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Almost Sure Exponential Stability of the θ-Euler-Maruyama Method for Neutral Stochastic Differential Equations with Time-Dependent Delay when $\theta \in [0, \frac{1}{2}]$

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Abstract. This paper represents a generalization of the stability result on the Euler-Maruyama solution, which is established in the paper *M. Milošević, Almost sure exponential stability of solutions to highly nonlinear neutral stochastics differential equations with time-dependent delay and Euler-Maruyama approximation, Math. Comput. Model. 57 (2013) 887 – 899.* The main aim of this paper is to reveal the sufficient conditions for the global almost sure asymptotic exponential stability of the θ-Euler-Maruyama solution ($\theta \in [0, \frac{1}{2}]$), for a class of neutral stochastic differential equations with time-dependent delay. The existence and uniqueness of solution of the approximate equation is proved by employing the one-sided Lipschitz condition with respect to the both present state and delayed arguments of the drift coefficient of the equation. The technique used in proving the stability result required the assumption $\theta \in (0, \frac{1}{2}]$, while the method is defined by employing the parameter θ with respect to the both drift coefficient and neutral term. Bearing in mind the difference between the technique which will be applied in the present paper and that used in the cited paper, the Euler-Maruyama case ($\theta = 0$) is considered separately. In both cases, the linear growth condition on the drift coefficient is applied, among other conditions. An example is provided to support the main result of the paper.

1. Introduction and Preliminary Results

Stochastic differential equations are well-known for describing those phenomena which are influenced by some random factors. Often the investigation of such phenomena requires more complex models based on (neutral) stochastic differential delay equations or (neutral) stochastic functional differential equations. In most cases these equations cannot be solved explicitly, so it is necessary to study the approximate solutions. There is an extensive literature based on the analysis of different properties of the exact and approximate solutions of stochastic differential equations (see, for example [1, 4–9, 12, 13, 17]).

Very important issue in the analysis of stochastic differential equations is to determine the conditions under which the exact and approximate solutions share some stability properties. There are many papers, such as [2, 10, 14, 16, 18–20], where the authors studied a.s. exponential stability of different approximate solutions for several classes of stochastic differential equations.

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The subject of this paper is consideration of the θ-Euler-Maruyama method, when $\theta \in [0, \frac{1}{2}]$, for a class of neutral stochastic differential equations with time-dependent delay under the linear growth condition on the drift coefficient of the equation, among other conditions. Because of the presence of the time-dependent delay, this method could be implicit with respect to the present-state argument of the drift, as well as, with respect to the delayed arguments of the drift and neutral term. For that reason, the technique which is used in this paper differs from those used in the context of some other classes of stochastic differential equations. Significant contribution to the analysis of the θ-Euler-Maruyama method for neutral stochastic differential delay equations is given in [2], where authors considered the case when the delay is constant and $\theta \in (\frac{1}{2}, 1]$. In this case, the method is implicit only with respect to the present-state argument of the drift. Moreover, in [19] and [20], one can find the analysis of the θ-methods for stochastic differential equations with constant delay and for neutral stochastic differential equations with constant delay, respectively.

As usual, we first present some standard notations and definitions which are necessary for further consideration. The initial assumption is that all random variables and processes considered here are defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all P-null sets). Let $w = \{w(t), t \geq 0\}$ be an m-dimensional standard Brownian motion. Let $|x|$ stand for the Euclidean norm of $x \in R^d$ and, for simplicity, $|A|^2 = trace(A^T A)$ for matrix A, where A^T is the transpose of a vector or a matrix.

For a given $\tau > 0$, denote by $C([-\tau, 0]; R^d)$ the family of continuous functions $\varphi : [-\tau, 0] \to R^d$ with the supremum norm $\|\varphi\| = \sup_{-\tau\leq t\leq 0} |\varphi(t)|$. Also, denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ the family of $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^d)$-valued bounded random variables.

Let $\delta : R_+ \to [0, \tau]$ be the delay function which is Borel-measurable. We consider the following neutral stochastic differential equation with time-dependent delay

$$d[x(t) - u(x(t - \delta(t)), t)] = f(x(t), x(t - \delta(t)), t)dt + g(x(t), x(t - \delta(t)), t)dw(t), \ \ t \geq 0, \tag{1}$$

satisfying the initial condition

$$x_0 = \varphi = \{\varphi(t) : t \in [-\tau, 0]\} \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^d), \tag{2}$$

where the functions

$$f : R^d \times R^d \times R_+ \to R^d, \ g : R^d \times R^d \times R_+ \to R^{d\times m}, \ u : R^d \times R_+ \to R^d$$

are all Borel-measurable and $x(t)$ is a d-dimensional state process.

A d-dimensional stochastic process $\{x(t), t \geq -\tau\}$ is said to be a solution to Eq.(1) if it is a.s. continuous, $\mathcal{F}_t$-adapted, $\int_0^\infty |f(x(t), x(t - \delta(t)), t)|dt < \infty$ a.s., $\int_0^\infty |g(x(t), x(t - \delta(t)), t)|^2 dt < \infty$ a.s, $x_0 = \varphi$ a.s, and for every $t \geq 0$ the integral form of Eq.(1) holds a.s.

For the purpose of the following consideration we impose the assumptions which will be used explicitly in the paper.

$\mathcal{A}_1$: There exists a positive constant K such that, for all $x, y \in R^d$ and all $t \geq 0$,

$$|f(x, y, t)|^2 \leq K(|x|^2 + |y|^2). \tag{3}$$

$\mathcal{A}_2$: There exists a constant $\beta \in (0, 1)$ such that, for all $x, y \in R^d$ and all $t \geq 0$,

$$|u(x, t) - u(y, t)| \leq \beta|x - y|. \tag{4}$$

Additionally, if $u(0, t) = 0$, $t \geq 0$, then (4) implies that for all $x \in R^d$,

$$|u(x, t)| \leq \beta|x|. \tag{5}$$

$\mathcal{A}_3$: The delay function $\delta : R_+ \to [0, \tau]$ is differentiable and $|\delta'(t)| \leq \eta$, $t \geq 0$, where $\eta \in (0, 1)$.

It should be stressed that $\mathcal{A}_3$ implies that

$$|\delta(t) - \delta(s)| \leq \eta|t - s|, \ t, s \geq 0. \tag{6}$$

$\mathcal{A}_4$: Assume that there exist constants α_1 and α_2, for which $\alpha_1 > \frac{\alpha_2}{1-\eta} > 0$, such that, for all $x, y \in R^d$ and all $t \geq 0$,

$$2(x - u(y,t))^T f(x,y,t) + |g(x,y,t)|^2 \leq -\alpha_1 |x|^2 + \alpha_2 |y|^2. \tag{7}$$

Also, we will assume that

$$f(0,0,t) = g(0,0,t) = u(0,t) = 0,\ t \geq 0. \tag{8}$$

On the basis of the papers [12] and [13], one can conclude that hypotheses $\mathcal{A}_2 - \mathcal{A}_4$, together with the local Lipschitz condition on f and g and the assumption (8), guarantee the existence and uniqueness of the global solution of Eq.(1), which is almost surely exponentially stable. Results from [13] suggested to employ $\mathcal{A}_4$, as well as the linear growth condition $\mathcal{A}_1$ on the drift coefficient of Eq. (1), in order to prove the almost sure exponential stability of the θ-Euler-Maruyama solution.

It should be pointed out that this paper represents a generalization of the stability result on the Euler-Maruyama solution, which is established in [13], due to the fact that the consideration in this paper is based on the assumption that $\theta \in [0, \frac{1}{2}]$. However, the technique which will be used in the present paper is different from the one applied in [13] and it is mainly addressed to the case when $\theta \in (0, \frac{1}{2}]$. Since it could not be applied in its original form in the Euler-Maruyama case ($\theta = 0$), we will consider that case separately.

Since the equations which determines the θ-Euler-Maruyama solution are implicit, the first item that need to be considered is the existence and uniqueness of solutions of these equations. In that sense, we will employ the one-sided Lipschitz conditions in the first and second argument of the function f, which are given within the assumption C_1.

