$ for all $(c, d) \in$ un-cl$((T\times T)(U))$. Hence $\mu \in (\text{un-cl}((T\times T)(U)))^\circ = ((T\times T)(U))^\circ$. Thus $\mu \in W^\circ$ and so $(a, b) \notin W$. □

Proposition 2.13. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones and the polars of the subsets of $\mathcal{P}^2$ be in $\mathcal{P}^*$.*

- (a) *If T is nearly continuous, then its adjoint T^* maps an equicontinuous subset of $\mathcal{Q}^*$ into an equicontinuous subset of $\mathcal{P}^*$.*
- (b) *If T is invertible and its adjoint T^* maps equicontinuous subsets of $\mathcal{Q}^*$ into equicontinuous subsets of $\mathcal{P}^*$, then T is nearly continuous.*

Proof. (a) Let $B \subseteq \mathcal{Q}^*$ be an equicontinuous set. There exists $W \in \mathcal{W}$ such that $B \subseteq W^\circ$. Because T is nearly continuous, there is $U \in \mathfrak{U}$ such that $U \subseteq$ un-cl$((T \times T)^{-1}(W))$. Thus by Lemma 2.1 and Lemma 2.3(b),

$$T^*(B) \subseteq T^*(W^\circ) \subseteq ((T \times T)^{-1}(W))^\circ = (\text{un-cl}((T \times T)^{-1}(W)))^\circ \subseteq U^\circ.$$

(b) Let $W \in \mathcal{W}$. Then W° is an equicontinuous subset of $\mathcal{Q}^*$. By the hypothesis, $T^*(W^\circ)$ is an equicontinuous subset of $\mathcal{P}^*$. Thus there exists $U \in \mathfrak{U}$ such that $T^*(W^\circ) \subseteq U^\circ$. We claim that $U \subseteq$ un-cl$((T \times T)^{-1}(W))$. Indeed, if $(a, b) \notin$ un-cl$((T \times T)^{-1}(W))$, there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in$ un-cl$((T \times T)^{-1}(W))$, because un-cl$((T \times T)^{-1}(W))$ has (CP). We have $\mu \in (\text{un-cl}((T \times T)^{-1}(W)))^\circ$. Since T is invertible, by Lemma 2.3(b),

$$(\text{un-cl}((T \times T)^{-1}(W)))^\circ = ((T \times T)^{-1}(W))^\circ = T^*(W^\circ) \subseteq U^\circ.$$

Therefore $\mu \in U^\circ$ and so $(a, b) \notin U$. □

We shall say that the subset F of $\mathcal{P}^2$ is u-bounded if it is absorbed by each $U \in \mathfrak{U}$. The subset B of $\mathcal{P}$ is called bounded below (or above) whenever $\{0\} \times B$ (or $B \times \{0\}$) is u-bounded. The subset B is called bounded if it is bounded below and above. An element $a \in \mathcal{P}$ is called bounded below (or above) whenever $\{a\}$ is so (see [2]).

Theorem 2.14. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. If all of the elements of $\mathcal{Q}$ are bounded and $\mathcal{P}$ is upper-barrelled, then:*

(a) *Any linear mapping T of $\mathcal{P}$ into $\mathcal{Q}$ is nearly continuous.*
(b) *Any linear mapping S of $\mathcal{Q}$ onto $\mathcal{P}$ is nearly open.*

Proof. (a) Let $W \in \mathcal{W}$. We show that un-cl$((T\times T)^{-1}(W))$ is a barrel in $\mathcal{P}$, since $\mathcal{P}$ is upper-barrelled. Because un-cl$((T\times T)^{-1}(W))$ has (CP), it is sufficient to prove condition $(B1)$ for un-cl$((T\times T)^{-1}(W))$. Let $b \in \mathcal{P}$. There is a $\lambda > 0$ such that $(0, T(b)) \in \lambda W$, since $\mathcal{Q}$ is a locally convex cone. Let $U \in \mathfrak{U}$ be arbitrary. If $a \in U(b)U$, then by the hypothesis, $T(a)$ is bounded above, therefore there is a $\rho > 0$ such that $(T(a), 0) \in \rho W$. Then $(T(a), T(b)) \in (\lambda W) \circ (\rho W) \subseteq (\lambda + \rho)W$. Hence $(a, b) \in (\lambda + \rho)$un-cl$((T \times T)^{-1}(W))$. Thus there is $V \in \mathfrak{U}$ such that $V \subseteq$ un-cl$((T\times T)^{-1}(W))$, and so T is nearly continuous.

