Filomat 31:16 (2017), 5111–5116
https://doi.org/10.2298/FIL1716111A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Finite Dimensional Locally Convex Cones

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Abstract. We study the locally convex cones which have finite dimension. We introduce the Euclidean convex quasiuniform structure on a finite dimensional cone. In special case of finite dimensional locally convex topological vector spaces, the symmetric topology induced by the Euclidean convex quasiuniform structure reduces to the known concept of Euclidean topology. We prove that the dual of a finite dimensional cone endowed with the Euclidean convex quasiuniform structure is identical with it's algebraic dual.

1. Introduction

The theory of locally convex cones as developed in [5] and [7] uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the latter. For recent researches see [1–4].

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.

Let $\mathcal{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:
(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a, a) : a \in \mathcal{P}\}$);
(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone if

(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0, a) \in \rho U$.

We say that the convex subset E of $\mathcal{P}^2$ is uniformly convex whenever E has properties (U1) and (U3). The uniformly convex subsets play an important role in the construction of a convex quasiuniform structure. With every collection of uniformly convex subsets we can obtain a convex quasiuniform structure (see

2010 *Mathematics Subject Classification*. 46A03, 46A30
Keywords. Locally convex cones, finite dimensional cones.
Received: 30 December 2016; Accepted: 16 January 2017
Communicated by Teresa Alvarez Seco
This work was supported by the Young Researchers and Elite Club, Ahar Branch, Islamic Azad University, Ahar, Iran.
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[1], Proposition 2.2). With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in P : (a, b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ if for every $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

In locally convex cone $(\mathcal{P}, \mathfrak{U})$ the *closure* of $a \in \mathcal{P}$ is defined to be the set

$$\bar{a} = \bigcap_{U \in \mathfrak{U}} U(a)$$

(see [5], chapter I). The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called *separated* if $\bar{a} = \bar{b}$ implies $a = b$ for $a, b \in \mathcal{P}$. It is proved in [5] that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ is separated if and only if its symmetric topology is Hausdorff.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha.(+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. We set $\tilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a, b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}.$$

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathcal{P}$ and Q, a mapping $T : \mathcal{P} \to Q$ is called a *linear operator* if $T(a+b) = T(a)+T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(Q, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We denote the cone all linear functional on $\mathcal{P}$, by $L(\mathcal{P})$ and call it the algebraic dual of $\mathcal{P}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$. For $U \in \mathfrak{U}$, we set $U^\circ = \{\mu \in L(\mathcal{P}) : \mu(a) \leq \mu(b) + 1 \ \textit{if} \ (a, b) \in U\}$. We have $\mathcal{P}^* = \bigcup_{U \in \mathfrak{U}} U^\circ$.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. We shall say that the subset F of $\mathcal{P}^2$ is u-bounded if it is absorbed by each $U \in \mathfrak{U}$. The subset B of $\mathcal{P}$ is called bounded below (or above) whenever $\{0\} \times B$ (or $B \times \{0\}$) is u-bounded. The subset B is called bounded if it is bounded below and above. An element $a \in \mathcal{P}$ is called bounded below (or above) whenever $\{a\}$ is so.

Let $(\mathcal{P}, \mathfrak{U})$ and $(Q, \mathcal{W})$ be locally convex cones. The linear operator $T : \mathcal{P} \to Q$ is called u-bounded whenever for every u-bounded subset B of $\mathcal{P}^2$, $(T \times T)(B)$ is u-bounded. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called bornological if every u-bounded linear operator from $(\mathcal{P}, \mathfrak{U})$ into any locally convex cone is continuous. The linear operator $T : \mathcal{P} \to Q$ is called bounded below whenever for every bounded below subset A of $\mathcal{P}$, $T(A)$ is bounded below. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called b-bornological if every bounded below linear operator from $(\mathcal{P}, \mathfrak{U})$ into any locally convex cone is continuous (see [1]). Since every u-bounded linear operator is bounded below, every b-bornological cone is bornological.

The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called a *uc-cone* whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$ (see [1]). It is proved in [1] that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ is a *uc*-cone if and only if $\mathfrak{U}$ has a u-bounded element.

For a subset F of $\mathcal{P}^2$ we denote by $uch(F)$, the smallest uniformly convex subset of $\mathcal{P}^2$, which contains F and call it the uniform convex hull of F(such a subset of $\mathcal{P}^2$ obviously exists, since the properties (U_1) and (U_3) are preserved by arbitrary intersections).

