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Generalized α-Meir-Keeler Contraction Mappings on Branciari b-metric Spaces

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Abstract. In this paper, α-Meir-Keeler and generalized α-Meir-Keeler contractions on Branciari b-metric spaces are introduced. Existence and uniqueness of fixed points of such contractions are discussed and related theorems are proved. Various consequences of the main results are also presented.

1. Introduction and Preliminaries

One recent generalization in metric space theory has been defined by means of combining the Branciari (or rectangular) metric and b-metric. This new metric type is referred to as rectangular b-metric or Branciari b-metric. Accordingly, the analogs of contraction mappings on metric spaces have been studied on the Branciari b-metric spaces and some fixed point results have been proved [6–8, 12].

In this paper we consider the problem of existence and uniqueness of fixed points for contraction mappings of Meir-Keeler type defined on Branciari b-metric spaces. We present results which provide the conditions for existence and uniqueness of fixed points of such mappings.

In what follows, we describe the essentials on Branciari b-metric spaces and Meir-Keeler type contractions.

The concept of b-metric spaces have been introduced by Czerwik [5] and Bakhtin [3].

Definition 1.1. *[3, 5] Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a mapping satisfying the following conditions for all $x, y, z \in X$:*

(M_b1) $d(x, y) = 0$ if and only if $x = y$;

(M_b2) $d(x, y) = d(y, x)$;

(M_b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for some real number $s \geq 1$.

Then the mapping d is called a b-metric and the pair (X, d) is called a b-metric space (M_bS) with a constant $s \geq 1$.

On the other hand, Branciari [4] proposed a generalization of the metric in which he replaced the triangular inequality by a rectangular inequality. This new metric have been referred to by different names such as generalized metric, rectangular metric and Branciari metric. Following the paper by Aydi *et al.* [2], we will call this metric Branciari metric.

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Definition 1.2. *[4] Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:*
(BM1) $d(x,y) = 0$ if and only if $x = y$;
(BM2) $d(x,y) = d(y,x)$;
(BM3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Combining these two types of metric makes it possible to introduce a new metric as in the following definition.

Definition 1.3. *[6] Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:*
(BM_b1) $d(x,y) = 0$ if and only if $x = y$;
(BM_b2) $d(x,y) = d(y,x)$;
(BM_b3) $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ for some real number $s \geq 1$.
The map d is called a Branciari b-metric and the pair (X,d) is called a Branciari b-metric space (BM_bS).

We give next an important warning concerning Branciari metric spaces and Branciari b-metric spaces.
If an open ball of radius r centered at $x \in X$ is denoted and defined as

$$B_r(x) = \{y \in X \mid d(x,y) < r\},$$

such an open ball is not necessarily an open set in Branciari metric spaces or in Branciari b-metric space.
Let $\mathcal{T}$ be the collection of all subsets $\mathcal{A}$ of X with the following property: For each $a \in \mathcal{A}$ there exist $r > 0$ such that $B_r(a) \subseteq \mathcal{A}$. Then $\mathcal{T}$ defines a topology for the BM_bS (X,d), which is not necessarily Hausdorff.
The next definition gives the concepts of convergent sequence, Cauchy sequence, completeness and continuity on Branciari b-metric space.

Definition 1.4. *[6] Let (X,d) be a Branciari b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

1. *A sequence $\{x_n\} \subset X$ converges to a point $x \in X$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$. The limit notation is the same as that in metric spaces, that is,*
$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$
2. *A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$, in other words, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.*
3. *(X,d) is called a complete Branciari b-metric space if every Cauchy sequence in X converges to some $x \in X$.*
4. *A mapping $T : X \to X$ is said to be continuous with respect to the Branciari b-metric d if, for any sequence $\{x_n\} \subset X$ which converges to some $x \in X$, that is $\lim_{n\to\infty} d(x_n, x) = 0$ we have $\lim_{n\to\infty} d(Tx_n, Tx) = 0$.*

Some more warnings about the Branciari and Branciari b-metric spaces are listed below.