C_1: Let $f \in C(R^d \times R^d \times R_+; R^d)$ and suppose that there exist constants $\mu_1, \mu_2 > 0$ such that, for all $x, y, z \in R^d$ and all $t \geq 0$,

$$\langle x - y, f(x,z,t) - f(y,z,t) \rangle \leq \mu_1 |x - y|^2, \tag{9}$$
$$\langle x - y, f(z,x,t) - f(z,y,t) \rangle \leq \mu_2 |x - y|^2. \tag{10}$$

Moreover, we introduce the following lemmas which will be used in proving the stability results. The first one represents an elementary inequality, while the second one is proved in [13] (see Lemma 3).

Lemma 1.1. *For all $a, b > 0$, $p \geq 1$, $c > 0$, we have that*

$$(a + b)^p \leq (1 + c)^{p-1}(a^p + c^{1-p} b^p).$$

From Lemma 1.1 and assumption (5), if $c = \beta$, then we conclude that for all $x, y \in R^d$, $t \geq 0$,

$$|x - u(y,t)|^p \leq (1 + \beta)^{p-1}(|x|^p + \beta |y|^p). \tag{11}$$

In the sequel, we will use $[\cdot]$ to denote the integer part function.

Lemma 1.2. *Assume that (6) holds. For any $i \in \{0, 1, 2, \ldots\}$, let $i - [\delta(i\Delta)/\Delta] = a$, where $a \in \{-n_*, -n_*+1, ..., 0, 1, ..., i\}$. Then,*

$$\#\{j \in \{0, 1, 2, \ldots\} : j - [\delta(i\Delta)/\Delta] = a\} \leq [(1 - \eta)^{-1}] + 1,$$

where #S denotes the number of elements of the set S.

2. Almost Sure Exponential Stability of the θ-Euler-Maruyama Solution when $\theta \in (0, \frac{1}{2}]$

First, let us present the autonomous version of the initial equation (1), that is,

$$x(t) = \varphi(0) + u(x(t - \delta(t))) - u(x(-\delta(0))) + \int_0^t f(x(s), x(s - \delta(s)))ds \tag{12}$$
$$+ \int_0^t g(x(s), x(s - \delta(s)))dw(s), \quad t \geq 0,$$

satisfying the initial condition $x(t) = \varphi(t)$, $t \in [-\tau, 0]$.

In the sequel we will assume that, instead of the assumptions $\mathcal{A}_1$–$\mathcal{A}_4$, (8) and C_1, their autonomous versions hold.

Choose a step size $\Delta \in (0,1)$ such that $\Delta = \tau/n_*$ for some integer $n_* > \tau$. We will define the discrete θ-Euler-Maruyama approximate solution q corresponding to Eq.(12) on the equidistant partition $k\Delta, k = -(n_*+1), -n_*, ..., -1, 0, 1, ...$ In that sense, set

$$\delta(-\Delta) = \delta(0), \quad q_{-(n_*+1)\Delta} = \varphi(-n_*\Delta). \tag{13}$$

Define

$$q_k = \varphi(k\Delta),\ k = -n_*, -n_*+1, ..., 0, \tag{14}$$

while, for $k \in \{0, 1, 2, ...\}$,

$$\begin{aligned} q_{k+1} &= q_k + \theta u(q_{k+1-[\delta((k+1)\Delta)/\Delta]}) + (1-\theta)u(q_{k-[\delta(k\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}) - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) \\ &\quad + \theta f(q_{k+1}, q_{k+1-[\delta((k+1)\Delta)/\Delta]})\Delta + (1-\theta) f(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta + g(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta w_k, \end{aligned} \tag{15}$$

where $\Delta w_k = w((k+1)\Delta) - w(k\Delta)$. For simplicity, denote

$$\begin{aligned} z_k &= q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}) - \theta f(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta, \\ f_k &= f(q_k, q_{k-[\delta(k\Delta)/\Delta]}), \\ g_k &= g(q_k, q_{k-[\delta(k\Delta)/\Delta]}), \end{aligned}$$

such that

$$z_{k+1} = z_k + f_k\Delta + g_k\Delta w_k, \quad k \in \{0, 1, 2, ...\}. \tag{16}$$

As mentioned in the introduction, first of all, we are interested in the conditions under which there exists unique θ-Euler-Maruyama approximate solution of Eq.(15). In other words, we are interested in establishing the existence and uniqueness of solution to the equation of the form

$$x = \tilde{d} + \theta\Big(\Delta f(x, a) I_{A^c} + \Delta f(x, x) I_A + u(x) I_A\Big), \quad x \in R^d, \tag{17}$$

for given $a, \tilde{d} \in R^d$, where $I_A = 1$ if $[\delta((k+1)\Delta)/\Delta] = 0$ and $I_A = 0$, otherwise. In that sense, we impose the next theorem without proof. The proof can be found in [11] or in [15].

Theorem 2.1 (Brouwer's fixed point theorem). *Assume that $K \subset R^d$ is a compact and convex set and that $f : K \to K$ is continuous function. Then there exists a fixed point of f, i.e., $x \in K$ such that $f(x) = x$.*

Now, we present a lemma which establishes the existence and uniqueness of solution to Eq.(17) for any $\theta \in (0, 1]$. It should be mentioned that the proof corresponding to the case when $\theta = 1$ can be found in [14].

Lemma 2.2. *Assume that the condition (4) and the hypothesis C_1 hold. If $\theta((\mu_1 + \mu_2)\Delta + \beta) < 1$, then, there exists unique solution to Eq. (17).*

Proof. The uniqueness of solution to Eq. (17) is proved straightforwardly. Namely, if we suppose that x and y are both solutions of Eq. (17), then on the basis of conditions (4), (9) and (10), for any $a, \tilde{d} \in R^d$, we have that

$$\begin{aligned} |x-y|^2 &= \theta\Big[\Delta I_{A^c}\langle x-y, f(x,a) - f(y,a)\rangle + \Delta I_A \langle x-y, f(x,x) - f(y,y)\rangle + I_A\langle x-y, u(x) - u(y)\rangle\Big] \\ &\le \theta\Big[\Delta\mu_1 I_{A^c}|x-y|^2 + \Delta I_A\langle x-y, f(x,x) - f(x,y)\rangle + \Delta I_A\langle x-y, f(x,y) - f(y,y)\rangle + I_A\beta|x-y|^2\Big] \\ &\le \theta\Big[\Delta\mu_1 I_{A^c}|x-y|^2 + \Delta\mu_2 I_A|x-y|^2 + \Delta\mu_1 I_A|x-y|^2 + I_A\beta|x-y|^2\Big] \\ &\le \theta((\mu_1+\mu_2)\Delta + \beta)|x-y|^2. \end{aligned}$$

Bearing in mind the assumption $\theta((\mu_1+\mu_2)\Delta+\beta)<1$, we conclude that $x=y$.

In order to prove the existence of solution to Eq. (17), denote that

$$R=\frac{\theta|\Delta f(\tilde{d},a)I_{A^c}+\Delta f(\tilde{d},\tilde{d})I_A+u(\tilde{d})I_A|}{1-\theta((\mu_1+\mu_2)\Delta+\beta)}.$$

Then, define a ball $B=\{x\in R^d:|x-\tilde{d}|\le R\}$ and functions $H:R^d\to B,\ G:B\to B$, such that

$$H(x)=\tilde{d}+R\frac{x-\tilde{d}}{R\vee|x-\tilde{d}|},\ x\in R^d, \tag{18}$$

$$G(x)=H\Big(\tilde{d}+\theta(\Delta f(x,a)I_{A^c}+\Delta f(x,x)I_A+u(x)I_A)\Big),\ x\in B. \tag{19}$$

Since B is a compact and convex set and G is continuous function on B, it follows from Theorem 2.1 that there exists a fixed point $x^*=G(x^*)$.

If we assume that

$$\theta|\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A|>R, \tag{20}$$

then

$$x^*=G(x^*)=H\Big(\tilde{d}+\theta(\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A)\Big)=\tilde{d}+R\frac{\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A}{|\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A|}, \tag{21}$$

implying that $|x^*-\tilde{d}|=R$.

