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Polynomial Inequalities in Regions with Corners in the Weighted Lebesgue Spaces

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Abstract. In this work, we investigate the order of the growth of the modulus of orthogonal polynomials over a contour and also arbitrary algebraic polynomials in regions with corners in a weighted Lebesgue space, where the singularities of contour and the weight functions satisfy some condition.

1. Introduction

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $L \subset \mathbb{C}$ be a closed rectifiable Jordan curve, $G := intL$, with $0 \in G$, $\Omega := extL$. Let $h(z)$ be a nonnegative, summable on a L and nonzero except possibly on a set of measure zero function. The systems of polynomials $\{K_n(z)\}$, $K_n(z) = a_n z^n + ...$, $\deg K_n = n$, $n = 0, 1, 2, ...$, satisfying the condition

$$\int_L h(z) K_n(z) \overline{K_m(z)} \, |dz| = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}$$

are called orthonormal polynomials for the pair (L, h). These polynomials are determined uniquely if the coefficient $a_n > 0$.

These polynomials were first studied by G. Szegö [36], [37]. V.I. Smirnov [32], P.P. Korovkin [19], Ya. L. Geronimus [16] investigated these polynomials under various conditions on the weight function $h(z)$ and contour L. In [34], P.K. Suetin investigated many properties of the polynomials $\{K_n(z)\}$ for smooth contour and weight function $h(z)$ that is zero or infinite at finite number points on the contour L. A.L. Kuz'mina [20] and G. Fauth [14] have considered some properties of the polynomials $\{K_n(z)\}$ for piecewise analytic contour L with finite number of corners. In [35], P. . Suetin obtained several estimates for the rate of growth of the polynomials $\{K_n(z)\}$ on the contour L, depending of the singularity of the weight function $h(z)$ on L and of the contour L.

Let L be a rectifiable Jordan curve with the natural parametrization $z = z(s), 0 \leq s \leq l := mesL$. We say that $L \in C(1, \alpha)$, $0 < \alpha < 1$, if $z(s)$ is continuously differentiable and $z'(s) \in Lip\alpha$. Let L belong to $C(1, \alpha)$ everywhere except for a single point $z_1 \in L$, i.e., the derivative $z'(s)$ satisfies the Lipschitz condition on the

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$[0,l]$ and $z(0)=z(l)=z_1$, but $z'(0)\neq z'(l)$. Assume that L has a corner at z_1 with exterior angle $\nu\pi,\ 0<\nu\leq 2$, and denote the set of such curves by $C(1,\alpha,\nu)$.

Denoted by $w=\Phi(z)$, the univalent conformal mapping of Ω onto $\Delta:=\{w:|w|>1\}$ with normalization $\Phi(\infty)=\infty,\ \lim_{z\to\infty}\frac{\Phi(z)}{z}>0$ and $\Psi:=\Phi^{-1}$. For $t\geq 1$, we set

$$L_t:=\{z:\ |\Phi(z)|=t\},\ L_1\equiv L,\ G_t:=intL_t,\ \Omega_t:=extL_t.$$

Let $\left\{z_j\right\}_{j=1}^{m}$ be the fixed system of distinct points on curve L. For some fixed R_0, $1<R_0<\infty$, and $z\in\overline{G}_{R_0}\backslash G$, consider the generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z):=h_0(z)\prod_{j=1}^{m}\left|z-z_j\right|^{\gamma_j}, \tag{1}$$

where $\gamma_j>-1$, for all $j=1,2,...,m$, and h_0 is uniformly separated from zero in L, i.e. there exists a constant $c_0(L)>0$ such that for all $z\in G_{R_0}$

$$h_0(z)\geq c_0(L)>0.$$

P.K. Suetin [35] investigated this problem for $K_n(z)$ with the weight function $h(z)$ defined as in (1) and for the curve $L\in C(1,\alpha,\nu)$. He showed that the condition of "pay off" singularity curve and weight function at the points z_1 can be given as following:

$$(1+\gamma_1)\,\nu_1=1, \tag{2}$$

and, under this conditions, for $K_n(z)$ provided the following estimation:

$$|K_n(z)|\leq c(L)\sqrt{n+1},\ z\in L, \tag{3}$$

where $c(L)>0$ is a constant independent on n.

In this work we study the estimations of the (3)-type for more general contours of the complex plane and we obtain the analog of the equality (2) corresponding to the general case. In parallel, we also study the growth of arbitrary algebraic polynomials with respect to their seminorm in the weighed Lebesgue space, under the (2)-type conditions.

2. Definitions and Main Results

Throughout this paper, $c,c_0,c_1,c_2,...$ are positive and $\varepsilon_0,\varepsilon_1,\varepsilon_2,...$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n\in\mathbb{N}_0:=\{1,2,...\}\cup\{0\}$.

Without loss of generality, the number R_0 in the definition of the weight functions, we can take $R_0=2$. Otherwise the natural number n can be chosen $n\geq\left[\frac{\varepsilon_0}{R_0-1}\right]$, where ε_0, $0<\varepsilon_0<1$, some fixed small constant.

Let $0<p\leq\infty$. For a rectifiable Jordan curve L, we denote

$$\|P_n\|_{\mathcal{L}_p}\ :=\ \|P_n\|_{\mathcal{L}_p(h,L)}:=\left(\int_L h(z)\,|P_n(z)|^p\,|dz|\right)^{1/p},\ 0<p<\infty,$$

$$\|P_n\|_{\mathcal{L}_\infty}\ :=\ \|P_n\|_{\mathcal{L}_\infty(1,L)}:=\max_{z\in L}|P_n(z)|,\ p=\infty.$$

Clearly, $\|\cdot\|_{\mathcal{L}_p}$ is a quasinorm (i.e. a norm for $1\leq p\leq\infty$ and a $p-$norm for $0<p<1$).

For any $k\geq 0$ and $m>k$, notation $i=\overline{k,m}$ means that $i=k,k+1,...,m$.

Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w : |w| < 1\}$ onto the $G = intL$ normalized by $\psi(0) = 0$, $\psi'(0) > 0$. By [28, pp.286-294], we say a bounded Jordan region G is called κ *-quasidisk*, $0 \le \kappa < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, the homeomorphism of the plane $\overline{\mathbb{C}}$ on plane $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a κ *-quasicircle*. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is κ*-quasidisk* (κ *-quasicircle)* for some $0 \le \kappa < 1$.

We denoted the class of κ *-quasicircle* by $Q(\kappa)$, $0 \le \kappa < 1$, and denote by $L \in Q$, if $L \in Q(\kappa)$, for some $0 \le \kappa < 1$. It is well-known that the quasicircle may not even be locally rectifiable in [21, p.104].

Definition 2.1. We say that $L \in \widetilde{Q}(\kappa)$, $0 \le \kappa < 1$, if $L \in Q(\kappa)$ and L is rectifiable. Analogously, $L \in \widetilde{Q}$, if $L \in \widetilde{Q}(\kappa)$, for some $0 \le \kappa < 1$.