1. The limit of a sequence in a Branciari or a Branciari b-metric spaces is not necessarily unique.
2. A convergent sequence in a Branciari or a Branciari b-metric spaces may not be a Cauchy sequence.
3. A Branciari or a Branciari b-metric may not be continuous.

These facts can be seen in the next example inspired by [6].

Example 1.5. *Let $X = A \cup B$ where $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ and $B = \{2,3,4,5,\ldots\}$. Let the function $d : X \times X \to [0,\infty)$ satisfying $d(x,y) = d(y,x)$ be defined as follows.*

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 2\alpha & \text{if} \quad x, y \in A, \\ \dfrac{\alpha}{2n} & \text{if} \quad x \in A, y \in \{2,3\}, \\ \alpha & \quad \text{otherwise}. \end{cases}$$

for some $\alpha > 0$. Observe that

$$d(\frac{1}{2},\frac{1}{3}) = 2\alpha > d(\frac{1}{2},2) + d(2,\frac{1}{3}) = \frac{5\alpha}{12},$$

so, d is not a metric,

$$d(\frac{1}{2},\frac{1}{3}) = 2\alpha > d(\frac{1}{2},2) + d(2,5) + d(5,\frac{1}{3}) = \frac{17\alpha}{12},$$

hence, d is not a Branciari metric. In addition

$$d(\frac{1}{m},\frac{1}{n}) = 2\alpha > s[d(\frac{1}{n},2) + d(2,\frac{1}{m})] = 2\alpha s\frac{m+n}{4mn},$$

for $n, m \in \mathbb{N}$ satisfying $\frac{4mn}{m+n} > s$. Therefore, d is not a b-metric.

However, it is Branciari b-metric with $s = 2$ because we have

$$d(x, y) \leq 2[d(x, u) + d(u, v) + d(v, y)],$$

for all $x, y, u, v \in X$ such that u, v are different from each other and from x and y.

It is easy to see that

$$\lim_{n\to\infty} d(\frac{1}{n},2) = \lim_{n\to\infty}\frac{\alpha}{2n} = 0,$$

and

$$\lim_{n\to\infty} d(\frac{1}{n},3) = \lim_{n\to\infty}\frac{\alpha}{2n} = 0,$$

that is, the sequence $\{\frac{1}{n}\}$ converges to both 2 and 3. On the other hand, the sequence $\{\frac{1}{n}\}$ is convergent but not Cauchy. Indeed,

$$\lim_{n\to\infty} d(\frac{1}{n},\frac{1}{n+k}) = \lim_{n\to\infty} 2\alpha \neq 0.$$

Another fact is that although the open ball $B_{\alpha/2}(\frac{1}{2}) = \{2, 3, \frac{1}{2}\}$ contains 2, there is no positive r for which $B_r(2) \subset B_{\alpha/2}(\frac{1}{2})$. Also, there are no $r_1, r_2 > 0$ such that $B_{r_1}(2) \cap B_{r_2}(3) = \phi$. Hence, (X, d) is not Hausdorff.

All these facts about Branciari b-metric create troubles when dealing with convergent and Cauchy sequences. However, these troubles can be overcome with the help of the following property.

Proposition 1.6. *[10] Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X, d) such that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X$. Then $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.*

Remark 1.7. *The Proposition 1.6 is valid if we replace Branciari metric space by a Branciari b-metric space.*

We will investigate the fixed points of Meir-Keeler type contractions on Branciari b-metric spaces. Therefore, we first recall the definition of the classical Meir-Keeler contraction.

Definition 1.8. *[11] Let (X, d) be a metric space. Let $T : X \to X$ be a mapping satisfying the following.*

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon, \tag{1}$$

for all $x, y \in X$. Then T is called Meir-Keeler contraction.

The following remark can be observed immediately.

Remark 1.9. *If T is Meir-Keeler contraction then,*

$$d(Tx,Ty) < d(x,y),$$

for all $x,y \in X$ when $x \neq y$. Also, if $x = y$ then $d(Tx,Ty) = 0$, hence,

$$d(Tx,Ty) \leq d(x,y),$$

for all $x,y \in X$.