On the other hand, (21) yields

$$\frac{|\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A|}{R}(x^*-\tilde{d})=\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A.$$

Consequently, by repeating the corresponding procedure from [14] (Lemma 1), we obtain

$$|\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A|\le R,$$

which is a contradiction with respect to the assumption (20), since $\theta\in(0,1]$. Thus, we conclude that $\theta|\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A)|\le R$. Then, definitions (18) and (19) give

$$x^*=H\Big(\tilde{d}+\theta(\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A)\Big)=\tilde{d}+\theta(\Delta f(x^*,a)I_{A^c}+\Delta f(x^*,x^*)I_A+u(x^*)I_A),$$

that is, x^* is a unique solution of Eq. (17). □

Our main goal is to reveal the conditions under which the θ-Euler-Maruyama solution, defined by (13)-(15), is almost surely asymptotically exponentially stable in the sense of the following definition.

Definition 2.3. *The solution q_k of Eq.(15) is globally almost surely asymptotically exponentially stable if there exists a constant $\varepsilon>0$ such that*

$$\limsup_{k\to\infty}\frac{\log|q_k|}{k\Delta}\le-\varepsilon\ a.s.$$

for any bounded initial condition φ.

The next theorem establishes the global almost sure asymptotic exponential stability of the discrete θ-Euler-Maruyama solution when $\theta\in(0,\frac{1}{2}]$, for small enough step-size Δ.

Theorem 2.4. *Assume that the conditions of Lemma 2.2 hold, together with the hypotheses $\mathcal{A}_1$–$\mathcal{A}_4$. Additionally, let $\theta \in (0, \frac{1}{2}]$ and suppose that*

$$\beta \in \left(0, \frac{-1+\sqrt{3}}{2}\right), \tag{22}$$

$$\alpha_1 > \max\left\{K + \Big(K + \alpha_2 + 4\beta^2(1-\theta)^2\Big)([(1-\eta)^{-1}] + 1), 2\theta K + \alpha_2 + 4\beta^2\frac{(1-\theta)^2}{\theta}\right\}. \tag{23}$$

Then, there exists a $\Delta^* \in (0,1)$ such that the θ-Euler-Maruyama approximate solution defined by (13)-(15) is almost surely asymptotically exponentially stable, whenever $\Delta \in (0, \Delta^*)$.

Proof. On the basis of (15) and (16) we have that

$$\begin{aligned}|z_{k+1}|^2 &= |z_k|^2 + [2(q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k + |g_k|^2 + (1-2\theta)|f_k|^2\Delta]\Delta + m_k,\\ &= |z_k|^2 + [2(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k + |g_k|^2 + (1-2\theta)|f_k|^2\Delta]\Delta\\ &\quad +2(1-\theta)(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))^T f_k\Delta + m_k,\end{aligned} \tag{24}$$

where

$$m_k = |g_k\Delta w_k|^2 - |g_k|^2\Delta + 2(z_k + f_k\Delta)^T g_k\Delta w_k. \tag{25}$$

Applying $\mathcal{A}_4$ and $\mathcal{A}_2$, as well as the assumption $\mathcal{A}_1$ on (24), for $\alpha_1 > \frac{\alpha_2}{1-\eta} > 0$ and $\theta \in (0, \frac{1}{2}]$, we get

$$\begin{aligned}|z_{k+1}|^2 &\le |z_k|^2 + (-\alpha_1|q_k|^2 + \alpha_2|q_{k-[\delta(k\Delta)/\Delta]}|^2 + (1-2\theta)|f_k|^2\Delta)\Delta\\ &\quad +2\beta(1-\theta)|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}||f_k|\Delta + m_k\\ &\le |z_k|^2 + (-\alpha_1|q_k|^2 + \alpha_2|q_{k-[\delta(k\Delta)/\Delta]}|^2 + (1-2\theta)|f_k|^2\Delta)\Delta\\ &\quad +(\beta^2(1-\theta)^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + |f_k|^2)\Delta + m_k\\ &\le |z_k|^2 - \alpha_1|q_k|^2\Delta + \alpha_2|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta + (1-2\theta)K|q_k|^2\Delta^2 + (1-2\theta)K|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta^2\\ &\quad +2\beta^2(1-\theta)^2|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta + 2\beta^2(1-\theta)^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + K|q_k|^2\Delta + K|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta + m_k.\end{aligned} \tag{26}$$

Thus, the estimate (26) can be written as

$$\begin{aligned}|z_{k+1}|^2 &\le |z_k|^2 + \Big(\alpha_2 + (1-2\theta)K\Delta + 2\beta^2(1-\theta)^2 + K\Big)|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta\\ &\quad +2\beta^2(1-\theta)^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + \Big(-\alpha_1 + (1-2\theta)K\Delta + K\Big)|q_k|^2\Delta + m_k.\end{aligned} \tag{27}$$

Then, for an arbitrary constant $A > 1$, we find that

$$\begin{aligned}A^{(k+1)\Delta}|z_{k+1}|^2 - A^{k\Delta}|z_k|^2 &\le A^{(k+1)\Delta}|z_k|^2(1 - A^{-\Delta}) + [-\alpha_1 + (1-2\theta)K\Delta + K]\,\Delta A^{(k+1)\Delta}|q_k|^2\\ &\quad + \Big[\alpha_2 + (1-2\theta)K\Delta + 2\beta^2(1-\theta)^2 + K\Big]\,\Delta A^{(k+1)\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2\\ &\quad +2\beta^2(1-\theta)^2\Delta A^{(k+1)\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + A^{(k+1)\Delta}m_k.\end{aligned} \tag{28}$$

For simplicity, denote

$$\begin{aligned}R_1(\Delta) &= 1 - A^{-\Delta},\\ R_2(\Delta) &= -\alpha_1 + (1-2\theta)K\Delta + K,\\ R_3(\Delta) &= \alpha_2 + (1-2\theta)K\Delta + 2\beta^2(1-\theta)^2 + K.\end{aligned}$$

Consequently, we see from (28) that

$$\begin{aligned}A^{k\Delta}|z_k|^2 &\le |z_0|^2 + R_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|z_i|^2 + R_2(\Delta)\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + R_3(\Delta)\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2\\ &\quad +2\beta^2(1-\theta)^2\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2 + M_k,\end{aligned} \tag{29}$$

where

$$M_k = \sum_{i=0}^{k-1} A^{(i+1)\Delta} m_i$$

is a local martingale with $M_0 = 0$.

By the definition of z_k and Lemma 1.1, for $c = \beta$, we get

$$|z_k|^2 \le (1+\beta)|q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 + \frac{1+\beta}{\beta}|\theta f_k \Delta|^2.$$

Further on, applying the assumption $\mathcal{A}_1$, Lema 1.1, inequality (11) and then assumption $\mathcal{A}_2$, we find that

$$\begin{aligned}
|z_k|^2 &\le (1+\beta)|q_k - u(q_{k-[\delta(k\Delta)/\Delta]}) + (1-\theta)(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))|^2 \\
&\quad + \frac{1+\beta}{\beta}\theta^2 K\Big(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2\Big)\Delta^2 \\
&\le (1+\beta)^2|q_k - u(q_{k-[\delta(k\Delta)/\Delta]})|^2 + \frac{(1+\beta)^2}{\beta}(1-\theta)^2|u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))|^2 \\
&\quad + \frac{1+\beta}{\beta}\theta^2 K\Big(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2\Big)\Delta^2 \\
&\le (1+\beta)^3\left(|q_k|^2 + \beta|q_{k-[\delta(k\Delta)/\Delta]}|^2\right) + \beta(1+\beta)^2(1-\theta)^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2 \\
&\quad + \frac{1+\beta}{\beta}\theta^2 K\Big(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2\Big)\Delta^2 \\
&\le (1+\beta)^3\left(|q_k|^2 + \beta|q_{k-[\delta(k\Delta)/\Delta]}|^2\right) + 2\beta(1+\beta)^2(1-\theta)^2\left(|q_{k-[\delta(k\Delta)/\Delta]}|^2 + |q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2\right) \\
&\quad + \frac{1+\beta}{\beta}\theta^2 K(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2)\Delta^2 \\
&= \left((1+\beta)^3 + \frac{1+\beta}{\beta}\theta^2 K\Delta^2\right)|q_k|^2 + \left(\beta(1+\beta)^3 + 2\beta(1+\beta)^2(1-\theta)^2 + \frac{1+\beta}{\beta}\theta^2 K\Delta^2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad + 2\beta(1+\beta)^2(1-\theta)^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2. \qquad (30)
\end{aligned}$$