In [8], the authors obtained the following result for $L \in \widetilde{Q}(\kappa)$, $0 \le \kappa < 1$:

Theorem A. *Let $p > 0$. Suppose that $L \in \widetilde{Q}(\kappa)$, for some $0 \le \kappa < 1$ and $h(z)$ defined in (1) for $\gamma_j = 0$, for all $j = \overline{1, m}$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}_0$, there exists $c_1 = c_1(L, p) > 0$ such that:*

$$\|P_n\|_{\mathcal{L}_\infty} \le c_1 (n+1)^{\frac{1+\kappa}{p}} \|P_n\|_{\mathcal{L}_p(h_0, L)}. \tag{4}$$

Corollary A.

$$\|K_n\|_{\mathcal{L}_\infty} \le c_1 (n+1)^{\frac{1+\kappa}{2}}. \tag{5}$$

Thus, Theorem A provides an opportunity to observe the growth of $|P_n(z)|$ on the curve L. Note that, Theorem A for $L := \{z : |z| = 1\}$ (i.e. $\kappa = 0$) provided in [18]. The other classical results are similar to (4) we can find in [38]. The evaluations of (4)-type for $0 < p < \infty$, $h(z) \equiv 1$ (or $h(z) \ne 1$) was also investigated in [33], [23], [24], [26, pp.122-133], [30], [1]-[8] and others (see also the references cited therein), for different Jordan curves having special properties. In [11, Theorem 6] obtained identical inequalities for more general curves and for another weighed function. There are more references regarding the inequality of (4)-type, we can find in Milovanović et al. [25, Sect.5.3].

From the conditions of the theorem, we see that, it holds for $k-$quasidisks with $0 \le k < 1$. But calculating the coefficient of quasiconformality κ for some curves is not an easy task. Therefore, we define a more general class of curves with another characteristic. One of them is the following:

Definition 2.2. We say that $L \in Q_\alpha$, $0 < \alpha \le 1$, if $L \in Q$ and $\Phi \in Lip\alpha$, $z \in \overline{\Omega}$.

We note that the class Q_α is sufficiently wide. A detailed account on it and the related topics are contained in [29], [22], [40] and the references cited therein. We consider only some cases:

Remark 2.3. a) If $L = \partial G$ is a Dini-smooth curve [29, p.48], then $L \in Q_1$.

b)If $L = \partial G$ is a piecewise Dini-smooth curve and largest exterior angle at L has opening $\alpha\pi$, $0 < \alpha \le 1$, [29, p.52], then $L \in Q_\alpha$.

c)If $L = \partial G$ is a smooth curve having continuous tangent line, then $L \in Q_\alpha$ for all $0 < \alpha < 1$.

d) If L is quasismooth (in the sense of Lavrentiev), that is, for every pair z_1, $z_2 \in L$, if $s(z_1, z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on L, there exists a constant $c > 1$ such that $s(z_1, z_2) \le c\,|z_1 - z_2|$, then $\Phi \in Lip\ \alpha$ for $\alpha = \frac{1}{2}(1 - \frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$ [40].

e) If L is "c-quasiconformal" (see, for example, [22]), then $\Phi \in Lip\ \alpha$ for $\alpha = \frac{\pi}{2(\pi - \arcsin\frac{1}{c})}$. Also, if L is an asymptotic conformal curve, then $\Phi \in Lip\ \alpha$ for all $0 < \alpha < 1$ [22].

Definition 2.4. It is said that $L \in \widetilde{Q}_\alpha$, $0 < \alpha \le 1$, if $L \in Q_\alpha$ and L is rectifiable.

In this case, we have the following:

Theorem B. ([27]) *Let $p > 0$. Suppose that $L \in \widetilde{Q}_\alpha$, for some $0 < \alpha \leq 1$ and $h(z)$ defined as in (1) with $\gamma_j = 0$, for all $j = \overline{1,m}$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}_0$, there exists $c_2 = c_2(L,p) > 0$ such that*

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_2 \|P_n\|_{\mathcal{L}_p(h_0,L)} \begin{cases} (n+1)^{\frac{1}{\alpha p}}, & \frac{1}{2} \leq \alpha \leq 1, \\ (n+1)^{\frac{\delta}{p}}, & 0 < \alpha < \frac{1}{2}, \end{cases} \tag{6}$$

where $\delta = \delta(L)$, $\delta \in [1,2]$, is a certain number.

Corollary B.

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_2 \begin{cases} (n+1)^{\frac{1}{2\alpha}}, & \frac{1}{2} \leq \alpha \leq 1, \\ (n+1)^{\frac{\delta}{2}}, & 0 < \alpha < \frac{1}{2}, \end{cases} \tag{7}$$

Therefore, according to 2.3, we can calculate α in the right parts of estimations (6) and (7) for each case, respectively.

Now, let's introduce "special" singular points on the curve L.

Definition 2.5. We say that $L \in \widetilde{Q}[\nu]$, $0 < \nu < 2$, if

a) $L \in \widetilde{Q}$,

b) For $\forall z \in L$, there exists a $r := r(L,z) > 0$ and $\nu := \nu(L,z)$, $0 < \nu < 2$, such that for some $0 \leq \theta_0 < 2$ a closed maximal circular sector $S(z;r,\nu) := \left\{\zeta : \zeta = z + re^{i\theta\pi},\ \theta_0 < \theta < \theta_0 + \nu\right\}$ of radius r and opening $\nu\pi$ lies in $\overline{G} = \overline{intL}$ with vertex at z.

It is well known that each quasicircle satisfies the condition b). Nevertheless, this condition imposed on L gives a new geometric characterization of the curve. For example, if the contour L^* defined by

$$L^* := [0,i] \cup \left\{z : z = e^{i\theta\pi},\ \frac{1}{2} < \theta < 2\right\} \cup [1,0],$$

then the coefficient of quasiconformality k of the L^* does not obtain so easily, whereas $L^* \in Q\left[\frac{3}{2}\right]$.

Definition 2.6. We say that $L \in \widetilde{Q}_\alpha[\nu_1,...,\nu_m]$, $0 < \nu_1,...,\nu_m < 2$, $0 < \alpha \leq 1$, if there exists a system of points $\{\zeta_i\} \in L$, $i = \overline{1,m}$, such that $L \in \widetilde{Q}[\nu_i]$ for any points $\zeta_i \in L$, $i = \overline{1,m}$, and $\Phi \in Lip\alpha$, $0 < \alpha \leq 1$, $z \in \overline{\Omega} \setminus \{\zeta_i\}$.

It is clear from Definition 2.5 (2.6), that each contour $L \in \widetilde{Q}_\alpha[\nu_1,...,\nu_m]$, $0 < \nu_1,...,\nu_m < 2$, $0 < \alpha \leq 1$, $i = \overline{1,m}$, may have "singularity" at the points $\{\zeta_i\}_{i=1}^m \in L$. If a contour L does not have such "singularity", i.e. if $\nu_i = 1$, $i = \overline{1,m}$, then it is written as $L \in \widetilde{Q}_\alpha$, $0 < \alpha \leq 1$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^m \in L$ are defined in (1) and $\{\zeta_i\}_{i=1}^m \in L$ are defined in Definitions 2.4 coincides. Without the loss of generality, we also will assume that the points $\{z_i\}_{i=1}^m$ are ordered in the positive direction on the curve L.

