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Discontinuity of Control Function in the (F, φ, θ)-Contraction in Metric Spaces

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Abstract. In this paper, we improve very recent results of Kumrod *et al.* [2] with discontinuity of control function in the (F, φ, θ)-contraction in metric spaces. Illustrative examples and an application in nonlinear integral equation are presented.

1. Introduction and Preliminaries on φ-fixed points and (F, φ)-contraction mappings

In 2014, Jleli *et al.* [1] introduced the concepts of φ-fixed points, φ-Picard mappings and weakly φ-Picard mappings. After that Kumrod *et al.* [2] extended the concepts of (F, φ, θ)-contraction mapping and (F, φ, θ)-weak contraction mapping in metric spaces and established φ-fixed point results for such mappings. Their results were combined with the continuous control function F.

Here we review basic definitions and theorems.

Let X be a nonempty set, $\varphi : X \to [0, \infty)$ be a given function and $T : X \to X$ be a mapping. We denote the set of all fixed points of T by

$$F_T := \{x \in X : Tx = x\}$$

and denote the set of all zeros of the function φ by

$$Z_\varphi := \{x \in X : \varphi(x) = 0\}.$$

Definition 1.1. *Let X be a nonempty set and $\varphi : X \to [0, \infty)$ be a given function. An element $z \in X$ is called φ-fixed point of the mapping $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$.*

Definition 1.2. *Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function. A mapping $T : X \to X$ is said to be a φ-Picard mapping if and only if*

- *$F_T \cap Z_\varphi = \{z\}$, where $z \in X$,*
- *$T^n x \to z$ as $n \to \infty$, for each $x \in X$.*

Definition 1.3. *Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function. We say that the mapping $T : X \to X$ is a weakly φ-Picard mapping if and only if*

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- *T has at least one φ-fixed point,*
- *the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is a φ-fixed point of T.*

Also, Jleli *et al.* introduced the new concept of control function $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

(F1) $\max\{a,b\} \le F(a,b,c)$ for all $a,b,c \in [0,\infty)$;

(F2) $F(0,0,0) = 0$;

(F3) F is continuous.

The class of all functions satisfying the conditions (F1)-(F3) is denoted by $\mathcal{F}$.

Example 1.4. *Let $F_1, F_2, F_3 : [0,\infty) \to [0,\infty)$ be defined by*

1. $F_1(a,b,c) = a + b + c$;
2. $F_2(a,b,c) = \max\{a,b\} + c$;
3. $F_3(a,b,c) = a + a^2 + b + c$;

for all $a,b,c \in [0,\infty)$. Then $F_1, F_2, F_3 \in \mathcal{F}$.

By using the control function in $\mathcal{F}$, Jleli *et al.* defined the new contractive conditions and proved the φ-fixed point results as follows:

Definition 1.5. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ) -contraction with respect to the metric d if and only if there is $k \in (0,1)$ such that*

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \le kF(d(x,y), \varphi(x), \varphi(y)) \tag{1}$$

for all $x,y \in X$.

Definition 1.6. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ) -weak contraction with respect to the metric d if and only if there is $k \in (0,1)$ and $L \ge 0$ such that*

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \le kF(d(x,y), \varphi(x), \varphi(y)) + L[F(d(y,Tx), \varphi(y), \varphi(Tx)) - F(0, \varphi(y), \varphi(Tx)))] \tag{2}$$

for all $x,y \in X$.

In this paper, we introduce the concepts of (F,φ,θ)-contraction mapping and (F,φ,θ)-weak contraction mapping in metric spaces and establish φ-fixed point results for such mappings with discontinuous control function F. Presented theorems extend the φ-fixed point results of Kumrod *et al.* [1, 2]. Here are examples of expressing highlight the validity of our results. Numerical experiments are given for approximating the φ-fixed point with examples in [2]. Finally, as an application, the fixed point results are verified from our main results and we prove the existence and uniqueness of a solution of a nonlinear integral equation.

2. Main results

Let J be the set of all functions $\theta : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(j1) θ is a nondecreasing function, i.e., $t_1 < t_2$ implies $\theta(t_1) \le \theta(t_2)$;

(j2) θ is continuous;

(j3) $\sum_{n=0}^{\infty} \theta^n(t) < \infty$ for all $t > 0$.

Note that (j4) implies (j3).

We introduce the new concept of control function $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions without continuity:

(F_M1) $\max\{a,b\} \le F(a,b,c)$ for all $a,b,c \in [0,\infty)$;

(F_M2) $F(0,0,0) = 0$;

(F_M3) $\limsup_{n\to\infty} F(x_n, y_n, 0) \le F(x,y,0)$ when $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

The class of all functions satisfying the conditions $(F1) - (F3)$ is denoted by $\mathcal{F}_{\mathcal{M}}$.