Bornological and b-bornological locally convex cones are studied in [1]. Firstly, we review the construction of this structure briefly: Let $\mathcal{P}$ be a cone and U be a uniformly convex subset of $\mathcal{P}$. We set $\mathcal{P}_U = \{a \in \mathcal{P} : \exists\lambda > 0 \quad (0, a) \in \lambda U\}$ and $\mathfrak{U}_U = \{\alpha U : \alpha > 0\}$. Then $(\mathcal{P}_U, \mathfrak{U}_U)$ is a locally convex cone (a uc-cone). In [1] we proved that there is the finest convex quasiuniform structure $\mathfrak{U}_\tau$ (or $\mathfrak{U}_{b\tau}$) on locally convex cone $(\mathcal{P}, \mathfrak{U})$ such that $\mathcal{P}^2$ (or $\mathcal{P}$) has the same u-bounded (or bounded below) subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$(or $\mathfrak{U}_{b\tau}$). The locally convex cone $(\mathcal{P}, \mathfrak{U}_\tau)$ is the inductive limit of the uc-cones $(\mathcal{P}_U, \mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B}$ is the collection of all uniformly convex and u-bounded subsets of $\mathcal{P}^2$. Also $(\mathcal{P}, \mathfrak{U}_{b\tau})$ is the inductive limit of the uc-cones $(\mathcal{P}_U, \mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B} = \{uch(\{0\} \times B) : B \text{ is bounded below}\}$. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is bornological or b-bornological if and only if $\mathfrak{U}$ is equivalent to $\mathfrak{U}_\tau$ or $\mathfrak{U}_{b\tau}$, respectively.

Let $\mathcal{P}$ be a cone. There is the finest convex quasiuniform structure $\mathfrak{U}_\beta$ on $\mathcal{P}$ that makes $(\mathcal{P}, \mathfrak{U}_\beta)$ into a locally convex cone (see Proposition 2.2 from [1]). If $\mathcal{B}$ is the collection of all uniformly convex subsets of $\mathcal{P}^2$ such that for every $a \in \mathcal{P}$ and $U \in \mathcal{B}$ there is $\lambda > 0$ such that $(0, a) \in \lambda U$, then $\mathfrak{U}_\beta$ is the coarsest convex quasiuniform structure on $\mathcal{P}$ that contains $\mathcal{B}$.

2. Finite Dimensional Locally Convex Cones

We define the concepts of the base and dimension for cones. On the finite dimensional cones we introduce and investigate the concept of Euclidean convex quasiuniform structure.

Suppose $\mathcal{P}$ is a cone and $B \subseteq \mathcal{P}$. We set

$$span(B) = \Big\{ \sum_{i=1}^{n} \alpha_i b_i : n \in \mathbb{N}, b_1, \cdots, b_n \in B, \alpha_1, \cdots, \alpha_n \geq 0 \Big\}.$$

If $B = \emptyset$, then we set $span(B) = \{0\}$. The subset B of $\mathcal{P}$ is called linearly independent whenever $span(B) \neq span(E)$ for each $E \subsetneq B$. Obviously, for every $x \neq 0$, $\{x\}$ is linearly independent.

Definition 2.1. *Let $\mathcal{P}$ be a cone. We shall say that the subset B of $\mathcal{P} \setminus \{0\}$ is a base for $\mathcal{P}$ whenever*
(1) for every $a \in \mathcal{P}$ there are $n \in \mathbb{N}$, $b_1, ..., b_n \in B$ and $\alpha_1, ..., \alpha_n \geq 0$ such that $a = \sum_{i=1}^{n} \alpha_i b_i$, in the other words $\mathcal{P} = span(B)$,
(2) B is linearly independent.

Definition 2.2. *We shall say that the cone $\mathcal{P}$ is a n-dimensional cone whenever $\mathcal{P}$ has a finite base $B = \{b_1, b_2, \cdots, b_n\}$, $n \in \mathbb{N}$, and for every base B' for $\mathcal{P}$, $Card(B') \geq n$.*

Remark 2.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be cones of dimension $n \in \mathbb{N}$. Then the cones $\mathcal{P}$ and $\mathcal{Q}$ are not necessarily isomorphic. For example the cones $[0, +\infty)$ and $\{0, +\infty\}$ are one dimensional, but they are not algebraically isomorphic. Also, if the cone $\mathcal{P}$ is a vector space, then the dimensions of $\mathcal{P}$ are not equal as a vector space and as a cone. For example, $\mathbb{R}^2$ is 2-dimensional as a vector space and it is 3-dimensional as a cone. The set $\{(-1, 0), (1, 1), (1, -1)\}$ is a base for $\mathbb{R}^2$ as a cone and for every base B' for $\mathbb{R}^2$, we have $card(B') \geq 3$.*

The following lemma is an equivalent form of the axiom of choice.