Then, substituting (30) into (29) we get

$$\begin{aligned}
A^{k\Delta}|z_k|^2 &\le |z_0|^2 + K_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + K_2(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 \\
&\quad + K_3(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2 + M_k, \qquad (31)
\end{aligned}$$

where

$$\begin{aligned}
K_1(\Delta) &= R_1(\Delta)\left((1+\beta)^3 + \frac{1+\beta}{\beta}\theta^2 K\Delta^2\right) + R_2(\Delta)\Delta, \\
K_2(\Delta) &= R_1(\Delta)\left(\beta(1+\beta)^3 + 2\beta(1+\beta)^2(1-\theta)^2 + \frac{1+\beta}{\beta}\theta^2 K\Delta^2\right) + R_3(\Delta)\Delta, \\
K_3(\Delta) &= 2R_1(\Delta)\beta(1+\beta)^2(1-\theta)^2 + 2\beta^2(1-\theta)^2\Delta.
\end{aligned}$$

Noting that $K_3(\Delta) > 0$, in a view of (13), we conclude that

$$K_3(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2 \le K_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 + K_3(\Delta)A^{\Delta}\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2. \qquad (32)$$

Thus (31) becomes

$$A^{k\Delta}|z_k|^2 \le |z_0|^2 + K_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + (K_2(\Delta) + K_3(\Delta)A^{\Delta})\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2$$
$$+K_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 + M_k. \tag{33}$$

By applying Lemma 1.2 to the second sum on the right-hand side of (33), we obtain

$$\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 \le A^{n_*\Delta}\sum_{i=0}^{k-1} A^{(i-[\delta(i\Delta)/\Delta]+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2$$
$$\le ([(1-\eta)^{-1}]+1)A^{n_*\Delta}\sum_{i=-n_*}^{k-1} A^{(i+1)\Delta}|q_i|^2. \tag{34}$$

Bearing in the mind that $n_*\Delta = \tau, K_2(\Delta) > 0$ and $K_3(\Delta) > 0$, on the basis of (34) the expression (33) can be estimated as

$$A^{k\Delta}|z_k|^2 \le X + h(A,\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + M_k, \tag{35}$$

where, for any $\Delta \in (0,1)$, we have

$$X = |z_0|^2 + K_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 + (K_2(\Delta) + K_3(\Delta)A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau}\sum_{i=-n_*}^{-1} A^{(i+1)\Delta}|\varphi(i\Delta)|^2 < \infty, \tag{36}$$

and

$$h(A,\Delta) = K_1(\Delta) + (K_2(\Delta) + K_3(\Delta)A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau}$$
$$= (1-A^{-\Delta})\Big((1+\beta)^3 + \frac{1+\beta}{\beta}\theta^2K\Delta^2\Big) + (-\alpha_1 + (1-2\theta)K\Delta + K)\Delta$$
$$+\Big[(1-A^{-\Delta})\Big(\beta(1+\beta)^3 + 2\beta(1+\beta)^2(1-\theta)^2 + \frac{1+\beta}{\beta}\theta^2K\Delta^2\Big)$$
$$+\Big(\alpha_2 + (1-2\theta)K\Delta + 2\beta^2(1-\theta)^2 + K\Big)\Delta$$
$$+\Big(2(1-A^{-\Delta})\beta(1+\beta)^2(1-\theta)^2 + 2\beta^2(1-\theta)^2\Delta\Big)A^{\Delta}\Big]([(1-\eta)^{-1}]+1)A^{\tau}. \tag{37}$$

One can observe that

$$\frac{d}{dA}h(A,\Delta) = \Delta A^{-\Delta-1}\Big((1+\beta)^3 + \frac{1+\beta}{\beta}\theta^2K\Delta^2\Big)$$
$$+\Big[(1-A^{-\Delta})\Big(\beta(1+\beta)^3 + 2\beta(1+\beta)^2(1-\theta)^2 + \frac{1+\beta}{\beta}\theta^2K\Delta^2\Big)$$
$$+\Big(\alpha_2 + (1-2\theta)K\Delta + 2\beta^2(1-\theta)^2 + K\Big)\Delta$$
$$+\Big(2(1-A^{-\Delta})\beta(1+\beta)^2(1-\theta)^2 + 2\beta^2(1-\theta)^2\Delta\Big)A^{\Delta}\Big]([(1-\eta)^{-1}]+1)\tau A^{\tau-1}$$
$$+([(1-\eta)^{-1}]+1)A^{\tau}\Big[\Delta A^{-\Delta-1}\Big(\beta(1+\beta)^3 + 2\beta(1+\beta)^2(1-\theta)^2 + \frac{1+\beta}{\beta}\theta^2K\Delta^2\Big)$$
$$+\Big(2(1-A^{-\Delta})\beta(1+\beta)^2(1-\theta)^2 + 2\beta^2(1-\theta)^2\Delta\Big)\Delta A^{\Delta-1} + 2\Delta A^{-1}\beta(1+\beta)^2(1-\theta)^2\Big] > 0. \tag{38}$$

On the other hand, we have

$$h(1,\Delta) = (-\alpha_1 + (1-2\theta)K\Delta + K)\Delta + \Big((1-2\theta)K\Delta + K + \alpha_2 + 4\beta^2(1-\theta)^2\Big)([(1-\eta)^{-1}]+1)\Delta. \tag{39}$$

The application of the condition (23) yields the existence of

$$\Delta_1 = \frac{\alpha_1 - K - \Big(K + \alpha_2 + 4\beta^2(1-\theta)^2\Big)([(1-\eta)^{-1}] + 1)}{(1-2\theta)K([(1-\eta)^{-1}] + 2)}, \tag{40}$$

such that, for any $\Delta \in (0, \Delta^*)$, where $\Delta^* = \Delta_1 \wedge 1$, we have that $h(1, \Delta) < 0$. Moreover, for any step-size $\Delta \in (0, \Delta^*)$, the function $h(A, \Delta)$ is continuous with respect to $A \in (1, +\infty)$ and tends to $+\infty$ as $A \to +\infty$. So, there exists a unique $\bar{A} = \bar{A}(\Delta) > 1$, for which $h(\bar{A}, \Delta) = 0$ such that $h(A, \Delta) \le 0$ whenever $A \in (1, \bar{A}]$.

From (35) we conclude that, for any $\Delta \in (0, \Delta^*)$ and any $A \in (1, \bar{A}]$,

$$A^{k\Delta}|z_k|^2 \le X + M_k. \tag{41}$$

On the basis of the semi-martingale convergence theorem (see [3]), we find that

$$\limsup_{k\to\infty} A^{k\Delta}|z_k|^2 \le \limsup_{k\to\infty}(X + M_k) < \infty \ \ a.s. \tag{42}$$

By the definition of z_k, assumption $\mathcal{A}_4$ and Lemma 1.1, for $c = \beta$, we get

$$\begin{aligned}
|z_k|^2 &\ge |q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 \\
&\quad -2\theta\Delta(q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k \\
&= |q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 - 2\theta\Delta(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k \\
&\quad -2\theta(1-\theta)\Delta(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))^T f_k \\
&\ge \frac{1}{1+\beta}|q_k|^2 - \frac{1}{\beta}|(1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) + \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 + \alpha_1\theta\Delta|q_k|^2 \\
&\quad -\alpha_2\theta\Delta|q_{k-[\delta(k\Delta)/\Delta]}|^2 - (1-\theta)^2\Delta|u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2 - \theta^2\Delta|f_k|^2.
\end{aligned}$$