Remark 2.1. *Let F be defined by $F(a,b,c) = a+b+[c]$ or $F(a,b,c) = \max\{a,b\}+[c]$. Then F satisfies (F_M3) but F is not continuous.*

Lemma 2.2. ([2, Lemma 2.1]) *If $\theta \in J$, then $\theta(t) < t$ for all $t > 0$.*

Remark 2.3. ([2, Remark 2.2]) *If $\theta \in J$, then $\theta(0) = 0$.*

Here we define the new contractive condition in metric spaces as follows:

Definition 2.4. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}_{\mathcal{M}}$. The mapping $T : X \to X$ is said to be an (F,φ,θ)-contraction with respect to the metric d if and only if there is $k \in (0,1)$ such that*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \le \theta(F(d(x,y),\varphi(x),\varphi(y))) \tag{3}$$

for all $x,y \in X$.

Now we give the existence of φ-fixed point results for (F,φ,θ)-contraction mappings with control function F which is not continuous.

Theorem 2.5. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}_{\mathcal{M}}$. Assume that the following conditions are satisfied:*

(H1) φ is lower semi-continuous,

(H2) $T : X \to X$ is an (F,φ,θ)-contraction with respect to the metric d.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a φ-Picard mapping.

Proof. The frame of the proof is the same in Theorem 2.5 [2]. So for arbitrary point $x \in X$, $\{T^n x\}$ is Cauchy sequence, $\lim_{n\to\infty} d(T^n x, z) = \lim_{n\to\infty} \varphi(T^n x) = 0$ and $\varphi(z) = 0$ for some $z \in X$.

$$\begin{aligned} d(T^{n+1}x, Tz) &\le \max\{d(T^{n+1}x,Tz), \varphi(T^{n+1}x)\} \\ &\le F(d(T^{n+1}x,Tz), \varphi(T^{n+1}x), \varphi(Tz)) \\ &\le \theta(F(d(T^n x, z), \varphi(T^n x), \varphi(z))) \\ &< F(d(T^n x,z), \varphi(T^n x), \varphi(z))). \\ &= F(d(T^n x,z), \varphi(T^n x), 0). \end{aligned}$$

Thus

$$\limsup_{n\to\infty} d(T^{n+1}x, Tz) \le \limsup_{n\to\infty} F(d(T^n x,z), \varphi(T^n x), 0) \le F(0,0,0) = 0.$$

$$\lim_{n\to\infty} d(T^n x, Tz) = \lim_{n\to\infty} d(T^n x, z) = 0,$$

So $z = Tz$ and it is a unique fixed point of T. □

Next, we give some examples to illustrate Theorem 2.5.

Example 2.6. *Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete metric space.*

1. *Fix $n \in \mathbb{N}$ and assume that $T : X \to X$ is defined by $Tx = \frac{kx^n}{n}$, where $k \in [0,1)$;*
2. *the function $\varphi : X \to [0,\infty)$ is defined by $\varphi(x) = x$ for all $x \in X$;*
3. *the function $F : [0,\infty)^3 \to [0,\infty)$ defined by $F(a,b,c) = a + b + [c]$, where $[c]$ is the integer part of c*
4. *the function $\theta : [0,\infty) \to [0,\infty)$ is defined by $\theta(t) = kt$ for $t \in [0,\infty)$, where k [0,1).*

Note that $F \in \mathcal{F}_{\mathcal{M}}$, $\theta \in J$ and further F is discontinuous.

T is an (F,φ,θ)-contraction mapping, because

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= \left|\frac{kx^n}{n} - \frac{ky^n}{n}\right| + \frac{k^n x}{n} + \left[\frac{ky^n}{n}\right] \\ &= \left|\frac{kx^n}{n} - \frac{ky^n}{n}\right| + \frac{k^n x}{n} + 0 \\ &\le k\left(\frac{|x-y||x^{n-1} + \cdots + y^{n-1}|}{n} + k\frac{x^n}{n}\right) \\ &\le k(|x-y| + x + 0) \\ &= k(|x-y| + x + [y]) \\ &= k(d(x,y) + x + [y]) \\ &= k(F(d(x,y),\varphi(x),\varphi(y))) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

This shows that all conditions of Theorem 2.5 are satisfied and so T has a φ-fixed point in X.

Example 2.7. *Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete metric space.*

1. *Fix $n \in \mathbb{N}$ and assume that $T : X \to X$ is defined by $Tx = \frac{kx^n}{n}$, where $k \in [0,1)$;*
2. *the function $\varphi : X \to [0,\infty)$ is defined by $\varphi(x) = x$ for all $x \in X$;*
3. *the function $F : [0,\infty)^3 \to [0,\infty)$ defined by $F(a,b,c) = \max\{a,b\} + [c]$, where $[c]$ is the integer part of c*
4. *the function $\theta : [0,\infty) \to [0,\infty)$ is defined by $\theta(t) = kt$ for $t \in [0,\infty)$, where k [0,1).*

Note that $F \in \mathcal{F}_{\mathcal{M}}$, $\theta \in J$ and further F is discontinuous.