Lemma 2.4. *Let $\mathcal{E}$ be a set of subsets of a set E and let $\mathcal{L}$ be a chain contained in $\mathcal{E}$. Then there is a maximal chain $\mathcal{M}$ with $\mathcal{L} \subseteq \mathcal{M} \subseteq \mathcal{E}$.*

Theorem 2.5. *Let E be a linearly independent subset of a cone $\mathcal{P}$ and F be a subset of $\mathcal{P}$ such that $E \subseteq F$ and $\mathcal{P} = span(F)$. Then there is a base B of $\mathcal{P}$ such that $E \subseteq B \subseteq F$.*

Proof. Let $L = E$, $\mathcal{L} = \{L\}$ and let $\mathcal{E}$ be the collection of all linearly independent subsets of F. By lemma 2.4, there is a maximal chain $\mathcal{M}$ with $\mathcal{L} \subseteq \mathcal{M} \subseteq \mathcal{E}$. We set $B = \bigcup_{M\in\mathcal{M}} M$. Then $E \subseteq B \subseteq F$. We prove that B is a base for $\mathcal{P}$. We claim that every element of F is a linear combination of elements of B. Otherwise, there is $x \in F$ such that $x \notin span(B)$. Now we can add $B \cup \{x\}$ to $\mathcal{M}$, and this is a contradiction with the maximality of $\mathcal{M}$. Then $\mathcal{P} = span(F) = span(B)$. Now, we prove that B is linearly independent. Otherwise, there is a subset $A \subsetneq B$ such that $span(A) = span(B)$. Then there is $x \in B \setminus A$ such that $x = \sum_{i}^{n} \alpha_i a_i$, for some $n \in \mathbb{N}$, $a_1, \cdots, a_n \in A$ and $\alpha_1, \cdots, \alpha_n > 0$. Since $\mathcal{M}$ is a chain, there is $M' \in \mathcal{M}$ such that $x, a_1, \cdots, a_n \in M'$. Then $span(M') = span(M' \setminus \{x\})$. This is a contradiction, since M' is linearly independent. $\square$

Corollary 2.6. *Every linearly independent subset of a cone can be extended to a base.*

Corollary 2.7. *Every cone $\mathcal{P}$ has a base. In fact we can take $E = \emptyset$ and $F = \mathcal{P}$, then there is a base B for $\mathcal{P}$ by theorem 2.5.*

Let $\mathcal{P}$ be a cone with finite dimension and $B = \{b_1, \cdots, b_n\}$ be a base for $\mathcal{P}$. We set $U_B = uch(\{0\} \times B)$ and $\mathfrak{U}_B = \{\alpha U_B : \alpha > 0\}$. We claim that $(\mathcal{P}, \mathfrak{U}_B)$ is a *uc*-cone. Let $a \in \mathcal{P}$. There are $\alpha_1, \cdots, \alpha_n \geq 0$, such that $a = \sum_{i=1}^{n} \alpha_i b_i$. Now, we have

$$(0, a) = \sum_{i=1}^{n} \alpha_i (0, b_i) \in \sum_{i=1}^{n} \alpha_i U_B \subseteq (\sum_{i=1}^{n} \alpha_i) U_B.$$

This shows that $(\mathcal{P}, \mathfrak{U}_B)$ is a locally convex cone. Since $\mathfrak{U}_B$ is created by U_B, we conclude that $(\mathcal{P}, \mathfrak{U}_B)$ is a *uc*-cone.