(b) We prove that un-cl$((S \times S)(W))$ is a barrel in $\mathcal{P}$. Let $b \in \mathcal{P}$. There is $x \in \mathcal{Q}$ such that $S(x) = b$, since S is onto. There is a $\lambda > 0$ such that $(0, x) \in \lambda W$. Thus $(0, S(x)) \in \lambda(S \times S)(W)$. Let $U \in \mathfrak{U}$ be arbitrary. If $a \in U(b)U$, then there is a $y \in \mathcal{Q}$ such that $S(y) = a$. Now y is bounded above, thus there is a $\rho > 0$ such that $(y, 0) \in \rho W$. Hence $(S(y), 0) \in \rho(S \times S)(W)$. Therefore

$$(a, b) = (S(y), S(x)) \in (\lambda + \rho)(S \times S)(W) \subseteq (\lambda + \rho)\text{un-cl}((S \times S)(W)).$$

□

3. Open Mapping Type Theorems

Walter Roth proved a Uniform Boundedness theorem and Hahn-Banach type Theorems for locally convex cones in [10] and [11] respectively. In this section of this paper, we prove some open mapping type theorems for locally convex cones.

Definition 3.1. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. The linear operator $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is called (uniformly) open* if for every $U \in \mathfrak{U}$ one can find $W \in \mathcal{W}$ such that $W \subseteq (T \times T)(U)$.

It is easy to see that if $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is open, then it is open under the upper, lower and symmetric topologies. If $T : \mathcal{P} \to \mathcal{Q}$ is open, then T is surjective. Indeed, let $q \in \mathcal{Q}$. For every $U \in \mathfrak{U}$ there is $W \in \mathcal{W}$ such that $W \subseteq (T \times T)(U)$. By (U_5), there is a $\lambda > 0$ such that $(0, q) \in \lambda W$. Thus $(0, q) \in (T \times T)(\lambda U)$. Hence there is a $p \in \mathcal{P}$ such that $(0, q) = (T \times T)(0, \lambda p)$, that is $q = T(\lambda p)$.

We say that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ has the *strict separation property* if the following holds:

(SP) For all $a, b \in \mathcal{P}$ and $U \in \mathfrak{U}$ such that $(a, b) \notin \rho U$ for some $\rho > 1$, there is a linear functional $\mu \in U^\circ$ such that $\mu(a) > \mu(b) + 1$ (see [4], II.2.12).

Theorem 3.1. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones.*

(i) If T from $\mathcal{P}$ into $\mathcal{Q}$ is nearly continuous, $\mathcal{Q}$ has (SP) and the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$, then T is continuous.
(ii) If T from $\mathcal{P}$ onto $\mathcal{Q}$ is nearly open, $\mathcal{P}$ has (SP) and the polars of the subsets of $\mathcal{Q}^2$ are in $\mathcal{Q}^*$, then T is open.

Proof. Let $W \in \mathcal{W}$ be arbitrary. Since T is nearly continuous, there exists $U' \in \mathfrak{U}$ such that $U' \subseteq$ un-cl$((T \times T)^{-1}(\frac{1}{2}W))$. Then (un-cl$((T \times T)^{-1}(\frac{1}{2}W)))^\circ \subseteq U'^\circ$. By Lemma 2.1 and Lemma 2.3(b), we have

$$((T \times T)^{-1}(\frac{1}{2}W))^\circ = (\text{un-cl}((T \times T)^{-1}(\frac{1}{2}W)))^\circ$$

and

$$T^*((\frac{1}{2}W)^\circ) \subseteq ((T \times T)^{-1}(\frac{1}{2}W))^\circ.$$

We set $U = \frac{1}{2}U'$, then $(T \times T)(U) \subseteq W$. For, if $(a, b) \in U$ and $(T(a), T(b)) \notin W$, then there exists $\mu \in (\frac{1}{2}W)^\circ$ such that $\mu(T(a)) > \mu(T(b)) + 1$, since $\mathcal{Q}$ has (SP). But $T^*(\mu) \in U'^\circ$. Therefore $T^*(\mu)(2a) \le T^*(\mu)(2b) + 1$, that is $(\mu \circ T)(a) \le (\mu \circ T)(b) + \frac{1}{2}$, which is a contraction. The proof of (ii) is similar to (i). □