T is an (F,φ,θ)-contraction mapping, because

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= \max\left\{\left|\frac{kx^n}{n} - \frac{ky^n}{n}\right|, \frac{kx^n}{n}\right\} + \left[\frac{ky^2}{2}\right] \\ &= \max\left\{\left|\frac{kx^n}{n} - \frac{ky^n}{n}\right|, \frac{kx^n}{n}\right\} + 0 \\ &\le k(\max\{|x-y|, x\} + [y]) \\ &= k(\max\{d(x,y), x\} + [y]) \\ &= k(F(d(x,y),\varphi(x),\varphi(y))) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

This shows that all conditions of Theorem 2.5 are satisfied and so T has a φ-fixed point in X.

Example 2.8. *Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete metric space.*

1. *Assume that* $T : X \to X$ *is defined by* $Tx = k\sin x$, *where* $k \in [0,1)$;
2. *the function* $\varphi : X \to [0,\infty)$ *is defined by* $\varphi(x) = x$ *for all* $x \in X$;
3. *the function* $F : [0,\infty)^3 \to [0,\infty)$ *defined by* $F(a,b,c) = a + b + [c]$, *where* $[c]$ *is the integer part of* c
4. *the function* $\theta : [0,\infty) \to [0,\infty)$ *is defined by* $\theta(t) = kt$ *for* $t \in [0,\infty)$, *where k [0,1).*

Note that $F \in \mathcal{F}_{\mathcal{M}}$, $\theta \in J$ *and further F is discontinuous.*

T is an (F,φ,θ)*-contraction mapping, because*

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= |k\sin x - k\sin y| + k\sin x + [k\sin y] \\ &\leq k|x-y| + kx + 0 \\ &= k(|x-y| + x + [y]) \\ &= k(d(x,y) + x + [y]) \\ &= k(F(d(x,y),\varphi(x),\varphi(y))) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

This shows that all conditions of Theorem 2.5 are satisfied and so T has a φ*-fixed point in X.*

Example 2.9. *Let* $X = [0,3]$ *and* $d : X \times X \to \mathbb{R}$ *be defined by* $d(x,y) = |x-y|$ *for all* $x,y \in X$. *Then* (X,d) *is a complete metric space.*

1. *Assume that* $T : X \to X$ *is defined by* $Tx = 0$ *if* $0 \leq x < 2.5$ *and* $Tx = k\ln\frac{x}{2}$ *if* $2.5 \leq x \leq 3$ *where* $k \in [0,1)$;
2. *The function* $\varphi : X \to [0,\infty)$ *is defined by* $\varphi(x) = x$ *for all* $x \in X$;
3. *the function* $F : [0,\infty)^3 \to [0,\infty)$ *defined by* $F(a,b,c) = a + b + [c]$ *where* $[c]$ *is the integer part of* c;
4. *the function* $\theta : [0,\infty) \to [0,\infty)$ *is defined by* $\theta(t) = 0$ *if* $0 \leq t \leq 1$ *and* $\theta(t) = k\ln(t)$ *if* $t \geq 1$, *where k [0,1).*

Note that F is $\mathcal{F}_{\mathcal{M}}$ *and further F is discontinuous.*

When $2.5 \leq x,y \leq 3$, *without loss of generality, we may suppose that* $x \geq y$. *Then we get*

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= \left|k\ln\frac{x}{2} - k\ln\frac{y}{2}\right| + k\ln\frac{x}{2} + \left[k\ln\frac{y}{2}\right] \\ &\leq \left|k\ln\frac{x}{2} - k\ln\frac{y}{2}\right| + k\ln\frac{x}{2} + k\ln\frac{y}{2} \\ &\leq 2k\ln\left(\frac{3}{2}\right) \\ &= k\ln 2.25 \\ &\leq k\ln(d(x,y) + x + [y]) \\ &= k\ln(F(d(x,y),\varphi(x),\varphi(y))) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

If $x \in [2.5,3]$ *and* $y \in [0,2.25]$, *then*

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= \left|k\ln\frac{x}{2} - 0\right| + k\ln\frac{x}{2} + [0] \\ &\leq 2k\ln\left(\frac{3}{2}\right) \\ &= k\ln 2.25 \\ &\leq k\ln(d(x,y) + x + [y]) \\ &= k\ln(F(d(x,y),\varphi(x),\varphi(y))) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

The other cases are clear. This shows that all conditions of Theorem 2.5 are satisfied and so T has a φ*-fixed point in X.*

Now by $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{M}}$, we have:

Corollary 2.10. ([2, Theorem 1.11]) *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. Suppose that the following conditions hold:*

(H1) φ is lower semi-continuous,

(H2) $T : X \to X$ is an (F,φ)-contraction with respect to the metric d.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a φ-Picard mapping;

(iii) if $x \in X$ and $z \in F_T$, then

$$d(T^n x, z) \leq \frac{k^n}{1-k} F(d(tx,x), \varphi(Tx), \varphi(x)),$$

for all $n \in \mathbb{N}$.