We state our new results. Our first results is related to the general case: assume that the curve L have "singularity" on the boundary points $\{z_i\}_{i=1}^m$, i.e., $\nu_i \neq 1$, for all $i = \overline{1,m}$, and the weight function h have "singularity" at the same points, i.e., $\gamma_i \neq 0$ for some $i = \overline{1,m}$. In this case, we have the following:

Theorem 2.7. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1,...,\nu_m]$, for some $0 < \nu_1,...,\nu_m < 1$, $\frac{1}{2-\nu_i} \leq \alpha \leq 1$; $h(z)$ defined as in (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}_0$, there exists $c_3 = c_3(L,p,\gamma_i,\alpha) > 0$ such that*

$$|P_n(z_i)| \leq c_3 (n+1)^{\frac{\gamma_i+1}{p}(2-\nu_i)} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{8}$$

and

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_3 (n+1)^{\frac{\widetilde{\gamma}+1}{p}(2-\widetilde{\nu})} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{9}$$

where $\widetilde{\gamma} := \max\{0,\gamma_i\}$ and $\widetilde{\nu} := \min\{0,\nu_i\}$, $i = \overline{1,m}$.

Corollary 2.8. *For any $K_n,\ n \in \mathbb{N}_0$, there exists $c_4 = c_4(L, \gamma_i, \alpha) > 0$ such that*

$$|K_n(z_i)| \leq c_4 (n+1)^{\frac{\gamma_i+1}{2}(2-\nu_i)}, \tag{10}$$

and

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_4 (n+1)^{\left(\widetilde{\gamma}+1\right)\left(1-\frac{\widetilde{\nu}}{2}\right)}, \tag{11}$$

where $\widetilde{\gamma} := \max\{0, \gamma_i\}$ and $\widetilde{\nu} := \min\{0, \nu_i\}$, $i = \overline{1,m}$.

The following result show the condition of "pay off" of singularity of curve and weight function at the points $\{z_i\}_{i=1}^m$:

Theorem 2.9. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1, ..., \nu_m]$, for some $0 < \nu_1, ..., \nu_m < 1$, $\frac{1}{(2-\nu_i)} \leq \alpha \leq 1$; $h(z)$ defined as in (1) and*

$$\gamma_i + 1 = \frac{1}{\alpha(2-\nu_i)}, \tag{12}$$

for each points $\{z_i\}_{i=1}^m$. Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}_0$, there exists $c_5 = c_5(L, p, \gamma_i, \alpha) > 0$ such that

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_5 (n+1)^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{13}$$

Corollary 2.10. *Suppose that $L \in \widetilde{Q}_\alpha[\nu_1, ..., \nu_m]$, for some $0 < \nu_1, ..., \nu_m < 1$, $\frac{1}{(2-\nu_i)} \leq \alpha \leq 1$; $h(z)$ defined as in (1) and*

$$\gamma_i + 1 = \frac{1}{\alpha(2-\nu_i)}, \tag{14}$$

for each points $\{z_i\}_{i=1}^m$. Then, for any $K_n,\ n \in \mathbb{N}_0$, there exists $c_5 = c_5(L, p, \gamma_i, \alpha) > 0$ such that

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_5 (n+1)^{\frac{1}{2\alpha}}. \tag{15}$$

Comparing Theorem 2.9 with Theorem B, it is seen that, if the equality (14) is satisfied, then the growth of rate of the polynomials $P_n(z)$ and, consequently, $K_n(z)$ on L does not depend on whether the weight function $h(z)$ and the boundary contour L have singularity or not. The condition (14) is called the condition of "interference of singularity" of weight function h and contour L at the "singular" points $\{z_i\}_{i=1}^m$.

Corollary 2.11. *Suppose that $L \in C(1, \alpha, \lambda_1)$, for some $1 < \lambda_1 < 2$, and $\frac{1}{\lambda_1} \leq \alpha < 1$. $h(z)$ defined as in (1) and*

$$(\gamma_1 + 1)\lambda_1 = 1,$$

for each points $\{z_i\}_{i=1}^m$. Then

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_5 \sqrt{n+1}. \tag{16}$$

The estimation (16) coincides from (3) for $\frac{1}{\lambda_1} \leq \alpha < 1$. Therefore, Theorem 2.9 generalizes the one result [35, Th1] for $1 < \lambda_1 < 2$ and $\frac{1}{\lambda_1} \leq \alpha < 1$.

Theorem 2.9 is true under the condition $0 < \nu_1 < 1$. On the other hand, from $\frac{1}{2-\nu_1} \leq \alpha \leq 1$ we see that the $\nu_1 = 1 - \varepsilon$ true only for $\alpha \geq 1 - \varepsilon, \forall \varepsilon > 0$. Therefore, for the $1 \leq \nu_1 \leq 2$, we can consider only curves L such that $\Phi(z) \in Lip(1-\varepsilon), \forall \varepsilon > 0, z \in \overline{\Omega}$. For this purpose, let's give a following definition.

Let S be rectifiable Jordan curve or arc and let $z = z(s)$, $s \in [0, |S|]$, $|S| := mes\ S$, denote the natural representation of S.

Definition 2.12. We say that a Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$.

Now, we shall define a new class of curves L, which have a exterior corners (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^m \in L$.

Definition 2.13. We say that a Jordan region $L \in PC_\theta(\lambda_1, \lambda_2, ..., \lambda_m)$, $0 < \lambda_i \leq 2$, $i = \overline{1,m}$, if $L = \partial G$ consists of the union of finite C_θ(smooth)- arcs $\{L_i\}_{i=1}^m$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_i \pi$, $0 < \lambda_i \leq 2$, at the corner points $\{z_i\}_{i=1}^m \in L$, where two arcs meet.

According to the "three-point" criterion [10, p.100], every piecewise smooth curve (without cusps) is quasiconformal.

In this case, we have the following:

Theorem 2.14. *Let $p > 0$. Suppose that $L \in PC_\theta(\lambda_1, ..., \lambda_m)$, for some $0 < \lambda_i \leq 2$, $i = \overline{1,m}$; $h(z)$ defined as in (1). Then, for any K_n, $n \in \mathbb{N}_0$, there exists $c_6 = c_6(L, p, \gamma_i, \varepsilon) > 0$ such that*

$$|P_n(z_i)| \leq c_6 (n+1)^{\frac{\gamma_i+1}{p}\widetilde{\lambda_i}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \forall \varepsilon > 0. \tag{17}$$

If

$$\gamma_i + 1 = \frac{1}{\lambda_i}, \tag{18}$$

is satisfies for each points $\{z_i\}_{i=1}^m$, then

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_6 (n+1)^{\frac{1}{p}+\varepsilon} \cdot \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{19}$$

where $\widetilde{\lambda_i} := \begin{cases} \lambda_i + \varepsilon, & \text{if } 0 < \lambda_i < 2, \\ 2, & \text{if } \lambda_i = 2, \end{cases}$ *for arbitrary small $\varepsilon > 0$.*

Corollary 2.15. *Suppose that $L \in PC_\theta(\lambda_1, \lambda_2, ..., \lambda_m)$, for some $0 < \lambda_i \leq 2$, $i = \overline{1,m}$; $h(z)$ defined as in (1). Then, for any K_n, $n \in \mathbb{N}_0$, there exists $c_7 = c_7(L, \gamma_i, \varepsilon) > 0$ such that*

$$|K_n(z_i)| \leq c_7 (n+1)^{\frac{\gamma_i+1}{2}\lambda_i+\varepsilon}, \ \forall \varepsilon > 0. \tag{20}$$

If

$$\gamma_i + 1 = \frac{1}{\lambda_i}, \tag{21}$$

for each points $\{z_i\}_{i=1}^m$,then

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_7 (n+1)^{\frac{1}{2}+\varepsilon}, \ \forall \varepsilon > 0. \tag{22}$$

The number $\varepsilon > 0$ on the right side of the estimations (17) (19) and, consequently, (20) and (22) can be removed. For this, we introduce the following definitions:

Definition 2.16. ([29, p.48]; see also [13]) We say that a Jordan curve or arc S called Dini-smooth (DS), if it has a parametrization $z = z(s)$, $0 \leq s \leq |S|$, such that $z'(s) \neq 0$, $0 \leq s \leq |S|$ and $\left|z'(s_2) - z'(s_1)\right| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

Now, we shall define a new class of curves, which at the finite number points have exterior corners and interior cusps simultaneously.