A locally convex cone $(\mathcal{P}, \mathfrak{U})$ is said to be *tightly covered by its bounded elements* if for all $a, b \in \mathcal{P}$ and $U \in \mathfrak{U}$ such that $a \notin U(b)$ there is some bounded element $a' \in \mathcal{P}$ such that $(a', a) \in V$ for all $V \in \mathfrak{U}$ and $a' \notin U(b)$ ([4], II.2.13).

Corollary 3.2. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones.*

(i) If T from $\mathcal{P}$ into $\mathcal{Q}$ is nearly continuous, $(\mathcal{Q}, \mathcal{W})$ is tightly covered by its bounded elements and the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$, then T is continuous.

(ii) If T from $\mathcal{P}$ onto $\mathcal{Q}$ is nearly open, $(\mathcal{P}, \mathfrak{U})$ is tightly covered by its bounded elements and the polars of the subsets of $\mathcal{Q}^2$ are in $\mathcal{Q}^*$, then T is open.

Proof. Since by the separation Theorem ([4], II.2.14), every locally convex cone $(\mathcal{Q}, \mathcal{W})$ which is tightly covered by its bounded elements has strict separation property (SP). □

We note that every locally convex cone which elements are bounded, is tightly covered by its bounded elements. Thus Theorem 2.14 and Corollary 3.2 give immediately:

Theorem 3.3. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. Let all of the elements of $\mathcal{Q}$ be bounded and $\mathcal{P}$ be upper-barrelled.*

(a) If the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$, then any linear mapping T of $\mathcal{P}$ into $\mathcal{Q}$ is continuous.

(b) If the polars of the subsets of $\mathcal{Q}^2$ are in $\mathcal{Q}^*$, then any linear mapping S of $\mathcal{Q}$ onto $\mathcal{P}$ is open.

Lemma 3.4. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones, $T : \mathcal{P} \to \mathcal{Q}$ be a linear operator and the polars of the subsets of $\mathcal{P}^2$ be in $\mathcal{P}^*$. Then for every $W \in \mathcal{W}$,*

$$\textit{un-cl}((T \times T)^{-1}(W)) \subseteq (T \times T)^{-1}(\textit{un-cl}(W)).$$

Proof. Let$(a, b) \notin (T \times T)^{-1}(\text{un-cl}(W))$. Since un-cl$(W)$ has (CP), there exists $\mu \in \mathcal{Q}^*$ such that $\mu(T(a)) > \mu(T(b)) + 1$ and $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in$ un-cl(W). We have $\mu \in (\text{un-cl}(W))^\circ = W^\circ$. Hence by Lemma 2.3$(b)$ and Lemma 2.1,

$$T^*(\mu) \in T^*(W^\circ) \subseteq ((T \times T)^{-1}(W))^\circ = (\text{un-cl}((T \times T)^{-1}(W)))^\circ.$$

Now if $(a, b) \in \text{un-cl}((T \times T)^{-1}(W))$, then $T^*(\mu)(a) \le T^*(\mu)(b) + 1$ and so $(\mu \circ T)(a) \le (\mu \circ T)(b) + 1$, which is a contradiction. □

It was proved in [6] that in a locally convex cone $(\mathcal{P}, \mathfrak{U})$, the set of all barrels $\mathfrak{B}$ is a convex quasiuniform structure on $\mathcal{P}$.