We insert the notion of α-admissibility to the Meir-Keeler contraction mappings in order to obtain more general results. The notion of α-admissible mappings have been introduced by Samet *et al.* [14] as follows.

Definition 1.10. *[14] A mapping $T : X \to X$ is called α-admissible if for all $x,y \in X$ we have*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1, \tag{2}$$

where $\alpha : X \times X \to [0,\infty)$ is a given function.

2. Main Results

In this paper we will generalize the classical Meir-Keeler contraction mappings by inserting the α-admissibility and replacing the metric $d(x,y)$ in the definition by a more general term. Moreover, we will define these mappings on Branciari b-metric spaces.

Definition 2.1. *Let (X,d) be a Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \ \text{ implies } \ \alpha(x,y)d(Tx,Ty) < \frac{\varepsilon}{s}, \tag{3}$$

for all $x,y \in X$. Then T is called α-Meir-Keeler contraction.

Remark 2.2. *If T is an α-Meir-Keeler contraction, then*

$$\alpha(x,y)d(Tx,Ty) \leq \frac{d(x,y)}{s},$$

for all $x,y \in X$, where the equality holds for $x = y$.

Further generalization on Meir-Keeler mappings can be done as follows.

Definition 2.3. *Let (X,d) be a Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \ \text{ implies } \ \alpha(x,y)d(Tx,Ty) < \frac{\varepsilon}{s}, \tag{4}$$

where

$$M(x,y) = \max\{d(x,y), d(Tx,x), d(Ty,y)\}, \tag{5}$$

for all $x,y \in X$, then T is called generalized α-Meir-Keeler contraction.

Remark 2.4. *Let $T : X \to X$ be a generalized α-Meir Keeler contraction. Then*

$$\alpha(x,y)d(Tx,Ty) \leq \frac{M(x,y)}{s}$$

for all $x,y \in X$, where the equality may hold only when $x = y$.

We will first prove the following lemma which will be used in the proof of existence and uniqueness theorems.

Lemma 2.5. *Let (X,d) be a Branciari b-metric space with a constant $s \geq 1$. Let $\{x_n\}$ be a sequence in X satisfying*

1. $x_m \neq x_n$ *for all* $m \neq n$, $m,n \in \mathbb{N}$,
2. $d(x_n, x_{n+1}) \leq \frac{1}{s} d(x_{n-1}, x_n)$, *for all* $n \in \mathbb{N}$,
3. $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$.

Then $\{x_n\}$ is a Cauchy sequence in (X,d).

Proof. To show that $\{x_n\}$ is a Cauchy sequence, it suffices to show that for any $k \in \mathbb{N}$

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0.$$

First, we observe that from the condition (2) it follows that

$$d(x_n, x_{n+1}) \leq \frac{1}{s^n} d(x_0, x_1) \tag{6}$$

for all $n \in \mathbb{N}$. We use the notations $a_n = d(x_n, x_{n+1})$ and $b_n = d(x_n, x_{n+2})$ for brevity. Note that taking limit as $n \to \infty$ in (6) we obtain

$$0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{1}{s^n} a_0 = 0. \tag{7}$$

Therefore, regarding (7) and the condition (3), for $k = 1$ or $k = 2$ we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0.$$

Consider $d(x_n, x_{n+k})$ with $k = 2p + 1 \geq 3$. Since $x_n \neq x_m$ for all $m \neq n$, $m,n \in \mathbb{N}$, we may apply the rectangular inequality (BM_b3) repeatedly to obtain

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})] \\ & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+k})] \\ & \leq & \\ & \vdots & \\ & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] + \ldots \\ & + & s^{(k-1)/2}[d(x_{n+k-3}, x_{n+k-2}) + d(x_{n+k-2}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k})] \\ & \leq & sa_n + s^2 a_{n+1} + s^3 a_{n+2} + \ldots + s^{k-1} a_{n+k-2} + s^k a_{n+k-1}. \end{array}$$

Taking (6) into account we have

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s\frac{1}{s^n} a_0 + s^2 \frac{1}{s^{n+1}} a_0 + s^3 \frac{1}{s^{n+2}} a_0 + \ldots + s^{k-1} \frac{1}{s^{n+k-2}} a_0 \\ & = & \frac{(k-1)a_0}{s^{n-1}}, \end{array}$$

which implies

$$\lim_{n\to\infty} d(x_n, x_{n+k}) \leq \lim_{n\to\infty} \frac{(k-1)a_0}{s^{n-1}} = 0,$$

that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

for any $k = 2p + 1 \in \mathbb{N}$.