Definition 2.17. We say that a Jordan curve $L \in PDS(\lambda_1, \lambda_2, ..., \lambda_m)$, $0 < \lambda_i \leq 2$, $i = \overline{1,m}$, if $L = \partial G$ consists of a union of finite number of Dini-smooth arcs $\left\{L_j\right\}_{j=0}^{m}$, connecting at the points $\left\{z_j\right\}_{j=0}^{m} \in L$ such that for every $z_i \in L$, $i = \overline{1,m}$, they have exterior (with respect to $\overline{G}$) angles $\lambda_i\pi$, $0 < \lambda_i \leq 2$, at the corner z_i.

In this case, we have the following:

Theorem 2.18. *Let $p > 0$. Suppose that $L \in PDS(\lambda_1, ..., \lambda_m)$, for some $0 < \lambda_i \leq 2$, $i = \overline{1,m}$; $h(z)$ defined as in (1). Then, for any K_n, $n \in \mathbb{N}_0$, there exists $c_8 = c_8(L, p, \gamma_i) > 0$ such that*

$$|P_n(z_i)| \leq c_8 (n+1)^{\frac{\gamma_i+1}{p}\lambda_i} \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{23}$$

If

$$\gamma_i + 1 = \frac{1}{\lambda_i}, \tag{24}$$

is satisfies for each points $\{z_i\}_{i=1}^{m}$, then

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_8 (n+1)^{\frac{1}{p}} \cdot \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{25}$$

Corollary 2.19. *Suppose that $L \in PDS(\lambda_1, \lambda_2, ..., \lambda_m)$, for some $0 < \lambda_i \leq 2$, $i = \overline{1,m}$; $h(z)$ defined as in (1). Then, for any K_n, $n \in \mathbb{N}_0$, there exists $c_9 = c_9(L, p, \gamma_i) > 0$ such that*

$$|K_n(z_i)| \leq c_9 (n+1)^{\frac{\gamma_i+1}{2}\lambda_i}. \tag{26}$$

If

$$\gamma_i + 1 = \frac{1}{\lambda_i}, \tag{27}$$

for each points $\{z_i\}_{i=1}^{m}$, then

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_9 \sqrt{n+1}. \tag{28}$$

Note that, $C(1, \alpha, \lambda_1) \subset PDS(\lambda_1) \subset PC_\theta(\lambda_1)$ for each fixed $0 < \lambda_1 \leq 2$ and $PC_\theta(\lambda_1) \subset \widetilde{Q}_\alpha[\lambda_1]$, for each fixed $0 < \lambda_1 < 1$. In this, (27) and (28) coincides with (2) and (3). Thus, the Corollary 2.19 generalizes the corresponding result in [35].

The sharpness of the estimations (4), (6),(0.3.3) (13) (17) and (23) for some special cases can be discussed by comparing them with the following results:

Remark 2.20. a)For any $n \in \mathbb{N}$, there exists a polynomials $P_n^* \in \wp_n$, and constants $c_{10} = c_{10}(L) > 0$ and $c_{11} = c_{11}(L, p, \gamma) > 0$ such that, for $L := \{z : |z| = 1\}$ a) $h^*(z) \equiv 1$ and b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$. we have:

$$a) \quad \left\|P_n^*\right\|_{\mathcal{L}_\infty} \geq c_{10} n^{\frac{1}{p}} \left\|P_n^*\right\|_{\mathcal{L}_p(h^*,\, L)}, \quad p > 1;$$

$$b) \quad \left\|P_n^*\right\|_{\mathcal{L}_\infty} \geq c_{11} n^{\frac{\gamma+1}{p}} \left\|P_n^*\right\|_{\mathcal{L}_p(h^{**},\, L)}, \quad p > \gamma + 1.$$

As can be seen from these considerations, some estimates for arbitrary polynomials exact, but estimates for ortogonal polynomials obtained as special case, are not exact. Now, we can give one condition where this estimation also will be exact. Suppose that on the one of singular points of weight function h, say z_1, satisfies the following relation:

$$\|K_n\|_{\mathcal{L}_\infty} \asymp |K_n(z_1)|,$$

Then, according to [3], for $L \in \widetilde{Q}_\alpha$ the following is true:

$$|K_n(z_1)| \leq (n+1)^{s_1-\frac{1}{2}}, \tag{29}$$

where $s_1 = \frac{\gamma_1+1}{2\alpha}$.

The inequality (29) is sharp. For $h_1(z) = |z-1|^2$, $L := \{z : |z| = 1\}$ and

$$K_n^*(z) = \frac{1}{\sqrt{(n+1)(n+2)}}\left[1 + 2z + ... + (n+1)z^n\right],$$

we have:

$$\left\|K_n^*\right\|_{\mathcal{L}_\infty} = \left|K_n^*(1)\right| = \frac{\sqrt{(n+1)(n+2)}}{2} \asymp n.$$

3. Some Auxiliary Results

For $a > 0$ and $b > 0$, we shall use the notations "$a \leq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well-known (see, for example, [10], [21, p.97] and [31]):

Definition 3.1. The Jordan arc (or curve) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).

Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and lets define

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

According to [31], we have the following facts:

Corollary 3.2. *If $S \in C_\theta$, then S is $(1+\varepsilon)-$quasiconformal for arbitrary small $\varepsilon > 0$.*

Corollary 3.3. *If S is an analytic curve or arc, then S is $1-$quasiconformal.*

Remark 3.4. It is well-known that, if we are not interested with the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are identical. However, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider that if the curve L is $K-$quasiconformal, then it is $\kappa-$quasicircle with $\kappa = \frac{K^2-1}{K^2+1}$.

By the following Remark 3.4, for simplicity, we will use both terms, depending on the situation.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set

$$d(z, M) = dist(z, M) := \inf\{|z-\zeta| : \zeta \in M\}.$$

For $\delta > 0$ and $z \in \mathbb{C}$ let us set: $B(z,\delta) := \{\zeta : |\zeta - z| < \delta\}$, $\Omega(z,\delta) := \Omega \cap B(z,\delta)$.