Proposition 3.5. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathfrak{B})$ be locally convex cones, where $\mathfrak{B}$ is the convex qusiuniform structure of all barrels. If $T : \mathcal{P} \to \mathcal{Q}$ is a linear operator and the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$, then T is continuous.*

Proof. Let $B \in \mathfrak{B}$. Then there is $U \in \mathfrak{U}$ such that $U \subseteq \text{un-cl}((T \times T)^{-1}(B))$. Therefore by Lemma 3.4, we have $U \subseteq (T \times T)^{-1}(\text{un-cl}(B)) = (T \times T)^{-1}(B)$. □

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone and $\mathcal{P}^*$ be its dual cone. We suppose that $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*)$ is the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $\mu \in \mathcal{P}^*$ continuous. The finite intersections of the sets $(\mu \times \mu)^{-1}(\tilde{\varepsilon})$ where $\mu \in \mathcal{P}^*$, $\varepsilon > 0$ and $\tilde{\varepsilon} = \{(a, b) \in \mathbb{R}^2 : a \le b + \varepsilon\}$, form a base for $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*)$. We call $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*)$ the weak convex quasiuniform structure on $\mathcal{P}$. We say that $T : \mathcal{P} \to \mathcal{Q}$ is weakly continuous or σ-continuous whenever it is continuous with respect to the weak convex quasiuniform structures on $\mathcal{P}$ and $\mathcal{Q}$.

Lemma 3.6. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and T be a linear operator from $\mathcal{P}$ into $\mathcal{Q}$ with adjoint T^*. If $T^*(\mathcal{Q}^*) \subseteq \mathcal{P}^*$, then T is weakly continuous.*

Proof. Let $W \in \mathcal{W}_\sigma(Q, Q^*)$. Then there are $n \in \mathbb{N}$ and $\mu_1, ..., \mu_n \in Q^*$ such that

$$\bigcap_{i=1}^{n} (\mu_i \times \mu_i)^{-1}(\tilde{1}) \subseteq W.$$

Since for every $i = 1, ..., n$, $T^*(\mu_i) \in \mathcal{P}^*$, then

$$U = \bigcap_{i=1}^{n} (T^*(\mu_i) \times T^*(\mu_i))^{-1}(\tilde{1}) \in \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*).$$

We have $(T \times T)(U) \subseteq W$. For, if $(a, b) \in \bigcap_{i=1}^{n} (T^*(\mu_i) \times T^*(\mu_i))^{-1}(\tilde{1})$. Then we have $\mu_i(T(a)) \leq \mu_i(T(b)) + 1$ for every $i = 1, ..., n$, that is

$$(T(a), T(b)) \in \bigcap_{i=1}^{n} (\mu_i \times \mu_i)^{-1}(\tilde{1}) \subseteq W.$$

Thus T is weakly continuous. □

If $\mathcal{P}$ is a cone with a convex quasiuniform structure $\mathfrak{U}$, then one can find a preorder (a reflexive transitive relation $\leq$) compatible by the algebraic operations and an (abstract) 0-neighborhood system $\mathcal{V}$ (a subcone of $\mathcal{P}$ without 0 directed towards 0) such that the convex quasiuniform structure $\widetilde{\mathcal{V}} = \{\tilde{v} : v \in \mathcal{V}\}$ is equivalent to $\mathfrak{U}$, where $\tilde{v} = \{(a, b) : a \leq b + v\}$ (see [4], I.5.5).

A locally convex quotient cone was studied in [7] by means the ordered structure. Also locally convex quotient cones investigated by Walter Roth in [9]. First we review the instruction of a locally convex quotient cone from [7] briefly. Let $\mathcal{P}$ be a cone and $\mathcal{M}$ be a subcone of $\mathcal{P}$. We consider the relation $\sim$ on $\mathcal{P}$ as $x \sim y$ if and only if $x + \mathcal{M} = y + \mathcal{M}$. This equivalence relation means that $x \sim y$ if both $x = y + m'$ and $y = x + m$ for some $m', m \in \mathcal{M}$. The equivalence class $\tilde{x}$ is a subset of $\hat{x} = x + \mathcal{M}$. The mapping $k(x) = \hat{x}$ is a linear mapping, which is called the canonical mapping of $\mathcal{P}$ onto $\mathcal{P}/\mathcal{M}$. If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone (by a preorder relation constructed) then the set $\mathcal{P}/\mathcal{M} = \{\hat{x} : x \in \mathcal{P}\}$ with $\hat{\mathcal{V}} = \{\hat{v} : v \in \mathcal{V}\}$ is a locally convex quotient cone. $\hat{x} \leq \hat{y} + \hat{v}$ means that for each $m \in \mathcal{M}$ there is a $m' \in \mathcal{M}$ such that $x + m \leq y + m' + v$. This implies in particular that $x \leq y + m' + v$ for some $m' \in \mathcal{M}$. It was proved in [7] that

