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A New Application of Quasi Monotone Sequences and Quasi Power Increasing Sequences to Factored Infinite Series

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Abstract. In this paper, we generalize a known theorem under more weaker conditions dealing with the generalized absolute Cesàro summability factors of infinite series by using quasi monotone sequences and quasi power increasing sequences. This theorem also includes some new results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [2]). A sequence (d_n) is said to be δ-quasi monotone, if $d_n \to 0$, $d_n > 0$ ultimately, and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and δ= (δ_n) is a sequence of positive numbers (see [3]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma, \gamma)\} = \{n^\sigma(\log n)^\gamma, \gamma \geq 0, 0 < \sigma < 1\}$(see [12]). If we take γ=0, then we get a quasi-σ-power increasing sequence. It is known that every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [11]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [8])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \texttt{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

Let $(\theta_n^{\alpha,\beta})$ be a sequence defined by (see [4])

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1, \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{3}$$

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The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta \mid_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{4}$$

If we take $\beta = 0$, then $\mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [10]). Also, if we take $\beta = 0$ and $\alpha = 1$, then we obtain $\mid C, 1 \mid_k$ summability .

2. Known Result

The following theorem is known dealing with an application of δ-quasi monotone sequence and power increasing sequence.

Theorem 2.1 ([5]). Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (3). Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, and $\mid \Delta\lambda_n \mid \leq \mid A_n \mid$ for all n. If the condition

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{5}$$

satisfies, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha, \beta \mid_k$, $0 < \alpha \leq 1$, $\alpha + \beta > 0$, and $k \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem A under more weaker conditions. We shall prove the following theorem.

Theorem 3. 1 Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (3). Let (X_n) be a quasi-f-power increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\Delta A_n \leq \delta_n$, $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, and $\mid \Delta\lambda_n \mid \leq \mid A_n \mid$ for all n. If the condition

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{6}$$

satisfies, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha, \beta \mid_k$, $0 < \alpha \leq 1$, $(\alpha + \beta - 1) > 0$, and $k \geq 1$.

Remark 3. 2 It should be noted that the condition (6) is reduced to the condition (5) when k=1. When $k > 1$, condition (6) is weaker than condition (5) but the converse is not true. As in [13], we can show that if (5) is satisfied, then we get

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} = O(X_m).$$

To show that the converse is false when $k > 1$, as in [6], the following example is sufficient. We can take $X_n = n^\sigma$, $0 < \sigma < 1$, and then construct a sequence (u_n) such that

$$u_n = \frac{(\theta_n^{\alpha,\beta})^k}{n {X_n}^{k-1}} = X_n - X_{n-1},$$

whence

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n {X_n}^{k-1}} = \sum_{n=1}^{m} (X_n - X_{n-1}) = X_m = m^\sigma,$$

and so

$$\sum_{n=1}^{m}\frac{(\theta_n^{\alpha,\beta})^k}{n} = \sum_{n=1}^{m}(X_n - X_{n-1})X_n^{k-1} = \sum_{n=1}^{m}(n^\sigma - (n-1)^\sigma)n^{\sigma(k-1)}$$

$$\geq \sigma\sum_{n=1}^{m}n^{\sigma-1}n^{\sigma(k-1)} = \sigma\sum_{n=1}^{m}n^{\sigma k-1} \sim \frac{m^{\sigma k}}{k} \; as \; m\to\infty.$$

It follows that

$$\frac{1}{X_m}\sum_{n=1}^{m}\frac{(\theta_n^{\alpha,\beta})^k}{n} \to \infty \quad as \quad m\to\infty$$

provided $k>1$. This shows that (5) implies (6) but not conversely.
We need the following lemmas for the proof of our theorem.
Lemma 3. 3 (Abel transformation)([1]). Let (a_k), (b_k) be complex sequences, and write $s_n = a_1 + a_2 + ... + a_n$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} s_k \Delta b_k + s_n b_n. \tag{7}$$

Lemma 3. 4 ([4]). If $0<\alpha\leq 1$, $\beta>-1$, and $1\leq v\leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^{\beta}a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^{\beta}a_p \mid . \tag{8}$$