Now, we take $d(x_n, x_{n+k})$ with $k = 2p \geq 4$ and regarding the fact that $x_n \neq x_m$ for all $m \neq n$, $m, n \in \mathbb{N}$ we apply the rectangular inequality (BM_b3) repeatedly which gives

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})] \\ & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+k})] \\ & \leq & \\ & \vdots & \\ & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] + \ldots \\ & + & s^{(k-1)/2}[d(x_{n+k-4}, x_{n+k-3}) + d(x_{n+k-3}, x_{n+k-2}) + d(x_{n+k-2}, x_{n+k})] \\ & \leq & sa_n + s^2 a_{n+1} + s^3 a_{n+2} + \ldots + s^{k-3} a_{n+k-4} + s^{k-2} a_{n+k-3} + s^{k-1/2} b_{n+k-2}. \end{array}$$

Because of (6) we get

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s\dfrac{1}{s^n}a_0 + s^2\dfrac{1}{s^{n+1}}a_0 + s^3\dfrac{1}{s^{n+2}}a_0 \\ & + & \ldots + s^{k-2}\dfrac{1}{s^{n+k-3}}a_0 + s^{k-1/2} b_{n+k-2} \\ & = & \dfrac{(k-2)a_0}{s^{n-1}} + s^{k-1/2} b_{n+k-2}, \end{array}$$

and taking into account the condition (3) we see that

$$\lim_{n\to\infty} d(x_n, x_{n+k}) \leq \lim_{n\to\infty}\left[\frac{(k-2)a_0}{s^{n-1}} + s^{k-1/2} b_{n+k-2}\right] = 0,$$

and conclude

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

for any $k = 2p \in \mathbb{N}$, which completes the proof. □

We will first prove an existence theorem for the fixed points of generalized α-Meir-Keeler contractions on Branciari b-metric spaces.

Theorem 2.6. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a continuous generalized α-Meir-Keeler contraction, that is, T satisfies the conditions of Definition 2.3. If $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be the element for which $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. Define the sequence $\{x_n\}$ in X as

$$x_{n+1} = Tx_n \text{ for } n \in \mathbb{N}.$$

If for some $n_0 \in \mathbb{N}_0$ we have $x_{n_0} = x_{n_0+1}$, that is $d(x_{n_0}, x_{n_0+1}) = 0$ then x_{n_0} would be a fixed point of T. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. First, we note that since T is α-admissible, then we have

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \tag{8}$$

or, in general,

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N}. \tag{9}$$

Similarly, because of $\alpha(x_0, T^2x_0) \geq 1$ we deduce

$$\alpha(x_0, T^2x_0) = \alpha(x_0, x_2) \geq 1 \Rightarrow \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1, \tag{10}$$

and hence,

$$\alpha(x_n, x_{n+2}) \geq 1 \quad \forall n \in \mathbb{N}. \tag{11}$$

We will show that the sequence $\{x_n\}$ satisfies the conditions of the Proposition 1.6 and the Lemma 2.5. Define the sequences $\{a_n\}$ and $\{b_n\}$ as $a_n := d(x_n, x_{n+1})$ and $b_n := d(x_n, x_{n+2})$. If we put $x = x_n$ and $y = x_{n+1}$ in (4), we note that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x_n, x_{n+1}) < \varepsilon + \delta \Longrightarrow \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) < \frac{\varepsilon}{s}, \tag{12}$$

where

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

From the Remark 2.4 we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) \leq \frac{M(x_n, x_{n+1})}{s},$$

where due to the fact that $x_n \neq x_{n+1}$ we see that equality does not hold, hence

$$d(x_{n+1}, x_{n+2}) < \frac{M(x_n, x_{n+1})}{s}. \tag{13}$$

If $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$, then (13) implies

$$d(x_{n+1}, x_{n+2}) < \frac{d(x_{n+1}, x_{n+2})}{s}$$

which is not possible. Then $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, so that (13) yields

$$d(x_{n+1}, x_{n+2}) < \frac{d(x_n, x_{n+1})}{s}. \tag{14}$$

Thus, the condition (2) of Lemma 2.5 holds.