Moreover, applying the conditions (4) and (5) we obtain

$$\begin{aligned}
|z_k|^2 &\ge \frac{1}{1+\beta}|q_k|^2 - 2(1-\theta)^2\beta|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 - 2\theta^2\beta|q_{k-[\delta(k\Delta)/\Delta]}|^2 + \alpha_1\theta\Delta|q_k|^2 \\
&\quad -\alpha_2\theta\Delta|q_{k-[\delta(k\Delta)/\Delta]}|^2 - (1-\theta)^2\beta^2\Delta(2|q_{k-[\delta(k\Delta)/\Delta]}|^2 + 2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2) - \theta^2 K\Delta(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2) \\
&= \left(\frac{1}{1+\beta} + \alpha_1\theta\Delta - \theta^2 K\Delta\right)|q_k|^2 + \Big(-2\theta^2\beta - \alpha_2\theta\Delta - 2(1-\theta)^2\beta^2\Delta - \theta^2 K\Delta\Big)|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad + \Big(-2(1-\theta)^2\beta - 2(1-\theta)^2\beta^2\Delta\Big)|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2.
\end{aligned} \tag{43}$$

Substituting (43) into (41), we get

$$\begin{aligned}
\Big(\frac{1}{1+\beta} + \alpha_1\theta\Delta - \theta^2 K\Delta\Big)A^{k\Delta}|q_k|^2 &\le \Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big)A^{k\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad +2(1-\theta)^2\beta\,(1+\beta\Delta)\,A^{k\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + X + M_k,
\end{aligned} \tag{44}$$

for any $A \in (1, \bar{A}]$. So, for any $\gamma \in \big(0, \log \bar{A}\big)$, there exists an integer k_1 such that for any integer $k_2 > k_1$,

$$\begin{aligned}
&\Big(\frac{1}{1+\beta} + \alpha_1\theta\Delta - \theta^2 K\Delta\Big) \sup_{k_1 \le k \le k_2} e^{\gamma k\Delta}|q_k|^2 \\
&\quad \le \Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big) \sup_{k_1 \le k \le k_2} e^{\gamma k\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\qquad +2(1-\theta)^2\beta\,(1+\beta\Delta) \sup_{k_1 \le k \le k_2} e^{\gamma k\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + \sup_{k_1 \le k \le k_2} (X + M_k) \\
&\quad \le \Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big) e^{\gamma\tau} \sup_{k_1 \le k \le k_2} e^{\gamma(k-[\delta(k\Delta)/\Delta])\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\qquad +2(1-\theta)^2\beta\,(1+\beta\Delta)\, e^{\gamma(\tau+1)} \sup_{k_1 \le k \le k_2} e^{\gamma(k-1-[\delta((k-1)\Delta)/\Delta])\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + \sup_{k_1 \le k \le k_2} (X + M_k) \\
&\quad \le \Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big) \Bigg(e^{\gamma\tau} \sup_{k_1-n_* \le k \le k_1-1} e^{\gamma k\Delta}|q_k|^2 + e^{\gamma\tau} \sup_{k_1 \le k \le k_2} e^{\gamma k\Delta}|q_k|^2\Bigg) \\
&\qquad +2(1-\theta)^2\beta\,(1+\beta\Delta) \Bigg(e^{\gamma(\tau+1)} \sup_{k_1-n_*-1 \le k \le k_1-1} e^{\gamma k\Delta}|q_k|^2 + e^{\gamma(\tau+1)} \sup_{k_1 \le k \le k_2} e^{\gamma k\Delta}|q_k|^2\Bigg) + \sup_{k_1 \le k \le k_2} (X + M_k),
\end{aligned}$$

implying that

$$\begin{aligned}
l(\Delta) \sup_{k_1 \le k \le k_2} e^{\gamma k\Delta}|q_k|^2 \le & \Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big) e^{\gamma\tau} \sup_{k_1-n_* \le k \le k_1-1} e^{\gamma k\Delta}|q_k|^2 \\
& +2(1-\theta)^2\beta\,(1+\beta\Delta)\, e^{\gamma(\tau+1)} \sup_{k_1-n_*-1 \le k \le k_1-1} e^{\gamma k\Delta}|q_k|^2 + \sup_{k_1 \le k \le k_2} (X + M_k),
\end{aligned} \tag{45}$$

where

$$l(\Delta) = \frac{1}{1+\beta} + \alpha_1\theta\Delta - \theta^2 K\Delta - \Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big) e^{\gamma\tau} - 2(1-\theta)^2\beta\,(1+\beta\Delta)\, e^{\gamma(\tau+1)}. \tag{46}$$

Note that

$$l(\Delta) > \Delta\Big[\alpha_1\theta - \theta^2 K - \Big(\alpha_2\theta + 4(1-\theta)^2\beta^2 + \theta^2 K\Big) e^{\gamma(\tau+1)}\Big] + \frac{1}{1+\beta} - 2\beta[\theta^2 + (1-\theta)^2] e^{\gamma(\tau+1)}. \tag{47}$$

For any $\theta \in (0, \frac{1}{2}]$ and any

$$\gamma \in \Big(0, -\frac{1}{\tau+1} \log\Big(2\beta(1+\beta)(\theta^2 + (1-\theta)^2\Big) \wedge \log \bar{A}\Big),$$

the condition (22), that is $\beta \in \Big(0, \frac{-1+\sqrt{3}}{2}\Big)$, implies

$$\frac{1}{1+\beta} - 2\beta[\theta^2 + (1-\theta)^2] e^{\gamma(\tau+1)} > 0.$$

Moreover, on the basis of (23), one can find

$$\gamma \in \Big(0, -\frac{1}{\tau+1} \log\Big(2\beta(1+\beta)(\theta^2 + (1-\theta)^2)\Big) \wedge \frac{1}{\tau+1} \log \frac{\alpha_1\theta - \theta^2 K}{\alpha_2\theta + 4(1-\theta)^2\beta^2 + \theta^2 K} \wedge \log \bar{A}\Big), \tag{48}$$

such that

$$\alpha_1\theta - \theta^2 K - \Big(\alpha_2\theta + 4(1-\theta)^2\beta^2 + \theta^2 K\Big) e^{\gamma(\tau+1)} > 0.$$

Consequently, if γ satisfies (48), then $l(\Delta) > 0$ for any $\Delta \in (0, \Delta^*)$. Because of that, letting $k_2 \to +\infty$, from (45) we obtain

$$\sup_{k_1 \le k < \infty} e^{\gamma k\Delta}|q_k|^2 \le \frac{1}{l(\Delta)}\Bigg[\Big(2\theta^2\beta + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\Big)e^{\gamma\tau} \sup_{k_1-n_*\le k\le k_1-1} e^{\gamma k\Delta}|q_k|^2 \\ +2(1-\theta)^2\beta\,(1+\beta\Delta)\,e^{\gamma(\tau+1)} \sup_{k_1-n_*-1\le k\le k_1-1} e^{\gamma k\Delta}|q_k|^2 + \sup_{k_1\le k<\infty}(X+M_k)\Bigg].$$

This inequality and (42) yield

$$\limsup_{k\to\infty} e^{\gamma k\Delta}|q_k|^2 < \infty,$$

whenever γ satisfies (48) and $\Delta \in (0, \Delta^*)$. Thus, (43) gives

$$\limsup_{k\to\infty} e^{\gamma k\Delta}|q_k|^2 \le \frac{1}{l(\Delta)} \limsup_{k\to\infty}(X + M_k) < \infty.$$

Consequently, we have

$$\limsup_{k\to\infty} \frac{\log(e^{\gamma k\Delta}|q_k|^2)}{k\Delta} = 0.$$

This yields

$$\limsup_{k\to\infty} \frac{\log|q_k|}{k\Delta} \le -\frac{\gamma}{2}$$

for any $\Delta \in (0, \Delta^*)$ and any γ satisfying (48). This completes the proof. □

3. The Euler-Maruyama Case

For the reasons mentioned in the introduction, in order to complete the paper we will consider the case when $\theta = 0$, that is, the Euler-Maruyama case. Then, (15) becomes

$$q_{k+1} = q_k + u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) + f(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta + g(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta w_k,\ \ k \in \{0, 1, 2, ...\}, \quad (49)$$

such that (13), (14) and (49) determine the discrete Euler-Maruyama solution. In this case, we have that

$$z_k = q_k - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}).$$

In the sequel, we will establish the stability result for the solution of Eq. (49), analogous to the one from Theorem 2.4. So, we will give only a sketch of proof, by stressing those parts which are different to corresponding parts of the proof of Theorem 2.4.