Definition 2.8. *Let $\mathcal{P}$ be a n-dimensional cone, with the base $B = \{b_1, b_2, \cdots, b_n\}$. We call $\mathfrak{U}_B$, the Euclidean convex quasiuniform structure on $\mathcal{P}$.*

Theorem 2.9. *Let $\mathcal{P}$ be a n-dimensional cone. If B and B' are two bases for $\mathcal{P}$, then $\mathfrak{U}_B$ and $\mathfrak{U}_{B'}$ are equivalent.*

Proof. Let $B = \{b_1, \cdots, b_n\}$ and $B' = \{b'_1, \cdots, b'_m\}$. Since B' is a base for $\mathcal{P}$, for every $j \in \{1, \cdots, n\}$, we have $b_j = \sum_{i=1}^{m} \alpha_{ij} b'_i$ for some $\alpha_1, \cdots, \alpha_m \geq 0$. We set $\lambda_j = \sum_{i=1}^{m} \alpha_{ij}$. Then we have $(0, b_j) \in \lambda_j U_{B'}$. This shows that $U_B = uch(\{0\} \times B) \subseteq \lambda U_{B'}$, where $\lambda = \max\{\lambda_i : i = 1, \cdots, n\}$. Therefore $\mathfrak{U}_B$ is finer than $\mathfrak{U}_{B'}$. Similarly, we can prove that $\mathfrak{U}_{B'}$ is finer than $\mathfrak{U}_B$. □

Theorem 2.10. *Let $\mathcal{P}$ a finite dimensional cone with the base B. Then $\mathfrak{U}_B$ is the finest convex quasiuniform structure on $\mathcal{P}$ that makes it a locally convex cone.*

Proof. Let $B = \{b_1, \cdots, b_n\}$ and $\mathfrak{U}$ be an arbitrary convex quasiuniform structure on $\mathcal{P}$ that makes $\mathcal{P}$ into a locally convex cone. suppose $V \in \mathfrak{U}$. There is $\lambda \geq 0$ such that $\{0\} \times B \subseteq \lambda V$. This shows that $U_B = uch(\{0\} \times B) \subseteq uch(\lambda V) = \lambda V$. Then $\frac{1}{\lambda} U_B \subseteq V$. Therefore $\mathfrak{U}_B$ is finer than $\mathfrak{U}$. □

Corollary 2.11. *Let $\mathcal{P}$ a finite dimensional cone with the base B. Then $\mathfrak{U}_B$ is equivalent with the convex quasiuniform structure $\mathfrak{U}_\beta$.*

We denote the convex hall of the set A by $conv(A)$.

Lemma 2.12. *Let $\mathcal{P}$ be a cone and $A \subseteq \mathcal{P}$. Then we have*

$$uch(\{0\} \times A) = \{(b, b + a) : b \in \mathcal{P}, a \in conv(\{0\} \cup A)\}.$$

Proof. Suppose $G = \{(b, b + a) : b \in \mathcal{P}, a \in conv(\{0\} \cup A)\}$. Since $0 \in \mathcal{P}$ and $A \subseteq conv(\{0\} \cup A)$, then we have $\{0\} \times A \subseteq G$. We prove that G is uniformly convex. For convexity, let $t \in [0, 1]$ and $(b, b + a), (b', b' + a') \in G$ for some $b, b' \in \mathcal{P}$ and $a, a' \in conv(\{0\} \cup A)$. Then we have

$$t(b, b + a) + (1 - t)(b', b' + a') = (tb + (1 - t)b', tb + (1 - t)b' + ta + (1 - t)a') \in G,$$

since $ta + (1 - t)a' \in conv(\{0\} \cup A)$. It is clear that $\triangle \subset G$. For (U_3), Let $(h, l) \in \lambda G$ and $(l, k) \in \gamma G$. Then there are $b, b' \in \mathcal{P}$ and $a, a' \in conv(\{0\} \cup A)$ such that $(h, l) = (\lambda b, \lambda b + \lambda a)$ and $(l, k) = (\gamma b', \gamma b' + \gamma a')$. This shows that $\gamma b' = \lambda b + \lambda a$ and then $(h, k) = (\lambda b, \gamma b' + \gamma a') = (\lambda b, \lambda b + \lambda a + \gamma a')$. Then

$$\frac{1}{\lambda + \gamma}(h, k) = (\frac{\lambda}{\lambda + \gamma} b, \frac{\lambda}{\lambda + \gamma} b + \frac{\lambda}{\lambda + \gamma} a + \frac{\gamma}{\lambda + \gamma} a') \in G.$$