Lemma 3.5. ([4]) *Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent.
and similarly so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon = \varepsilon(L) < 1$, $c = c(L) > 1$, $0 < r_0 < 1$ are constants, depending on L and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Corollary 3.6. *Under the assumptions of Lemma 3.5, if $z_3 \in L_{r_0}$ ($z_3 \in L_{Rr_0}$), then*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}$$

Corollary 3.7. *If $L \in C_\theta$, then*

$$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon},$$

for all $\varepsilon > 0$.

Let $\left\{z_j\right\}_{j=1}^{m}$ be a fixed the system of the points on L and the weight function $h(z)$ defined as (1).

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\left\{\left|z_i - z_j\right| : i, j = 1, 2, ..., m,\ i \neq j\right\}$, we put $\Omega(z_j, \delta_j) := \Omega \cap \left\{z : \left|z - z_j\right| \preceq \delta_j\right\}$; $\delta := \min\limits_{1\leq j\leq m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{m} \Omega(z_j, \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j, \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$.

Throughout this work, we will take $R = 1 + \frac{\varepsilon_0}{n+1}$, for some fixed $0 < \varepsilon_0 < 1$. Further, we introduce:

$$w_j := \Phi(z_j), \varphi_j := \arg w_j,\ L^j := L \cap \overline{\Omega}^j,\ \ L_R^j := L_R \cap \Omega^j,\ j = \overline{1, m}, \tag{30}$$

where $\Omega^j := \Psi(\Delta_j^{'})$ and

$$\Delta_1^{'} \ : = \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_m + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},$$
$$\Delta_m^{'} \ : = \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{m-1} + \varphi_m}{2} \leq \theta < \frac{\varphi_m + \varphi_1}{2}\right\},$$

and, for $j = \overline{2, m-1}$

$$\Delta_j^{'} := \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2}\right\};$$

$L = \bigcup\limits_{j=1}^{m} L^j$; $L_R = \bigcup\limits_{j=1}^{m} L_R^j$.

Lemma 3.8. *Let $L \in Q(\kappa)$ for some $0 \leq \kappa < 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \geq |w_1 - w_2|^{1+\kappa},$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows from [28, p.287, Lemma 9.9] and the estimation for the Ψ' (see, for example, [12, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{31}$$

The following lemma is a consequence of the results given in [15], [40].

Lemma 3.9. *Let $L \in C_\theta(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, 2, ..., m$,. Then*
i) for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j + \varepsilon} \leq |\Psi(w) - \Psi(w_j)| \leq |w - w_j|^{\lambda_j - \varepsilon}$, $|w - w_j|^{\lambda_j - 1 + \varepsilon} \leq |\Psi'(w)| \leq |w - w_j|^{\lambda_j - 1 - \varepsilon}$,

ii) for any $w \in \overline{\Delta} \backslash \Delta_j$, $(|w| - 1)^{1+\varepsilon} \leq d(\Psi(w), L)| \leq (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \leq |\Psi'(w)| \leq (|w| - 1)^{-\varepsilon}$.

The following lemma is a consequence of the results given in [29], [13, pp.32-36], and estimation (31) (see, for example, [12, Th.2.8]):

Lemma 3.10. *Let a Jordan curve $L \in PDS(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j \leq 2$, $j = \overline{1, m}$. Then,*
i) for any $w \in \Delta_j$, $\left|\Psi(w) - \Psi(w_j)\right| \asymp \left|w - w_j\right|^{\lambda_j}$, $|\Psi'(w)| \asymp \left|w - w_j\right|^{\lambda_j - 1}$;
ii) for any $w \in \overline{\Delta} \backslash \Delta_j$, $\left|\Psi(w) - \Psi(w_j)\right| \asymp \left|w - w_j\right|$, $|\Psi'(w)| \asymp 1$.

Lemma 3.11. ([9]) *Let L be a rectifiable Jordan curve, $h(z)$ defined as in (1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \leq R^{n + \frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \; p > 0, \gamma^* := \max\{0; \gamma_i\}, \; i = \overline{1, m}. \tag{32}$$

Remark 3.12. In case of $h(z) \equiv 1$, the estimation (32) has been proved in [17].

4. Proofs of Theorems

Throughout proofs of all theorems, we will take $n \geq \left[\frac{\varepsilon_0}{R_0 - 1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant. In addition, in case when $n = 0$, the number n, participating in the all inequalities below will be changed to $(n + 1)$.

4.1. Proof of Theorem 2.9

Proof. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1, ..., \nu_m]$, for some $0 < \nu_1, ..., \nu_m < 1$, $\frac{1}{2 - \nu_i} \leq \alpha \leq 1$, $i = \overline{1, m}$, be given and $h(z)$ defined as in (1). For each $R > 1$, let $w = \varphi_R(z)$ denotes be a univalent conformal mapping G_R onto the B, normalized by $\varphi_R(0) = 0$, $\varphi_R'(0) > 0$, and let $\left\{\zeta_j\right\}$, $1 \leq j \leq m \leq n$, be a zeros of $P_n(z)$ lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} B_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)} \varphi_R(z)}, \tag{33}$$

denotes a Blashke function with respect to zeros $\left\{\zeta_j\right\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$ ([39]). Clearly,

$$\left|B_{m,R}(z)\right| \equiv 1, \; z \in L_R, \tag{34}$$

and

$$\left|B_{m,R}(z)\right| < 1, \; z \in G_R. \tag{35}$$

For any $p > 0$ and $z \in G_R$, let us set

$$T_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}. \tag{36}$$

The function $T_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_n(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i}\int\limits_{L_R} T_n(\zeta)\,\frac{d\zeta}{\zeta - z},\ z \in G_R, \tag{37}$$

or

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}\right| \le \frac{1}{2\pi}\int\limits_{L_R}\left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta - z|} \le \int\limits_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta - z|},$$

since $\left|B_{m,R}(\zeta)\right| = 1$, for $\zeta \in L_R$. Lets now $z \in L$. Multiplying the numerator and denominator of the integrand by $h^{1/2}(\zeta)$, by the Hölder inequality, we obtain

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2} \le \frac{1}{2\pi}\left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2} \tag{38}$$

$$\times\left(\int\limits_{L_R}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta - z_j\right|^{\gamma_j}|\zeta - z|^2}\right)^{1/2} =: \frac{1}{2\pi}J_{n,1}\times J_{n,2},$$

where

$$J_{n,1} := \left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2},\ J_{n,2} := \left(\int\limits_{L_R}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta - z_j\right|^{\gamma_j}|\zeta - z|^2}\right)^{1/2}.$$

Then, since $\left|B_{m,R}(z)\right| < 1$, for $z \in L$, from Lemma 3.11, we have:

$$|P_n(z)| \le (J_{n,1}\cdot J_{n,2})^{2/p} \le \|P_n\|_p\cdot(J_{n,2})^{2/p},\ z \in L. \tag{39}$$