Lemma 3. 5 ([7]). Let (X_n) be a quasi-f-power increasing sequence. If (A_n) is a δ-quasi-monotone sequence with $\Delta A_n \leq \delta_n$ and $\sum n\delta_n X_n < \infty$, then we have the following

$$\sum_{n=1}^{\infty} nX_n \mid \Delta A_n \mid < \infty, \tag{9}$$

$$nA_nX_n = O(1) \quad as \quad n\to\infty. \tag{10}$$

Lemma 3. 6 ([7]). Under the conditions regarding (λ_n) and (X_n) of the theorem, we have

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n\to\infty. \tag{11}$$

4. Proof of Theorem 3.1

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v \lambda_v.$$

First, applying Abel's transformation and then using Lemma 3. 4, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1}A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v,$$

$$
\begin{aligned}
\mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\end{aligned}
$$

To complete the proof of Theorem 3. 1, by Minkowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty} \frac{1}{n} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad \texttt{for} \quad r = 1, 2.
$$

Firstly, using (2) and then applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$ and $k > 1$, we obtain

$$
\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n} \mid T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} \frac{1}{n} \mid \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n} \mid \frac{1}{n^{\alpha+\beta}} \sum_{v=1}^{n-1} v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}} \left\{ \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} \mid A_v \mid^k (\theta_v^{\alpha,\beta})^k \right\} \times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \mid A_v \mid^k (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2+(\alpha+\beta-1)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \mid A_v \mid\mid A_v \mid^{k-1} (\theta_v^{\alpha,\beta})^k \int_v^{\infty} \frac{dx}{x^{2+(\alpha+\beta-1)k}} \\
&= O(1) \sum_{v=1}^{m} \mid A_v \mid v^{k-1} \frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1} X_v^{k-1}} = O(1) \sum_{v=1}^{m} v \mid A_v \mid \frac{(\theta_v^{\alpha,\beta})^k}{v X_v^{k-1}}
\end{aligned}
$$

Now, using Abel's transformation we have

$$
\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n} \mid T_{n,1}^{\alpha,\beta} \mid^k &= O(1) \sum_{v=1}^{m} v \mid A_v \mid \frac{(\theta_v^{\alpha,\beta})^k}{v X_v^{k-1}} = O(1) \sum_{v=1}^{m-1} \Delta(v \mid A_v \mid) \sum_{p=1}^{v} \frac{(\theta_p^{\alpha,\beta})^k}{p X_p^{k-1}} + O(1) m \mid A_m \mid \sum_{v=1}^{m} \frac{(\theta_v^{\alpha,\beta})^k}{v X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1) \Delta \mid A_v \mid - \mid A_v \mid\mid X_v + O(1) m \mid A_m \mid X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta A_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid A_v \mid X_v + O(1) m \mid A_m \mid X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

in view of hypotheses of Theorem 3. 1 and Lemma 3. 5. Again, we have

$$
\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n} \mid T_{n,2}^{\alpha,\beta} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid \mid \lambda_n \mid^{k-1} \frac{(\theta_n^{\alpha,\beta})^k}{n} = O(1) \sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n X_n^{k-1}} \sum_{v=n}^{\infty} \mid \Delta\lambda_v \mid \\
&= O(1) \sum_{v=1}^{\infty} \mid \Delta\lambda_v \mid \sum_{n=1}^{v} \frac{(\theta_n^{\alpha,\beta})^k}{n X_n^{k-1}} = O(1) \sum_{v=1}^{\infty} \mid \Delta\lambda_v \mid X_v = O(1) \sum_{v=1}^{\infty} \mid A_v \mid X_v < \infty,
\end{aligned}
$$

in view of the hypotheses of Theorem 3.1 and Lemma 3. 6. This completes the proof of Theorem 3. 1.
If we take $\beta = 0$, then we get a new result for $\mid C, \alpha \mid_k$ summability factors. Also, if we take $\beta = 0$ and $\alpha = 1$, then we obtain a result dealing with $\mid C, 1 \mid_k$ summability factors. Finally, if we take γ=0, then we get another new result dealing with quasi-σ-power increasing sequences.

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