Therefore $(h,k) \in (\lambda+\gamma)G$. Now, since G is uniformly convex and contains $\{0\}\times A$, we conclude that $uch(\{0\}\times A) \subseteq G$. On the other hand if $(b, b+a) \in G$ for some $b \in \mathcal{P}$ and $a \in conv(\{0\}\cup A)$, then for each $n \in \mathbb{N}$, we have $(b,b+a) = (b,b)+(0,a) \in \frac{1}{n} \triangle + uch(\{0\}\times A) \subseteq (1+\frac{1}{n})uch(\{0\}\times A)$. This shows that

$$G \subseteq \bigcap_{n\in\mathbb{N}} (1+\frac{1}{n})uch(\{0\}\times A) = uch(\{0\}\times A).$$

□

Remark 2.13. *Let $\mathcal{P}$ be a vector space over $\mathbb{R}$, with dimension n. If $G = \{b_1,\cdots,b_n\}$ is a base for $\mathcal{P}$ as a vector space, then $B = G\cup(-G)$ is a base for $\mathcal{P}$ as a cone. We prove that the symmetric topology induced on $\mathcal{P}$ by $\mathfrak{U}_B$ is the Euclidean topology. We consider on $\mathcal{P}$ the norm is defined by*

$$\|x\| = \sum_{i=1}^{n} |\alpha_i|,$$

where $x = \sum_{i=1}^{n}\alpha_i b_i$ and $\alpha_i \in \mathbb{R}$ for each $i = 1,\cdots,n$. Let T be the unit ball of $\mathcal{P}$. We prove that $T \subseteq U_B(0)U_B$. If $x \in T$, than $\sum_{i=1}^{n}|\alpha_i| \le 1$. Without loss of generality, we suppose $\alpha_1,\cdots,\alpha_m < 0$, where $m \le n$. Now we have

$$\begin{aligned}(0,x) = (0,\sum_{i=1}^{n}\alpha_i b_i) &= (0,\sum_{i=1}^{m}(-\alpha_i)(-b_i)+\sum_{i=m+1}^{n}\alpha_i b_i)\\ &\in \sum_{i=1}^{n}|\alpha_i|U_B \subseteq (\sum_{i=1}^{n}|\alpha_i|)U_B \subseteq U_B.\end{aligned}$$

Then $x \in (0)U_B$. Also, since $(x,0) = (x, x+(-x))$ and $-x \in conv(\{0\}\cup B)$, lemma 2.12 shows that $x \in U_B(0)$. Therefore $x \in U_B(0)U_B$. On the other hand if $x \in U_B(0)$, then by the lemma 2.12 there are $n \in \mathbb{N}$ and $\alpha_1,\cdots,\alpha_n$, with $\sum_{i=1}^{n}|\alpha_i| \le 1$ such that $-x = \sum_{i=1}^{n}\alpha_i b_i$. This shows that $\|x\| = \|-x\| = \sum_{i=1}^{n}|\alpha_i| \le 1$. Then $x \in T$. Therefore $U_B(0)U_B \subseteq U_B(o) \subseteq T$. Then the Euclidean topology on $\mathcal{P}$ is equivalent with the symmetric topology induced by the Euclidean convex quasiuniform structure.

Theorem 2.14. *Let $\mathcal{P}$ be a n-dimensional cone with the base B. Then $(\mathcal{P},\mathfrak{U}_B)$ is b-bornological and bornological.*

Proof. It is clear that B is bounded below in $(\mathcal{P},\mathfrak{U}_B)$. Now, Let $(\mathcal{Q},\mathcal{W})$ be a locally convex cone and T be a bounded below linear operator from $(\mathcal{P},\mathfrak{U}_B)$ into $(\mathcal{Q},\mathcal{W})$. If T is not continuous, then there is $W \in \mathcal{W}$ such that $(T\times T)(U_B) \nsubseteq \alpha W$ for all $\alpha > 0$. Thus $(T\times T)(\{0\}\times B) \nsubseteq \alpha W$ for all $\alpha > 0$ (if $(T\times T)(\{0\}\times B) \subseteq \alpha W$, then $(T\times T)(U_B) = uch((T\times T)(\{0\}\times B)) \subseteq uch(\alpha W) = \alpha W$). This shows that $T(B)$ is not bounded below. Then T is not bounded below. This contradiction proves our claim. Since every b-bornological locally convex cone is bornological, we conclude that $(\mathcal{P},\mathfrak{U}_B)$ is bornological. □

Corollary 2.15. *For the finite dimensional cone $\mathcal{P}$, with base B, $\mathfrak{U}_B$ is equivalent with $(\mathfrak{U}_B)_{b\tau}$ and $(\mathfrak{U}_B)_\tau$.*

Suppose that $L(\mathcal{P})$ is the algebraic dual of $\mathcal{P}$. In the following theorem we show that the dual cone of any finite dimensional cone endowed with the Euclidean convex quasiuniform structure $\mathfrak{U}_B$ is identical with $L(\mathcal{P})$.