(∗) if for each $v \in \mathcal{V}$ and each $m \in \mathcal{M}$, $m \leq v$, i.e. $(m, 0) \in \tilde{v}$,

then k is an open mapping under the lower, upper and symmetric topologies.

Lemma 3.7. *With the condition (∗), $k : (\mathcal{P}, \widetilde{\mathcal{V}}) \to (\mathcal{P}/\mathcal{M}, \widetilde{\hat{\mathcal{V}}})$ is even (uniformly) open.*

Proof. We show that for every $\tilde{v} \in \widetilde{\mathcal{V}}$ there is $\tilde{\hat{w}} \in \widetilde{\hat{\mathcal{V}}}$ such that $\tilde{\hat{w}} \subseteq (k \times k)(\tilde{v})$. Set $w = \frac{v}{2}$. If $(\hat{x}, \hat{y}) \in \tilde{\hat{w}}$, i.e. $\hat{x} \leq \hat{y} + \hat{w}$, then there is $m' \in \mathcal{M}$ such that $x \leq y + m' + w$. Since by the condition (∗), $m' \leq \frac{v}{2}$, we have $x \leq y + v$, i.e. $(x, y) \in \tilde{v}$ and so $(\hat{x}, \hat{y}) = (k(x), k(y)) \in (k \times k)(\tilde{v})$. □

Let $(\mathcal{P}, \mathfrak{U})$ and $(Q, \mathcal{W})$ be locally convex cones. The linear operator $T : \mathcal{P} \to Q$ is called u-bounded whenever for every u-bounded subset B of $\mathcal{P}^2$, $(T \times T)(B)$ is u-bounded. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called bornological if every u-bounded linear operator from $(\mathcal{P}, \mathfrak{U})$ into any locally convex cone is continuous. If $(Q, \mathcal{W})$ is a locally convex cone such that Q^2 has the same u-bounded subsets under $\mathcal{W}$ and $\mathcal{W}_\sigma(Q, Q^*)$ and $(\mathcal{P}, \mathfrak{U})$ is a bornological cone, then the weakly continuous linear operator $T : \mathcal{P} \to Q$ is continuous (see [2]).

Theorem 3.8. *Let $(\mathcal{P}, \mathfrak{U})$ and $(Q, \mathcal{W})$ be two locally convex cones.*

(*a*) *Let T be a linear operator of $\mathcal{P}$ into Q. If $\mathcal{P}$ is bornological, the polars of the subsets of $\mathcal{P}^2$ being taken in $\mathcal{P}^*$ and Q^2 has the same u-bounded subsets under $\mathcal{W}$ and $\mathcal{W}_\sigma(Q, Q^*)$, then T is continuous.*

(*b*) *Let T be a linear operator of $\mathcal{P}$ onto Q such that the condition (∗) holds for $T^{-1}(0)$ and $T(a) = T(b)$ imply $a + T^{-1}(0) = b + T^{-1}(0)$ for all $a, b \in \mathcal{P}$. If Q is bornological, the polars of the subsets of Q^2 being taken in Q^* and $\mathcal{P}^2$ has the same u-bounded subsets under $\mathfrak{U}$ and $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*)$, then T is open.*

Proof. Let T^* be the adjoint operator. Then by Lemma 2.3,

$$T^*(Q^*) = T^*(\bigcup_{W\in\mathcal{W}} W^\circ) \subseteq \bigcup_{W\in\mathcal{W}} T^*(W^\circ) \subseteq \bigcup_{W\in\mathcal{W}} ((T\times T)^{-1}(W))^\circ.$$

Since polars are in $\mathcal{P}^*$, $T^*(Q^*) \subseteq \mathcal{P}^*$. Hence by Lemma 3.6, T is weakly continuous. Since $\mathcal{P}$ is bornological by the hypothesis, T is continuous by the above description.