By using notations (30), for the integral $J_{n,2}$, we obtain

$$(J_{n,2})^2 = \sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta - z_j\right|^{\gamma_j}|\zeta - z|^2} \asymp \sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}|\zeta - z|^2} =: \sum_{i=1}^{m}J_{n,2}^i, \tag{40}$$

where

$$J_{n,2}^i := \int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}|\zeta - z|^2},\ i = \overline{1,m}, \tag{41}$$

since the points $\left\{z_j\right\}_{j=1}^{m} \in L$ are distinct. It remains to estimate the integrals $J_{n,2}^i$ for each $i = \overline{1,m}$. For simplicity of our next calculations, we assume that

$$m = 1. \tag{42}$$

Under this assumptions, $L \in \widetilde{Q}_{\alpha}[\nu_1]$, for some $0 < \nu_1 < 1,\ 0 < \alpha \leq 1$. Then, according to [22], $\psi \in Lip\nu_1$ and there exists the number $\delta,\ 0 < \delta < \delta_0 < diam\ \overline{G},$ such that

$$\Phi \in Lip\frac{1}{2-\nu_1},\ z \in \overline{\Omega(z_1,\delta)}. \tag{43}$$

We denote that,

$$\begin{aligned} L^1_{R,1} &:= L^1_R \cap \Omega(z_1,\delta),\ L^1_{R,2} := L_R \backslash L^1_{R,1};\ F^1_{R,i} := \Phi(L^1_{R,i}); \\ L^1_1 &:= L^1 \cap B(z_1,\delta),\ L^1_2 := L^1 \backslash L^1_1;\ F^1_i := \Phi(L^1_i),\ i = 1,2. \end{aligned} \tag{44}$$

By taking into consideration these designations and by replacing the variable $\tau = \Phi(\zeta)$, from (31), we have

$$\begin{aligned} J^1_{n,2} &\asymp \sum_{i=1}^{2} \int\limits_{F^1_{R,i}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2} \\ &\asymp \sum_{i=1}^{2} \int\limits_{F^1_{R,i}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \\ &= : \sum_{i=1}^{2} J(F^1_{R,i}). \end{aligned} \tag{45}$$

So, we need to evaluate the integrals $J(F^1_{R,i})$ for each $i = 1,2$. For this, we will continue in the following manner. Let

$$\|P_n\|_{\infty} =: |P_n(z')|,\ z' \in L, \tag{46}$$

and let $w' = \Phi(z')$. There are two possible cases: the point z' may lie on L^1 or L^2.

1) Suppose first that $z' \in L^1$. If $z' \in L^1_i$, then $w' \in F^1_i$, for $i = 1,2$. Let's $F^{1,1}_{R,j} := \left\{\tau \in F^1_{R,j} : |\tau - w_1| \geq |\tau - w'|\right\}$, $F^{1,2}_{R,j} := F^1_{R,j} \setminus F^{1,1}_{R,j},\ j = 1,2$. Consider the individual cases.

1.1) Let $z' \in L^1_1$. Applying Lemma 3.5, we have

$$J(F^1_{R,1}) = \int\limits_{F^1_{R,1}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \tag{47}$$

$$\begin{aligned} &\preceq n \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{\gamma_1+1}} + n \int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1+1}} \\ &\preceq n \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau-w'|^{(\gamma_1+1)(2-\nu_1)-\frac{1}{\alpha}}\,|\tau-w'|^{\frac{1}{\alpha}}} + n \int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\tau-w_1|^{(\gamma_1+1)(2-\nu_1)-\frac{1}{\alpha}}\,|\tau-w_1|^{\frac{1}{\alpha}}} \\ &\preceq n \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau-w'|^{\frac{1}{\alpha}}} + n \int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\tau-w_1|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}, \end{aligned}$$

$$J(F^1_{R,2}) = \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)}, \tag{48}$$

$$\preceq \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2diam\overline{G}\right)^{-\gamma_1}, & -1<\gamma_1<0 \end{cases}$$

$$\preceq \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|\,(|\tau|-1)}.$$

If $z' \in L^1_1 \cap \overline{B(z_1, \frac{\delta}{2})}$, then according to

$$|\zeta - z'| \geq |\zeta - z_1| - |z' - z_1| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2},$$

from (48), we obtain:

$$\begin{aligned} J(F^1_{R,2}) &\preceq n \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|} \\ &\leq n \cdot \frac{2}{\delta} \int\limits_{F^1_{R,2}} |d\tau| \preceq n \cdot \left|F^1_{R,2}\right| \preceq n. \end{aligned} \tag{49}$$

If $z' \in L^1_1 \setminus \overline{B(z_1, \frac{\delta}{2})}$, then $|\zeta - z'| \geq |\tau - w'|^{\frac{1}{\alpha}}$, since $\Phi \in Lip\alpha$, $z \in \overline{\Omega}\setminus\{z_1\}$, and from (48), we obtain:

$$J(F^1_{R,2}) \preceq n \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|} \preceq n \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}. \tag{50}$$

Therefore, in this case, combining (45)-(50), we have:

$$J^1_{n,2} \preceq n^{\frac{1}{\alpha}}. \tag{51}$$

1.2) Let $z' \in L^1_2$. According to (14), analogously to case 1.1, we have:

$$J(F^1_{R,1}) = \int\limits_{F^1_{R,1}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \tag{52}$$

$$\begin{aligned} &\preceq n \int\limits_{F^{1,1}_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{\gamma_1+1}} + n \int\limits_{F^{1,2}_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1+1}} \\ &\preceq n \int\limits_{F^{1,1}_{R,2}} \frac{|d\tau|}{|\tau - w'|^{(\gamma_1+1)\frac{1}{\alpha}}} + n \int\limits_{F^{1,2}_{R,2}} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1+1)(2-\nu_1)-\frac{1}{\alpha}}\,|\tau - w_1|^{\frac{1}{\alpha}}} \\ &\preceq n \int\limits_{F^{1,1}_{R,2}} \frac{|d\tau|}{|\tau - w'|^{\frac{(\gamma_1+1)\alpha}{\alpha^2}}} + n \int\limits_{F^{1,2}_{R,2}} \frac{|d\tau|}{|\tau - w_1|^{\frac{1}{\alpha}}} \\ &\preceq n \int\limits_{F^{1,1}_{R,2}} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha^2(2-\nu_1)}}} + n \cdot n^{\frac{1}{\alpha}-1} \\ &\preceq n \cdot n^{\frac{1}{\alpha^2(2-\nu_1)}-1} + n^{\frac{1}{\alpha}} \preceq n^{\frac{1}{\alpha}} + n^{\frac{1}{\alpha}} \preceq n^{\frac{1}{\alpha}}, \end{aligned}$$

and, for the integral $J(F^1_{R,2})$:

$$J(F^1_{R,2}) = \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \tag{53}$$

$$\preceq \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2diam\overline{G}\right)^{-\gamma_1}, & -1<\gamma_1<0, \end{cases}$$

$$\preceq n\int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|} \preceq n\int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\tau-w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}.$$

So, for $z' \in L^1_2$, from (45), (52) and (53), we get:

$$J^1_{n,2} \preceq n^{\frac{1}{\alpha}}. \tag{54}$$

Therefore, in case of $z' \in L^1$ for each $\gamma_1 > -1$ and for all $z \in L$, from (39), (40), (52) and (54), we have

$$|P_n(z)| \preceq n^{\frac{1}{p\alpha}}\,\|P_n\|_p\,. \tag{55}$$

If $z' \in L\backslash L^1$, then inequality (53) will be ensured is better. So, we completed the proof. □