Theorem 2.16. *Let $\mathcal{P}$ be a finite dimensional cone with the base B. Then*
(a) $(\mathcal{P},\mathfrak{U}_B)^* = L(\mathcal{P})$,
(b) $U_B^\circ = (0\times B)^\circ$,

Proof. For (a), we note that the Euclidean convex quasiuniform structure $\mathfrak{U}_B$ on $\mathcal{P}$ is equivalent with $\mathfrak{U}_\beta$ by Corollary 2.11. Then $(\mathcal{P},\mathfrak{U}_B)^* = (\mathcal{P},\mathfrak{U}_\beta)^* = L(\mathcal{P})$.
For (b), we have $U_B^\circ \subseteq (0\times B)^\circ$, since $0\times B \subseteq U_B$. For the converse inclusion suppose $\mu \in (0\times B)^\circ$. Then $\mu(b) \ge -1$ for all $b \in B$. Now, let $(a,b) \in U_B$. Lemma 2.12, shows that $b = a+\sum_{i=1}^{n}\lambda_i b_i$ for some $n \in \mathbb{N}$, $\lambda_1,\cdots,\lambda_n > 0$ with $\sum_{i=1}^{n}\lambda_i \le 1$. Therefore $\mu(b) = \mu(a)+\sum_{i=1}^{n}\lambda_i\mu(b_i) \ge \mu(a)-\sum_{i=1}^{n}\lambda_i \ge \mu(a)-1$. Therefore $\mu(a) \le \mu(b)+1$. Then $\mu \in U_B^\circ$. □

Example 2.17. *The extended real number system* $\overline{\mathbb{R}}$ *is a cone with dimension 3. The subset* $B = \{-1, 1, +\infty\}$ *is a base for* $\overline{\mathbb{R}}$*. Then the Euclidean convex quasiuniform structure on* $\overline{\mathbb{R}}$ *is* $\mathfrak{U}_B = \{\alpha U_B : \alpha > 0\}$*, where*

$$U_B = uch(\{0\} \times B) = \{(x, y) : x, y \in \mathbb{R}, x - 1 \leq y \leq x + 1\} \cup \{(x, \infty) : x \in \overline{\mathbb{R}}\}.$$

The Euclidean convex quasiuniform structure is strictly finer than $\tilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$*. In fact we have* $U_B \subset \tilde{1}$*. For* $a \in \mathbb{R}$ *and* $U_B \in \mathfrak{U}_B$*, the upper, lower and symmetric neighborhoods are as follows:* $U_B(a) = [a - 1, a + 1], (a)U_B = [a-1, a+1] \cup \{+\infty\}$ *and* $U_B(a)U_B = [a-1, a+1]$*. For* $+\infty$ *we have* $U_B(\infty) = \overline{\mathbb{R}}$*,* $(\infty)U_B = \{\infty\}$ *and* $U_B(\infty)U_B = \{\infty\}$*. The symmetric topologies induced by* $\tilde{\mathcal{V}}$ *and* $\mathfrak{U}_B$ *are equivalent.*

Example 2.18. *Suppose* $B = (-1, 1)$*. Let* Q *be the collection of all sets* $a + \rho B$*, where* $a \in \mathbb{R}$ *and* $\rho \geq 0$*. Then* Q *is a cone endowed with the usual addition and scaler multiplication. Its neutral element is* $\{0\}$*. Also,* $\mathcal{B} = \{-1+B, 1+B\}$ *is a base for* Q*. We have* $uch(\{0\}\times\mathcal{B}) = \{((a, b), (a-2t_1, b+2t_2)) : a < b, t_1, t_2 \geq 0, t_1+t_2 \leq 1\}$*. We set* $\tilde{\mathcal{B}} = uch(\{0\}\times\mathcal{B})$*, then* $\mathfrak{U}_{\mathcal{B}} = \{\alpha\tilde{\mathcal{B}} : \alpha > 0\}$ *is the Euclidean convex quasiuniform structure on* Q*.*

Acknowledgment: This work was supported by the Young Researchers and Elite Club, Ahar Branch, Islamic Azad University, Ahar, Iran.

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