Theorem 3.1. *Assume that the hypotheses $\mathcal{A}_1$–$\mathcal{A}_4$ hold. Additionally, let*

$$\beta \in \left(0, \frac{-1+\sqrt{5}}{2}\right), \quad (50)$$

$$\alpha_1 > K + (K + \alpha_2 + 4\beta^2)([(1-\eta)^{-1}] + 1). \quad (51)$$

Then, there exists a $\tilde{\Delta}^* \in (0, 1)$, such that the Euler-Maruyama approximate solution defined by (13), (14) and (49) is almost surely asymptotically exponentially stable, whenever $\Delta \in (0, \tilde{\Delta}^*)$.

Proof. On the basis of (49) we have

$$\begin{aligned}|z_{k+1}|^2 &= |z_k|^2 + [2(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k + |g_k|^2 + |f_k|^2\Delta]\Delta \\ &\quad + 2(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))^T f_k\Delta + m_k,\end{aligned} \tag{52}$$

where

$$m_k = |g_k\Delta w_k|^2 - |g_k|^2\Delta + 2(z_k + f_k\Delta)^T g_k\Delta w_k.$$

Applying the arguments which are used for obtaining the estimate (27), we get

$$|z_{k+1}|^2 \le |z_k|^2 + (\alpha_2 + K\Delta + 2\beta^2 + K)|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta + 2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + (-\alpha_1 + K\Delta + K)|q_k|^2\Delta + m_k. \tag{53}$$

Then, for an arbitrary constant $A > 1$, we find that

$$\begin{aligned}A^{(k+1)\Delta}|z_{k+1}|^2 - A^{k\Delta}|z_k|^2 &\le A^{(k+1)\Delta}|z_k|^2(1 - A^{-\Delta}) + [-\alpha_1 + K\Delta + K]\,\Delta A^{(k+1)\Delta}|q_k|^2 \\ &\quad + \left[\alpha_2 + K\Delta + 2\beta^2 + K\right]\Delta A^{(k+1)\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\ &\quad + 2\beta^2\Delta A^{(k+1)\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + A^{(k+1)\Delta}m_k.\end{aligned} \tag{54}$$

If we recall that $R_1(\Delta) = 1 - A^{-\Delta}$ and denote

$$\tilde{R}_2(\Delta) = -\alpha_1 + K\Delta + K, \quad \tilde{R}_3(\Delta) = \alpha_2 + K\Delta + 2\beta^2 + K,$$

then, from (54) follows that

$$\begin{aligned}A^{k\Delta}|z_k|^2 &\le |z_0|^2 + R_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|z_i|^2 + \tilde{R}_2(\Delta)\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 \\ &\quad + \tilde{R}_3(\Delta)\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 + 2\beta^2\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2 + M_k,\end{aligned} \tag{55}$$

where

$$M_k = \sum_{i=0}^{k-1} A^{(i+1)\Delta}m_i$$

is a local martingale with $M_0 = 0$.

By the definition of z_k and (11), we get

$$|z_k|^2 = |q_k - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2 \le (1+\beta)|q_k|^2 + \beta(1+\beta)|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2. \tag{56}$$

Then, substituting (56) into (55) we obtain that

$$\begin{aligned}A^{k\Delta}|z_k|^2 &\le |z_0|^2 + \tilde{K}_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + \tilde{K}_2(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 \\ &\quad + \tilde{K}_3(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2 + M_k,\end{aligned} \tag{57}$$

where

$$\tilde{K}_1(\Delta) = R_1(\Delta)(1+\beta) + \tilde{R}_2(\Delta)\Delta, \quad \tilde{K}_2(\Delta) = \tilde{R}_3(\Delta)\Delta, \quad \tilde{K}_3(\Delta) = R_1(\Delta)\beta(1+\beta) + 2\beta^2\Delta.$$

Taking into account (32) and the fact that $\tilde{K}_3(\Delta) > 0$, the estimate (57) becomes

$$\begin{aligned}A^{k\Delta}|z_k|^2 &\le |z_0|^2 + \tilde{K}_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + (\tilde{K}_2(\Delta) + \tilde{K}_3(\Delta)A^{\Delta})\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 \\ &\quad + \tilde{K}_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 + M_k.\end{aligned} \tag{58}$$

By applying Lemma 1.2 to the second sum on the right-hand side of (58) and bearing in the mind that $n_*\Delta = \tau$, the expression (58) can be estimated as

$$A^{k\Delta}|z_k|^2 \le \tilde{X} + \tilde{h}(A,\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + M_k. \tag{59}$$

In the previous expression $\tilde{X}$ has the form (36), where $K_2(\Delta)$ and $K_3(\Delta)$ are substituted by $\tilde{K}_2(\Delta)$ and $\tilde{K}_3(\Delta)$, respectively, while

$$\begin{aligned}\tilde{h}(A,\Delta) &= \tilde{K}_1(\Delta) + (\tilde{K}_2(\Delta) + \tilde{K}_3(\Delta)A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau}\\ &= (1-A^{-\Delta})(1+\beta) + (-\alpha_1 + K\Delta + K)\Delta\\ &\quad + \Big[(\alpha_2 + K\Delta + 2\beta^2 + K)\Delta + (A^{\Delta}-1)\beta(1+\beta) + 2\beta^2\Delta A^{\Delta}\Big]([(1-\eta)^{-1}]+1)A^{\tau}.\end{aligned} \tag{60}$$

Observe that $\tilde{h}$ increases with respect to A, that is

$$\begin{aligned}\frac{d}{dA}\tilde{h}(A,\Delta) &= \Delta A^{-\Delta-1}(1+\beta) + \Big[(\alpha_2 + K\Delta + 2\beta^2 + K)\Delta + (A^{\Delta}-1)\beta(1+\beta) + 2\beta^2\Delta A^{\Delta}\Big]([(1-\eta)^{-1}]+1)\tau A^{\tau-1}\\ &\quad + ([(1-\eta)^{-1}]+1)A^{\tau}\Delta A^{\Delta-1}\beta\Big((1+\beta)+2\beta\Delta\Big) > 0.\end{aligned} \tag{61}$$

On the basis of the assumption (51), we have

$$\tilde{h}(1,\Delta) = (-\alpha_1 + K\Delta + K)\Delta + \Big(K\Delta + K + \alpha_2 + 4\beta^2\Big)\Delta([(1-\eta)^{-1}]+1) < 0$$

for any $\Delta \in (0,\tilde{\Delta}^*)$, where $\tilde{\Delta}^* = \tilde{\Delta}_1 \wedge 1$, while

$$\tilde{\Delta}_1 = \frac{\alpha_1 - K - (K + \alpha_2 + 4\beta^2)([(1-\eta)^{-1}]+1)}{K([(1-\eta)^{-1}]+2)}.$$

So, for any $\Delta \in (0,\tilde{\Delta}^*)$, there exists a unique $\tilde{A} = \tilde{A}(\Delta) > 1$, for which $\tilde{h}(\tilde{A},\Delta) = 0$, such that $\tilde{h}(A,\Delta) \le 0$ whenever $A \in (1,\tilde{A}]$.