4.2. Proof of Theorem 2.7

Proof. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1,...,\nu_m]$, for some $0<\nu_1,...,\nu_m<1$, $\frac{1}{2-\nu_i} \leq \alpha \leq 1$, $i=\overline{1,m}$, be given and $h(z)$ defined as in (1). By using the notations where we used in beginning of the proof of Theorem 2.9 ((33)-(36)) from (37) for $z=z_j$, we get:

$$T_n\left(z_j\right) = \frac{1}{2\pi i}\int\limits_{L_R} T_n(\zeta)\,\frac{d\zeta}{\zeta-z_j}.$$

Therefore, multiplying the numerator and the denominator of the integrand by $h^{1/2}(\zeta)$, according to the Hölder inequality, from (34) and (35), we obtain

$$\left|P_n\left(z_j\right)\right| \leq \left(\frac{1}{2\pi}\right)^{\frac{2}{p}} \left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/p}, \tag{56}$$

$$\times\left(\int\limits_{L_R} \frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta-z_j\right|^{2+\gamma_j}}\right)^{1/p} =: \left(\frac{1}{2\pi}\right)^{\frac{2}{p}} I_{n,1}\times I_{n,2},$$

where

$$I_{n,1} := \|P_n\|_{\mathcal{L}_p(h,L_R)}\,,\ I_{n,2} := \left(\int\limits_{L_R} \frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta-z_j\right|^{2+\gamma_j}}\right)^{1/p}.$$

Then, by Lemma 3.11, for each point $\left\{z_j\right\}_{j=1}^{m} \in L$, we have

$$\left|P_n\left(z_j\right)\right| \leq \|P_n\|_{\mathcal{L}_p} \cdot I_{n,2}. \tag{57}$$

Since the points $\left\{z_j\right\}_{j=1}^{m} \in L$ are distinct, by using designations (30), we get

$$(I_{n,2})^p = \sum_{i=1}^{m} \int\limits_{L_R^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{m} \left|\zeta - z_j\right|^{2+\gamma_j}} \asymp \sum_{i=1}^{m} \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}} =: \sum_{i=1}^{m} I_{n,2}^i, \tag{58}$$

where

$$I_{n,2}^i := \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}},\ i = \overline{1,m}. \tag{59}$$

Therefore, it remains to estimate the integrals $I_{n,2}^i$ for each $i = \overline{1,m}$. In this case, we also assume that $m = 1$. Under the notations (44), we have:

$$I_{n,2}^1 := \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int\limits_{L_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} + \int\limits_{L_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}}. \tag{60}$$

By applying (43), we obtain:

$$\int\limits_{L_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int\limits_{F_{R,1}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{2+\gamma_1}\,(|\tau| - 1)} \tag{61}$$

$$\leq \int\limits_{F_{R,1}^1} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{1+\gamma_1}\,(|\tau| - 1)} \leq n \int\limits_{F_{R,1}^1} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1+1)(2-\nu_1)}} \leq n^{(\gamma_1+1)(2-\nu_1)},$$

$$\int\limits_{L_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} \leq (\delta)^{-2-\gamma_1} mes L_{R,1}^1 \leq 1. \tag{62}$$

Then, from (60), we get:

$$I_{n,2}^1 \leq n^{(\gamma_1+1)(2-\nu_1)}. \tag{63}$$

By combining the relations (57)-(63), we obtain:

$$|P_n(z_1)| \leq n^{\frac{(\gamma_1+1)(2-\nu_1)}{p}} \|P_n\|_{\mathcal{L}_p},$$

and, according to our assumption $m = 1$, we complete the proof. □

4.3. Proof of Theorem 2.14.

Proof. Analogously to beginning of proof of Theorem 2.9, in this case, from (39)-(41) and (58), we obtain:

$$\left|P_n\left(z_j\right)\right| \leq \|P_n\|_{\mathcal{L}_p} \cdot I_{n,2},\ \ z \in L. \tag{64}$$

where

$$(I_{n,2})^p = \sum_{i=1}^{m} \int\limits_{L_R^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{m} \left|\zeta - z_j\right|^{2+\gamma_j}} \tag{65}$$

$$\asymp \sum_{i=1}^{m} \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}} \asymp \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}},\ i = \overline{1,m},$$

since the points $\left\{z_j\right\}_{j=1}^{m} \in L$ are distinct. Therefore, for the proof of (64) sufficiently to evaluate the following integral for each $i = \overline{1,m}$:

$$I_{n,2}^i := \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}}.$$

For simplicity of our next calculations, we assume that $i = 1$. By using the notations (44) and setting $\delta := c_1 d_{1,R}$ for some $c_1 > 1$, where $d_{1,R} := d(z_1, L_R^1)$, $\left|L_{R,i}^1\right| := mesL_{R,i}^1$, $i = 1, 2$, we have:

$$I_{n,2}^1 := \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int\limits_{L_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} + \int\limits_{L_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}},$$

where

$$\int\limits_{L_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} \preceq \int\limits_{d_{1,R}}^{c_1 d_{1,R}} \frac{ds}{s^{2+\gamma_1}} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}; \int\limits_{L_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} \preceq \int\limits_{c_1 d_{1,R}}^{\left|L_{R,2}^1\right|} \frac{ds}{s^{2+\gamma_1}} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}.$$

According these estimations, from (64) and (65), we get:

$$|P_n(z_1)| \preceq \frac{1}{d_{1,R}^{1+\gamma_1}} \|P_n\|_{\mathcal{L}_p}. \tag{66}$$

On the other hand, by Lemma 3.9, for $0 < \lambda_1 < 2$, and [12], for arbitrary continuum with simple connected complement, we have:

$$d_{1,R} \geq \frac{1}{n^{\widetilde{\lambda}_1}}, \tag{67}$$

where $\widetilde{\lambda}_1 := \begin{cases} \lambda_1 + \varepsilon, & \text{if } 0 < \lambda_1 < 2, \\ 2, & \text{if } \lambda_1 = 2, \end{cases}$ for arbitrary small $\varepsilon > 0$. From (66) and (67), we get the proof of (17).

Now, under the conditions (18) we will show estimation (19). Analogously to above situation, from (39)-(41), we have:

$$|P_n(z)| \preceq (J_{n,1} \cdot J_{n,2})^{2/p} \preceq \|P_n\|_p \cdot \left(J_{n,2}^i\right)^{1/p},\ z \in L, \tag{68}$$

where

$$J_{n,2}^i := \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i} |\zeta - z|^2},\ i = \overline{1,m}. \tag{69}$$

It remains to estimate the integrals $J^i_{n,2}$ for each $i = \overline{1,m}$. As we have assumed in proof of the (41), for simplicity of calculations, we also assume that $m = 1$. Therefore, we can estimate the following integral:

$$J^1_{n,2} = \int\limits_{L^1_R} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1} |\zeta - z|^2}.$$

Lets denote by:

$$\begin{aligned} l^1_{R,1} &: = L^1_R \cap \Omega(z_1, c_1 d_{1,R}),\ c_1 > 1, \\ l^1_{R,2} &: = L^1_R \cap (\Omega(z_1, \delta_1) \setminus \Omega(z_1, c_1 d_{1,R}))\ l^1_{R,3} := L^1_R \backslash (l^1_{R,1} \cup l^1_{R,2}). \\ l^1_1 &: = L^1 \cap B(z_1, c_1 d_{1,R}),\ l^1_2 := L^1 \cap (B(z_1, \delta_1) \backslash B(z_1, c_1 d_{1,R}))\,,\ l^1_3 := \ L^1 \backslash \left(L^1_1 \cup L^1_2\right) \end{aligned}$$

Then

$$J^1_{n,2} = \sum_{j=1}^{3} \int\limits_{l^1_{R,j}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1} |\zeta - z|^2} = \sum_{j=1}^{3} J^1_{n,2}(l^1_{R,j}),$$

where

$$J^1_{n,2}(l^1_{R,j}) := \int\limits_{L^1_{R,j}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1} |\zeta - z|^2},\ \ j = 1, 2, 3. \tag{70}$$

So, we need to evaluate the integrals $J^1_{n,2}(l^1_{R,i})$ for each $i = 1, 2, 3$.