For the sequence $\{b_n\}$ we consider the condition (4) with $x = x_n$ and $y = x_{n+2}$. Then we have

$$\varepsilon \leq M(x_n, x_{n+2}) < \varepsilon + \delta \Longrightarrow \alpha(x_n, x_{n+2})d(Tx_n, Tx_{n+2}) < \frac{\varepsilon}{s}, \tag{15}$$

where

$$M(x_n, x_{n+2}) = \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3})\}.$$

From the Remark 2.4 we have

$$d(x_{n+1}, x_{n+3}) = d(Tx_n, Tx_{n+2}) \leq \alpha(x_n, x_{n+2})d(Tx_n, Tx_{n+2}) \leq \frac{M(x_n, x_{n+2})}{s}. \tag{16}$$

Clearly,

$$M(x_n, x_{n+2}) = \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3})\} = \max\{b_n, a_n, a_{n+2}\}$$

can be either a_n or b_n since $\{a_n\}$ is a decreasing sequence. Consider the sequence $\{M_n\}$ defined as $M_n = \max\{a_n, b_n\}$. We already have $a_{n+1} \leq a_n$ and from (16) $b_{n+1} \leq \dfrac{\max\{b_n, a_n\}}{s} \leq \max\{b_n, a_n\}$. Therefore, for all $n \in \mathbb{N}$

$$M_{n+1} = \max\{a_{n+1}, b_{n+1}\} \leq \max\{a_n, b_n\} = M_n.$$

This means that the positive decreasing sequence $\{M_n\}$ is convergent to some $M \geq 0$.

If $M > 0$, then we have

$$M = \lim_{n\to\infty} \max\{a_n, b_n\} = \max\{\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\} = \max\{0, \lim_{n\to\infty} b_n\} = \lim_{n\to\infty} b_n.$$

On the other hand, if we let $n \to \infty$ in (16), we obtain

$$M = \lim_{n\to\infty} b_{n+1} < \lim_{n\to\infty} \frac{M_n}{s} = \frac{M}{s},$$

which contradicts the assumption $M > 0$ and hence, $M = 0$. Then, we conclude

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} d(x_n, x_{n+2}) = 0,$$

that is, the condition (3) of the Lemma 2.5 is satisfied.

Next we will show that for all $n \neq m$,

$$x_n \neq x_m. \tag{17}$$

Suppose on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. We already have $d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}$, hence, without loss of generality we may take $m > n + 1$. Since we have $d(x_m, x_{m+1}) = d(x_n, x_{n+1})$, the inequality (14) implies

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) < \frac{d(x_{m-1}, x_m)}{s} < \frac{d(x_{m-2}, x_{m-1})}{s^2} < \ldots < \frac{d(x_n, x_{n+1})}{s^{m-n}} \tag{18}$$

which is not possible. Therefore, the assumption that $x_n = x_m$ for some $m \neq n$ is incorrect and we should have $x_n \neq x_m$ for all $m \neq n$.

By the Lemma 2.5, the sequence $\{x_n\}$ is Cauchy in the complete Branciari b-metric space (X, d), so it converges to a limit $z \in X$. In addition, by the Proposition 1.6 there exists a unique $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{19}$$

By the continuity of T we have

$$\lim_{n\to\infty} d(Tx_n, Tz) = \lim_{n\to\infty} d(x_{n+1}, Tz) = 0,$$

that is, the sequence $\{x_n\}$ converges to Tz as well. Since the limit is unique, we conclude that $Tz = z$ which completes the proof. □

In order to provide the uniqueness of the fixed point of α-admissible mappings, extra condition is required. There are different versions of the uniqueness condition, two of which are given below.