Corollary 2.11. ([2, Theorem 1.12]) *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. Suppose that the following conditions hold:*

(H1) φ is lower semi-continuous,

(H2) $T : X \to X$ is an (F,φ)-weak contraction with respect to the metric d.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a weakly φ-Picard mapping;

(iii) if $x \in X$ and $T^n x \to z \in F_T$ as $n \to \infty$ then

$$d(T^n x, z) \leq \frac{k^n}{1-k} F(d(tx,x), \varphi(Tx), \varphi(x)),$$

for all $n \in \mathbb{N}$.

Next we generalize the contractive condition (2) and prove the another main result in this work.

Definition 2.12. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ,θ)-weak contraction with respect to the metric d if and only if*

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq \theta(F(d(x,y), \varphi(y), \varphi(Tx))) + L[F(N(x,y), \varphi(y), \varphi(Tx)) - F(0, \varphi(y), \varphi(Tx)))] \quad (4)$$

for all $x, y \in X$, where $N(x,y) = \min\{d(x,Tx), d(y,Ty), d(y,Tx)\}$ and $L \geq 0$.

Theorem 2.13. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}_M$ and $\theta \in J$. Assume that the following conditions are satisfied:*

(H1) φ is lower semi-continuous,

(H2) $T : X \to X$ is an (F,φ,θ)-weak contraction with respect to the metric d.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a weakly φ-Picard mapping.

Proof. The framework of the proof is the same in proof of [2, Theorem 2.9]. □

Remark 2.14. *If we take $\theta(t) := kt$ for all $t \in [0,\infty)$, where $k \in [0,1)$, then by Theorems 2.5 and 2.13 we obtain previous results in [1, 2].*

3. An application

Consider the following nonlinear integral equation:

$$x(t) = \phi(t) + \int_a^t K(t, s, x(s))ds, \tag{5}$$

where $a \in \mathbb{R}$, $x \in C([a, b], \mathbb{R})$, $\phi[a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are two given functions.

Theorem 3.1. *Consider the nonlinear integral equation* (5)*. Suppose that the following condition holds:*

(i) K is continuous;

(ii) there is $\theta \in J$ such that

$$|K(t, s, x(s)) - K(t, s, y(s))| \leq \frac{\theta(|x(s) - y(s)|)}{b - a}$$

for all $x, y \in C([a, b], \mathbb{R})$ and for $t, s \in [a, b]$.

Then the nonlinear integral equation (5) *has a unique solution.*

Proof. Let $X := C([a, b], \mathbb{R})$, $T : X \to X$ defined by

$$(Tx)(t) = \phi(t) + \int_0^t K(t, s, x(s))ds, \quad \forall x \in X.$$

The metric d given by $d(x, y) = \max_{t\in[a,b]} |x(s) - y(s)|$ for all $x, y \in X$. Thus X is a complete metric space. Now define control function F by $F(a, b, c) = \max\{a, b\} + [c]$ for each $a, b, c \in [0, \infty)$. Also define $\varphi(x) = 0$ for all $x \in X$.

Let $x, y \in X$ and $t \in [a, b]$. therefore

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| \int_a^t K(t, s, x(s))ds - \int_a^t K(t, s, y(s))ds \right| \\ &\leq \int_a^t |K(t, s, x(s)) - K(t, s, y(s))|ds \\ &\leq \int_a^t \frac{\theta(|x(s) - y(s)|)}{b - a} ds \\ &\leq \frac{1}{b - a} \int_a^t \theta(d(x, y))ds \\ &\leq \theta(d(x, y)). \end{aligned}$$

So

$$\begin{aligned} d(Tx, Ty) &\leq \theta(d(x, y)) \\ \max\{d(Tx, Ty), \varphi(Tx)\} &\leq \theta(\max\{d(x, y), \varphi(x)\}) \\ \max\{d(Tx, Ty), \varphi(Tx)\} + [\varphi(Ty)] &\leq \theta(\max\{d(x, y), \varphi(x)\} + [\varphi(y)]), \end{aligned}$$

for all $x, y \in X$. Hence it satisfies the contraction (3).

Thus all the conditions of Theorem 2.5 are satisfied and hence T has a unique φ-fixed point in X. This implies that there exists a unique solution of the nonlinear integral equation (5). □

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