(*b*) We can write $T = s\circ k$, where k is the canonical mapping of $\mathcal{P}$ onto $\mathcal{P}/T^{-1}(0)$ and s is the linear mapping of $\mathcal{P}/T^{-1}(0)$ onto Q. The mapping s is one-to-one. For, if $s(a+T^{-1}(0)) = s(b+T^{-1}(0))$, then $(s\circ k)(a) = (s\circ k)(b)$, i.e. $T(a) = T(b)$. By the hypothesis, $a + T^{-1}(0) = b + T^{-1}(0)$. Since k is continuous, $\mathcal{P}/T^{-1}(0)$ has the same u-bounded subsets under $\hat{\mathfrak{U}}$ and $\hat{\mathfrak{U}}_\sigma$. Therefore, by (*a*), s^{-1} is continuous. Thus s is open. On the other hand k is open by Lemma 3.7. Therefore T is open. □

Corollary 3.9. *If μ is a linear functional on a bornological cone $(\mathcal{P},\mathfrak{U})$ and the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$, then μ is continuous.*

Proof. Since $(\overline{\mathbb{R}},\widetilde{\mathcal{V}})$ has the same u-bounded subsets under $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{V}}_\sigma(\overline{\mathbb{R}},\overline{\mathbb{R}}^*)$ (see [2]). □

The locally convex cone $(\mathcal{P},\mathfrak{U})$ is called a *uc*-cone whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$. The *uc*-cones in locally convex cones play the role of normed spaces in topological vector spaces. Every *uc*-cone is bornological (see [2]).

Corollary 3.10. *Let $(\mathcal{P},\mathfrak{U})$ and $(Q,\mathcal{W})$ be two locally convex cones.*

(*a*) *If T is a linear operator of $\mathcal{P}$ into Q, $\mathcal{P}$ is a uc-cone, the polars of the subsets of $\mathcal{P}^2$ are in $\mathcal{P}^*$ and Q^2 has the same u-bounded subsets under $\mathcal{W}$ and $\mathcal{W}_\sigma(Q,Q^*)$, then T is continuous.*

(*b*) *Let T be a linear operator of $\mathcal{P}$ onto Q such that the condition $(*)$ holds for $T^{-1}(0)$ and $T(a) = T(b)$ imply $a + T^{-1}(0) = b + T^{-1}(0)$ for all $a, b \in \mathcal{P}$. If Q is a uc-cone, the polars of the subsets of Q^2 are in Q^* and $\mathcal{P}^2$ has the same u-bounded subsets under $\mathfrak{U}$ and $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$, then T is open.*

We illustrate Theorems 2.14, 3.1 and 3.8 in the following example.

Example 3.2. Let $(E,\|.\|)$ be a normed vector space with unit ball B. Let $\mathcal{P}$ be the subcone of *Conv*(E) (the cone of all non-empty convex subsets of E) consisting of all sets $a+\rho B, a\in E, \rho\geq 0$. Then the dual cone of $\mathcal{P}$ may be easily discribed: It is the set of all $\mu\oplus r$, where $r\geq 0$ and μ is a linear functional on E such that $\|\mu\|\leq r$, if we define $(\mu\oplus r)(a+\rho B) = \mu(a)+r\rho$. The polar of B (as an abstract neighborhood for $\mathcal{P}$) consists of those $\mu\oplus r$ with $r\leq 1$ (see [4], II, Example 2.17). Set

$$\tilde{B} = \{(a+\rho_a B, c+\rho_c B)\in\mathcal{P}^2 \ : \ a+\rho_a B\subseteq c+\rho_c B+B\}$$

and $\mathfrak{U} = \{\alpha\tilde{B} \ : \ \alpha > 0\}$. The locally convex cone $(\mathcal{P},\mathfrak{U})$ is a *uc*-cone and has (*SP*), since it is tightly covered by its bounded elements (in fact all elements of $\mathcal{P}$ are bounded). The cone $\mathcal{P}$ is fully complete. In fact, all subcones of $\mathcal{P}^*$ are $\{0\oplus 0\}$, $\{0\oplus r, r\geq 0\}$ and $\mathcal{P}^*$, which all of them are nearly closed and $w(\mathcal{P}^*,\mathcal{P})$-closed.