($U1$) For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.

($U2$) For every pair x and y of fixed points of T, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

We give uniqueness theorem employing the condition ($U1$).

Theorem 2.7. *If the condition* ($U1$) *is added to the conditions of Theorem 2.6, then the mapping T has a unique fixed point.*

Proof. First, observe that the existence of a fixed point is proved in Theorem 2.6. Therefore, we need to prove only the uniqueness of the fixed point. Let the mapping T have two fixed points, say $z, w \in X$, such that $z \neq w$.

By the contractive condition (3) with the fixed points z and w we see that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(z, w) < \varepsilon + \delta \text{ implies } \alpha(z, w) d(Tz, Tw) < \frac{\varepsilon}{s}, \tag{20}$$

where

$$M(z,w) = \max\{d(z,w), d(z,Tz), d(w,Tw)\} = \max\{d(z,w), 0, 0\} = d(z,w). \tag{21}$$

Regarding the condition $(U1)$, that is, $\alpha(z,w) \geq 1$, since $d(z,w) > 0$, the Remark 2.4 implies

$$d(z,w) = d(Tz,Tw) \leq \alpha(z,w)d(Tz,Tw) < \frac{M(z,w)}{s} = \frac{d(z,w)}{s}, \tag{22}$$

which is a contradiction. Therefore, $d(z,w) = 0$, or, $z = w$ which completes the proof of the uniqueness. □

In order to weaken the conditions for existence of a fixed point often the continuity of the mapping T is being replaced by the so-called α-regularity condition of the space. The α-regularity on a Branciari b-metric space is defined as follows.

Definition 2.8. *A Branciari b-metric space (X,d) is called α-regular if for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n,x) = 0$ and satisfying $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Combining the Theorems 2.6 and 2.7 with the condition of the Definition 2.8, we state another theorem.

Theorem 2.9. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a generalized α-Meir-Keeler contraction. Assume that,*

1. *There exists $x_0 \in X$ for which $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$,*
2. *Either T is continuous or (X,d) is α-regular. Then T has a fixed point in X.*

If, in addition, the condition $(U1)$ holds, the fixed point of T is unique.

Proof. The case of continuity of the mapping T has been considered in Theorem 2.6. We will only prove the existence part for the case of α-regularity of (X,d). As in Theorem 2.6, starting with the point x_0 for which $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$ we construct the sequence $\{x_n\} = \{Tx_{n-1}\}$, for all $n \in \mathbb{N}$. This sequence converges to a unique limit $z \in X$. We will show that z is a fixed point of T.

Due to the fact that (X,d) is α-regular, the sequence $\{x_n\}$ satisfies $\alpha(x_n,z) \geq 1$ since it converges to z and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. We substitute $x = x_n$ and $y = z$ in the inequality (3) which gives

$$\varepsilon \leq M(x_n,z) < \varepsilon + \delta \text{ implies } \alpha(x_n,z)d(Tx_n,Tz) < \frac{\varepsilon}{s}, \tag{23}$$

where

$$M(x_n,z) = \max\{d(x_n,z), d(x_n,Tx_n), d(z,Tz)\}. \tag{24}$$

On the other hand, from the Remark 2.4 we have

$$d(x_{n+1},Tz) = d(Tx_n,Tz) \leq \alpha(x_n,z)d(Tx_n,Tz) < \frac{M(x_n,z)}{s}. \tag{25}$$

By the Proposition 1.6, we have

$$\lim_{n\to\infty} d(x_{n+1},Tz) = d(z,Tz).$$

Also,

$$\lim_{n\to\infty} M(x_n,z) = \lim_{n\to\infty} \max\{d(x_n,z), d(x_n,Tx_n), d(z,Tz)\} = d(z,Tz).$$

Taking limit as $n \to \infty$ of both sides of (25) we end up with

$$d(z,Tz) < \frac{d(z,Tz)}{s},$$

and thus conclude that $d(z,Tz) = 0$.