From (59), we conclude that, for any $\Delta \in (0,\tilde{\Delta}^*)$ and any $A \in (1,\tilde{A}]$,

$$A^{k\Delta}|z_k|^2 \le \tilde{X} + M_k, \tag{62}$$

and then, the semi-martingale convergence theorem yields

$$\limsup_{k\to\infty} A^{k\Delta}|z_k|^2 \le \limsup_{k\to\infty}(\tilde{X} + M_k) < \infty \ \ a.s.$$

By the definition of z_k, Lemma 1.1, for $c = \beta$, as well as the condition (5), we get

$$|z_k|^2 \ge \frac{1}{1+\beta}|q_k|^2 - \frac{1}{\beta}|u(q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2 \ge \frac{1}{1+\beta}|q_k|^2 - \beta|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2. \tag{63}$$

Substituting (63) into (62), we get

$$\frac{1}{1+\beta}A^{k\Delta}|q_k|^2 \le \beta A^{k\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + \tilde{X} + M_k. \tag{64}$$

So, for any $\tilde{\gamma} \in \Big(0,\log\tilde{A}\Big)$, there exists an integer k_1 such that for any integer $k_2 > k_1$,

$$\begin{aligned}\sup_{k_1\le k\le k_2} e^{\tilde{\gamma}k\Delta}|q_k|^2 &\le \beta\,(1+\beta)\sup_{k_1\le k\le k_2} e^{\tilde{\gamma}k\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + (1+\beta)\sup_{k_1\le k\le k_2}(\tilde{X}+M_k)\\ &\le \beta(1+\beta)e^{\tilde{\gamma}(\tau+1)}\sup_{k_1\le k\le k_2} e^{\tilde{\gamma}(k-1-[\delta((k-1)\Delta)/\Delta])\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + (1+\beta)\sup_{k_1\le k\le k_2}(\tilde{X}+M_k)\\ &\le \beta(1+\beta)\left(e^{\tilde{\gamma}(\tau+1)}\sup_{k_1-n_*-1\le k\le k_1-1} e^{\tilde{\gamma}k\Delta}|q_k|^2 + e^{\tilde{\gamma}(\tau+1)}\sup_{k_1\le k\le k_2} e^{\tilde{\gamma}k\Delta}|q_k|^2\right) + (1+\beta)\sup_{k_1\le k\le k_2}(\tilde{X}+M_k).\end{aligned}$$

On the basis of the condition (50), that is $\beta \in \Big(0, \frac{-1+\sqrt{5}}{2}\Big)$, for any

$$\tilde{\gamma} \in \Big(0, -\frac{1}{\tau+1}\log(\beta(1+\beta)) \wedge \log \tilde{A}\Big),$$

we have that

$$\sup_{k_1 \le k \le k_2} e^{\tilde{\gamma}k\Delta}|q_k|^2 \le \Big(1-\beta(1+\beta)e^{\tilde{\gamma}(\tau+1)}\Big)^{-1}\Bigg[\beta(1+\beta)\Bigg(e^{\tilde{\gamma}(\tau+1)}\sup_{k_1-n_*-1\le k\le k_1-1} e^{\tilde{\gamma}k\Delta}|q_k|^2\Bigg)+(1+\beta)\sup_{k_1\le k\le k_2}(\tilde{X}+M_k)\Bigg]. \tag{65}$$

Letting $k_2 \to +\infty$ in (65), we obtain

$$\sup_{k_1 \le k < \infty} e^{\tilde{\gamma}k\Delta}|q_k|^2 \le \Big(1-\beta(1+\beta)e^{\tilde{\gamma}(\tau+1)}\Big)^{-1}\Bigg[\beta(1+\beta)\Bigg(e^{\tilde{\gamma}(\tau+1)}\sup_{k_1-n_*-1\le k\le k_1-1} e^{\tilde{\gamma}k\Delta}|q_k|^2\Bigg)+(1+\beta)\sup_{k_1\le k<\infty}(\tilde{X}+M_k)\Bigg],$$

which yields

$$\limsup_{k\to\infty} e^{\tilde{\gamma}k\Delta}|q_k|^2 < \infty.$$

Following the procedure from the proof of Theorem 2.4, we obtain

$$\limsup_{k\to\infty}\frac{\log|q_k|}{k\Delta} \le -\frac{\tilde{\gamma}}{2}$$

for any $\Delta \in (0,\Delta^*)$ and any $\tilde{\gamma} \in \Big(0, -\frac{1}{\tau+1}\log(\beta(1+\beta)) \wedge \log\tilde{A}\Big)$. □

4. Numerical Simulations

In order to illustrate the previous theoretical result, when $\theta \in (0, \frac{1}{2}]$, we present an example.

Example 4.1. *Consider the following scalar neutral stochastic differential equation with time-dependent delay*

$$\begin{aligned} &d\Big[x(t)-\frac{1}{50}x(t-\delta(t))\Big] \\ &\quad= \Big(-\frac{1}{20}x(t)-\frac{1}{40}\sin x(t-\delta(t))\Big)dt+\frac{1}{10\sqrt{10}}\frac{x(t-\delta(t))}{1+x^4(t-\delta(t))}\cos x(t)dw(t), \quad t\in[0,50], \end{aligned} \tag{66}$$

with the initial condition $\varphi(t) = -1$, $t \in [-\tau, 0]$, where $\tau = 0.5$ and $\varphi \in C^b_{\mathcal{F}_0}([-\tau,0];R)$. Obviously, the drift coefficient $f(x,y) = -\frac{1}{20}x - \frac{1}{40}\sin y$ satisfies the linear growth condition $\mathcal{A}_1$ for $K = \frac{2}{20^2}$, while the function $u(x) = \frac{1}{50}x$, $x \in R$ satisfies the assumption $\mathcal{A}_2$ for $\beta = \frac{1}{50}$. Assume that the delay function is of the form $\delta(t) = \frac{1}{4} - \frac{1}{4}\sin t$, $t \in [0,50]$. Then,

$$\begin{aligned} &|\delta'(t)| = \Big|-\frac{1}{4}\cos t\Big| \le \frac{1}{4} = \eta, \\ &|\delta(t)-\delta(s)| \le \frac{1}{4}|t-s|, \quad t,s\in[0,50] \end{aligned}$$

and we find that $\mathcal{A}_3$ hold with $\eta = \frac{1}{4}$. In order to verify $\mathcal{A}_4$, note that

$$\begin{aligned} 2(x-u(y))f(x,y)+|g(x,y)|^2 &= -\frac{2}{20}x^2-\frac{2}{40}x\sin y+\frac{2}{20\cdot 50}xy+\frac{2}{40\cdot 50}y\sin y+\frac{1}{1\,000}\frac{y^2}{(1+y^4)^2}\cos^2 x \\ &\le -\frac{2}{20}x^2+\frac{1}{40}(x^2+y^2)+\frac{1}{1\,000}(x^2+y^2)+\frac{1}{1\,000}y^2+\frac{1}{1\,000}y^2 \\ &\le -\frac{37}{500}x^2+\frac{14}{500}y^2. \end{aligned}$$

So, $\alpha_1 = \frac{37}{500}$ *and* $\alpha_2 = \frac{14}{500}$. *Moreover, we have that*

$$\frac{\alpha_2}{1-\eta} = \frac{14}{375} < \frac{37}{500} = \alpha_1.$$

Thus we conclude that $\mathcal{A}_4$ *holds, as well as (8), that is,* $f(0,0) = g(0,0) = u(0) = 0$.

Observing that, for any $x_1, x_2, y \in R^d$,

$$\langle x_1 - x_2, f(x_1, y) - f(x_2, y)\rangle = -\frac{1}{20}\langle x_1 - x_2, x_1 - x_2\rangle = -\frac{1}{20}|x_1 - x_2|^2,$$
$$\langle x_1 - x_2, f(y, x_1) - f(y, x_2)\rangle = -\frac{1}{40}\langle x_1 - x_2, \sin x_1 - \sin x_2\rangle \leq \frac{1}{40}|x_1 - x_2|^2,$$

we conclude that C_1 *holds for any positive* μ_1 *and* $\mu_2 = \frac{1}{40}$. *So, we will choose* $\mu_1 = \mu_2 = \frac{1}{40}$, *such that* $\theta((\mu_1 + \mu_2)\Delta + \beta) < 1$ *for any* $\Delta \in (0, 1)$ *and any* $\theta \in (0, \frac{1}{2}]$. *Thus, Lemma 2.2 guarantees that the corresponding* θ*-Euler-Maruyama approximate equations have unique solutions. Bearing in mind (15), for* $\theta = \frac{1}{4}$ *and any* $\Delta \in (0, 1)$, *we have that* $q_k = \varphi(k\Delta)$, $k = -n_*, -n_* + 1, ..., 0$ *and for* $k = 0, 1, 2, ...$,

$$\begin{aligned} q_{k+1} = q_k &+ \frac{1}{200}q_{k+1-[\delta((k+1)\Delta)/\Delta]} + \frac{1}{100}q_{k-[\delta(k\Delta)/\Delta]} - \frac{3}{200}q_{k-1-[\delta((k-1)\Delta)/\Delta]} - \frac{1}{80}q_{k+1}\Delta - \frac{3}{80}q_k\Delta \\ &- \frac{1}{160}\sin q_{k+1-[\delta((k+1)\Delta)/\Delta]}\Delta - \frac{3}{160}\sin q_{k-[\delta(k\Delta)/\Delta]}\Delta + \frac{1}{10\sqrt{10}}\frac{q_{k-[\delta(k\Delta)/\Delta]}}{1 + q^4_{k-[\delta(k\Delta)/\Delta]}}\cos q_k\Delta w_k. \end{aligned} \tag{67}$$