Now, let $(F,\|.\|)$ be another normed vector space with unit ball U and $Q = \{x+\lambda U : \ x\in F, \ \lambda\geq 0\}$. Let $T : E\to F$ be a surjective linear operator, $\alpha > 0$, such that $\|T(a)\| = \alpha\|a\|$ for all $a\in E$. We extend T to an operator $\overline{T} : \mathcal{P}\to Q$ by $\overline{T}(a+\rho B) = T(a)+\alpha\rho U$ (see [4], II, Example 6.10(d)). Since $\mathcal{P}$ and Q are full (containing the abstract neighborhood systems $\{\alpha B : \ \alpha > 0\}$ and $\{\lambda U : \ \lambda > 0\}$ respectively), then they are upper barreled (see [6]). By the boundedness of the elements of these cones, the operator $\overline{T}$ is nearly continuous and nearly open (Theorem 2.14). Since $\mathcal{P}$ and Q have (*SP*), then $\overline{T}$ is continuous and open (Theorem 3.1). Also it is easy to check that $\mathcal{P}$, Q and $\overline{T}$ satisfy in the conditions of Theorem 3.8.

In the following example we show that a nearly continuous operator is not necessarily continuous.

Example 3.3. We consider the mapping $T : (\overline{\mathbb{R}}_+,\widetilde{\mathcal{V}})\to(\overline{\mathbb{R}}_+,\widetilde{\mathcal{V}})$ defined as

$$T(a) = \begin{cases} 0, & a = 0 \\ +\infty, & a \neq 0. \end{cases}$$

The cone $\overline{\mathbb{R}}_+ = [0, +\infty]$ with $\widetilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$ is a locally convex cone, where

$$\tilde{\varepsilon} = \{(a, b) \in \overline{\mathbb{R}}_+^{\,2} : a \leq b + \varepsilon\}.$$

The dual cone of $(\overline{\mathbb{R}}_+, \widetilde{\mathcal{V}})$ is the positive reals together with $\bar{0}$ which maps all $a \in \mathbb{R}_+$ into 0 and $+\infty$ into $+\infty$ (see [1]). The mapping $T : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a linear operator (we note that T is not linear from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$.). The linear operator T is not continuous, since for each $\varepsilon > 0$, $(\frac{\varepsilon}{2}, 0) \in \tilde{\varepsilon}$ but $(T(\frac{\varepsilon}{2}), T(0)) \notin \tilde{1}$. We show that T is nearly continuous. Let $\varepsilon > 0$ be arbitrary. Then

$$(T \times T)^{-1}(\tilde{\varepsilon}) = \{(a, b) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ : T(a) \leq T(b) + \varepsilon\} = \overline{\mathbb{R}}_+^{\,2} \setminus \{(a, 0) : a > 0\}$$

has not (CP) property. The smallest set containing this set which has (CP) is $\overline{\mathbb{R}}_+^{\,2}$, i.e. un-cl$((T \times T)^{-1}(\tilde{\varepsilon})) = \overline{\mathbb{R}}_+^{\,2}$, and $\tilde{1} \subseteq$ un-cl$((T \times T)^{-1}(\tilde{\varepsilon}))$. Therefore T is nearly continuous.

Also $T^*(\overline{\mathbb{R}}_+^*) \nsubseteq \overline{\mathbb{R}}_+^*$, for example for $\mu = \alpha > 0$, $T^*(\mu) = \mu \circ T = T$ which doesn't belong to $\overline{\mathbb{R}}_+^*$. We have

$$\{0\} = (\text{un-cl}((T \times T)^{-1}(\tilde{\varepsilon})))^\circ \neq ((T \times T)^{-1}(\tilde{\varepsilon}))^\circ = \{0, T\}.$$

This shows that if the polars are not in $\mathcal{P}^*$ ($T \in \mathcal{L}(\overline{\mathbb{R}}_+)$, but $T \notin \overline{\mathbb{R}}_+^*$), Lemma 2.1 dose not hold necessarily.

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