The uniqueness proof is identical to the proof of Theorem 2.7. □

3. Consequences

We next derive some consequences of the main results presented in the previous section. The generalized Meir-Keeler contraction considered in this work covers many particular types of contractions. We will discuss some of these cases below.

Corollary 3.1. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping satisfying the following conditions.*

(1) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq N(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y)d(Tx,Ty) < \frac{\varepsilon}{s}, \tag{26}$$

where

$$N(x,y) = \max\{d(x,y), \frac{1}{2}[d(Tx,x) + d(Ty,y)]\}, \tag{27}$$

for all $x, y \in X$.

(2) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.

If T is continuous or (X, d) is α-regular, then T has a fixed point. If, in addition, T satisfies the condition $(U1)$, then the fixed point is unique.

Proof. The proof is obvious from the Theorem 2.9 due to the fact that $N(x,y) \leq M(x,y)$ for all $x, y \in X$. □

Corollary 3.2. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-Meir-Keeler contraction, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y)d(Tx,Ty) < \frac{\varepsilon}{s}, \tag{28}$$

for all $x, y \in X$.

If T is continuous or (X, d) is α-regular, then T has a fixed point. If, in addition, T satisfies the condition $(U1)$, then the fixed point is unique.

Proof. The proof follows easily from the relation $d(x,y) \leq M(x,y)$ for all $x, y \in X$. □

Some more consequences are concluded from the main result given in Theorem 2.9 by taking $\alpha(x,y) = 1$.

Corollary 3.3. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be a continuous mapping satisfying the following:*
For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \frac{\varepsilon}{s}, \tag{29}$$

where

$$M(x,y) = \max\{d(x,y), d(Tx,x), d(Ty,y)\}, \tag{30}$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 3.4. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be a continuous mapping. Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq N(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \frac{\varepsilon}{s}, \tag{31}$$

where

$$N(x,y) = \max\{d(x,y), \frac{1}{2}[d(Tx,x) + d(Ty,y)]\}, \tag{32}$$

for all $x, y \in X$. Then T has a unique fixed point.

The last consequence is the classical Meir-Keeler contraction on Branciari b-metric spaces.

Corollary 3.5. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be a continuous Meir-Keeler contraction mapping, that is, given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \frac{\varepsilon}{s}, \tag{33}$$

for all $x, y \in X$. Then T has a unique fixed point.

The general nature of α-admissible mappings makes it possible to deduce fixed point theorems for cyclic mappings and mappings defined on partially ordered spaces. Assume that a partial ordering $\preceq$ is defined on a Branciari b-metric space (X, d) with a constant $s \geq 1$. Let $T : X \to X$ be an increasing mapping. Then, by choosing the function α as

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \preceq y \text{ or } y \preceq x \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

the following fixed point theorems easily follow from the main theorems.

Corollary 3.6. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Let $T : X \to X$ be an increasing mapping satisfying the following condition for all comparable pairs $x, y \in X$:*
Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \frac{\varepsilon}{s}, \tag{34}$$

where

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\}. \tag{35}$$

Assume that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$ for some $x_0 \in X$. Then T has a fixed point. If, in addition, any two fixed points of T are comparable then T has a unique fixed point.

Corollary 3.7. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Let $T : X \to X$ be an increasing mapping satisfying the following condition for all comparable pairs $x, y \in X$:*
Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq N(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \frac{\varepsilon}{s}, \tag{36}$$

where

$$N(x, y) = \max\{d(x, y), \frac{1}{2}[d(Tx, x) + d(Ty, y)]\}. \tag{37}$$

Assume that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$ for some $x_0 \in X$. Then T has a fixed point. If, in addition, any two fixed points of T are comparable then T has a unique fixed point.

Corollary 3.8. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Let $T : X \to X$ be an increasing mapping satisfying the following condition for all comparable pairs $x, y \in X$:*
Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \frac{\varepsilon}{s}. \tag{38}$$

Assume that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$ for some $x_0 \in X$. Then T has a fixed point. If, in addition, any two fixed points of T are comparable then T has a unique fixed point.

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