In Figure 1, we plotted several trajectories of the θ*-Euler-Maruyama solution (67).*

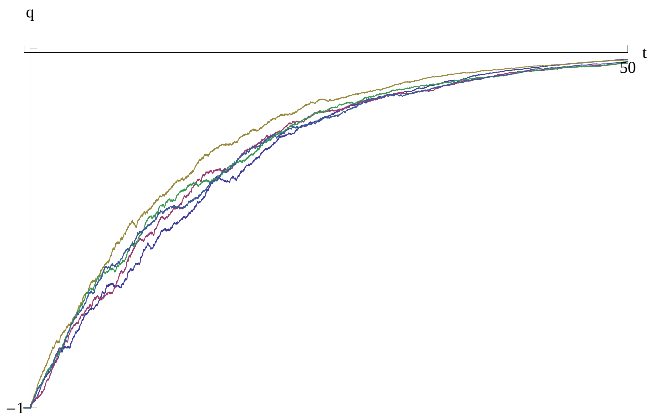

Figure 1: Trajectories of the θ-Euler-Maruyama solution with $\Delta = 0.01$

Moreover, the assumption (22) holds since $\frac{1}{50} = \beta \in \left(0, \frac{-1+\sqrt{3}}{2}\right)$. *Also,*

$$\frac{37}{500} = \alpha_1 > \max\left\{K + \left(K + \alpha_2 + 4\beta^2(1-\theta)^2\right)([(1-\eta)^{-1}] + 1), 2\theta K + \alpha_2 + 4\beta^2\frac{(1-\theta)^2}{\theta}\right\} = 0.0728,$$

that is, the condition (23) is fulfilled.

In order to find the scope of Δ *for which the* θ*-Euler-Maruyama solution (67) is a.s. exponentially stable, we need to calculate* Δ_1, *defined in (40). Since* $\Delta_1 = 0.16$, *in the sequel we will deal with* $\Delta \in (0, 0.16)$. *So, we may choose* $\Delta = 0.01$. *Then, we need to calculate* $\bar{A} = \bar{A}(0.01)$, *which is the unique solution of the equation* $h(A, 0.01) = 0$, *where* $h(A, \Delta)$ *is given in (37). By direct computation we obtain that* $\bar{A} = 1.0009$. *Bearing in mind (48), on the basis of Theorem 2.4, we conclude that for any*

$$\gamma \in \left(0, 1.0623 \wedge 0.2302 \wedge 0.0009\right),$$

that is, for $\gamma \in (0, 0.0009)$, *the* θ*-Euler-Maruyama solution satisfies*

$$\limsup_{k\to\infty} \frac{\log |q_k|}{k\Delta} \leq -\frac{\gamma}{2} \ a.s.$$

In order to illustrate that the previous inequality holds, we simulated the trajectories of the ratio on the left-hand side of the inequality, which correspond to the trajectories plotted in Figure 1. We plotted those trajectories against the line $z = -0.00045$, *that is the lower bound of the expression on the right-hand side of the inequality. The result of this simulation is presented in Figure 2.*

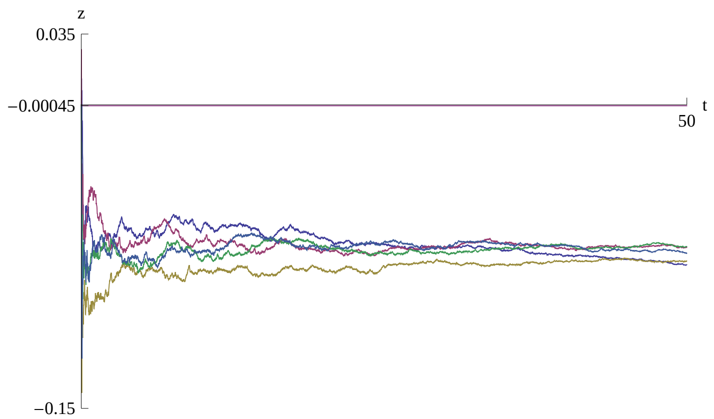

Figure 2: Trajectories of the ratio $\frac{\log |q_k|}{k\Delta}$, $k = 1, ..., 5\,000$, against the line $z = -0.00045$, with $\Delta = 0.01$

References

[1] M. Jovanović, S. Janković, Neutral stochastic functional differential equations with additive pertubations, Appl. Math. Comput. 213 (2009) 370–379.

[2] G. Lan, C. Yuan, Exponential stability of the exact solutions and θ–EM approximations to neutral SDDEs with Markov switching, J. Comput. Appl. Math. 285 (2015) 230–242.

[3] R.Sh. Liptser, A.N. Shiryaev, Theory of Martingales, Kluwer Academic Publishers, 1989.

[4] Q. Luo, X. Mao, Y. Shen, New criteria on exponential stability of neutral stochastic differential delay equations, Systems and Control Letters 55 (2006) 826–834.

[5] X. Mao, Stochastic Differential Equations and Applications, Horvood, Chichester, 1997.

[6] X. Mao, M.J. Rassias, Khasminskii-type theorems for stochastic differential delay equations, Stochastic analysis and applications 23 (2005) 1045–1069.

[7] X. Mao, Asymptotic properties of neutral stochastic differential delay equations, Stochastics and Stochastics Reports 68 (2000) 273–295.

[8] X. Mao, Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions, Appl. Math. Comput. 217 (2011) 5512–5524.

[9] X. Mao, Stability of stochastic differential equations with Markovian switching, Stochastic Processes and Applications 79 (1999) 45–67.

[10] X. Mao, L. Szpruch, Strong convergence and stability of implicit numerical methods for stochastics differential equations with non-globally Lipschitz continuous coefficients, J. Comput. Appl. Math. 238 (2013) 14–28.

[11] J. Milnor, Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem, Amer. Math. Monthly 85 (1978) 521–524.

[12] M. Milošević, Highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama method, Math. Comput. Model. 54 (2011) 2235–2251.

[13] M. Milošević, Almost sure exponential stability of solutions to highly nonlinear neutral stochastics differential equations with time-dependent delay and Euler-Maruyama approximation, Math. Comput. Model. 57 (2013) 887–899.

[14] M. Milošević, Implicit numerical methods for highly nonlinear neutral stochastics differential equations with time-dependent delay, Appl. Math. and Comput. 244 (2014) 741–760.

[15] C. A. Rogers, A less strange version of Milnor's proof of Brouwer's fixed-point theorem, Amer. Math. Monthly 87 (1980) 525–527.

[16] F. Wu, X. Mao, L. Szpruch, Almost sure exonential stability of numerical solutions for stochastic delay differential equations, Numeriche Mathematik 115 (2010) 681–697.
[17] M. Xue, S. Zhou, S. Hu, Stability of nonlinear neutral stochastic functional differential equations, J. Appl. Math. (2010)
[18] Z. Yu, Almost surely asqmptotic stability of exact and numerical solutions for neutral stochastic pantograph equations, Abstract and Applied Analysis, 2011, doi:10.1155/2011/14079.
[19] X. Zong, F. Wu, C. Huang, Theta schemes for SDDEs with non-globally Lipschitz continuous coefficients, J. Comput. Appl. Math. 278 (2015) 258-277.
[20] X. Zong, F. Wu, C. Huang, Exponential mean square stability of the theta approximations for neutral stochastic differential delay equations, J. Comput. Appl. Math. 286 (2015) 172-185.