Lets assume

$$\|P_n\|_{C(\overline{G})} =: |P_n(z')|\,,\ z' \in L^1 = l^1_1 \cup l^1_2 \cup l^1_3.$$

There are three possible cases: point z' may lie on l^1_1, or l^1_2 and or l^1_3.

1) Suppose first that $z' \in l^1_1$.

1.1) According Lemma 3.9, we get:

a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,1}) \ : \ = \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1} - 1} |\zeta - z'|^2} = \int\limits_{l^1_{R,1}} \frac{|\zeta - z_1|^{1 - \frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z'|^2} \tag{71}$$

$$\le d^{1-\frac{1}{\lambda_1}}_{1,R} \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z'|^2} \le d^{1-\frac{1}{\lambda_1}}_{1,R} \int\limits_{d(z',\, l^1_{R,1})}^{c_1 d(z',\, l^1_{R,1})} \frac{ds}{s^2} \le \frac{d^{1-\frac{1}{\lambda_1}}_{1,R}}{d(z',\, l^1_{R,1})} \le n^{1+\varepsilon};$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_{R,1}) := \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1} - 1} |\zeta - z'|^2} \tag{72}$$

$$= \int\limits_{l^1_{R,1} \cap \{\zeta : |\zeta - z_1| \ge |\zeta - z'|\}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^1_{R,1} \cap \{\zeta : |\zeta - z_1| < |\zeta - z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\le \int\limits_{d(z',\, L_R)}^{c_2 d(z',\, L_R)} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\, L_R)}^{c_3 d(z_1,\, L_R)} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \le d^{-\frac{1}{\lambda_1}}(z',\, L_R) + d^{-\frac{1}{\lambda_1}}(z_1,\, L_R) \le n^{1+\varepsilon}.$$

1.2) a) If $1 \le \lambda_1 \le 2$, then let us remember that $z' \in l_1^1$, and consequently, $|z_1 - z'| \le c_1 d_{1,R}$ for some $c_1 > 1$. Then, $|\zeta - z_1| \le |\zeta - z'| + |z' - z_1| \le |\zeta - z'| + c_1 d_{1,R}$, and, according to well-known inequality [39, p.121]

$$|A + B|^p \le |A|^p + |B|^p\,,\ 0 < p \le 1,\ A > 0,\ B > 0,$$

we get:

$$|\zeta - z_1|^{1-\frac{1}{\lambda_1}} \le |\zeta - z'|^{1-\frac{1}{\lambda_1}} + (d_{1,R})^{1-\frac{1}{\lambda_1}}\,.$$

Therefore, applying Lemma 3.9, we obtain:

$$J_{n,2}^1(l_{R,2}^1) := \int\limits_{l_{R,2}^1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} \tag{73}$$

$$\le \int\limits_{l_{R,2}^1 \cup l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z'|^{1+\frac{1}{\lambda_1}}} + d_{1,R}^{1-\frac{1}{\lambda_1}} \cdot \int\limits_{l_{R,2}^1 \cup l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z'|^2}$$

$$\le \int\limits_{d(z',\,L_R)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + d_{1,R}^{1-\frac{1}{\lambda_1}} \cdot \int\limits_{d(z',\,L_R)}^{c_2} \frac{ds}{s^2} \le d^{-\frac{1}{\lambda_1}}(z',\,L_R) + d_{1,R}^{1-\frac{1}{\lambda_1}} \cdot d^{-1}(z',\,L_R) \le n^{1+\varepsilon}.$$

b) If $0 < \lambda_1 < 1$, then

$$J_{n,2}^1(l_{R,2}^1) := \int\limits_{l_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{74}$$

$$= \int\limits_{l_{R,2}^1 \cap \{\zeta:\,|\zeta - z_1| \ge |\zeta - z'|\}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l_{R,2}^1 \cap \{\zeta:\,|\zeta - z_1| < |\zeta - z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\le \int\limits_{d(z',\,L_R)}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\,L_R)}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \le d^{-\frac{1}{\lambda_1}}(z',\,L_R) + d^{-\frac{1}{\lambda_1}}(z_1,\,L_R) \le n^{1+\varepsilon}.$$

1.3) a) If $1 \le \lambda_1 \le 2$, then

$$J_{n,2}^1(l_{R,3}^1) \quad = \int\limits_{l_{R,3}^1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} \tag{75}$$

$$\le (diamL)^{1-\frac{1}{\lambda_1}} \int\limits_{l_{R,3}^1} \frac{|d\zeta|}{|\zeta - z'|^2} \le \frac{(diamL)^{1-\frac{1}{\lambda_1}}}{(\delta_1 - c_1 d_{1,R})^2}\left|l_{R,3}^1\right| \le 1.$$

b) If $0 < \lambda_1 < 1$, then

$$J_{n,2}^1(l_{R,3}^1) := \int\limits_{l_{R,3}^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \le \frac{\left|l_{R,3}^1\right|}{\delta_1^{1-\frac{1}{\lambda_1}}\,(\delta_1 - c_1 d_{1,R})^2} \le 1. \tag{76}$$

2) Let $z' \in l_2^1$.

2.1) According to Lemma 3.9, we get:
a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,1}) \quad := \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{77}$$

$$= \int\limits_{l^1_{R,1}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} \le d_{1,R}^{1-\frac{1}{\lambda_1}} \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z'|^2} \le d_{1,R}^{1-\frac{1}{\lambda_1}} \int\limits_{d(z',\, L^1_R)}^{c_2} \frac{ds}{s^2} \le \frac{d_{1,R}^{1-\frac{1}{\lambda_1}}}{d(z',\, L^1_R)} \le n^{1+\varepsilon};$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_{R,1}) := \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{78}$$

$$\begin{aligned}
&= \int\limits_{l^1_{R,1}\cap\{\zeta:|\zeta-z_1|\ge|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^1_{R,1}\cap\{\zeta:|\zeta-z_1|<|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \\
&\le \int\limits_{l^1_{R,1}\cup l^1_{R,2}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^1_{R,1}\cup l^1_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \\
&\le \int\limits_{d(z',\, L_R)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \\
&\le d^{-\frac{1}{\lambda_1}}(z',\, L_R) + d^{-\frac{1}{\lambda_1}}(z_1,\, L_R) \le n^{1+\varepsilon}.
\end{aligned}$$

2.2) a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,2}) := \int\limits_{l^1_{R,2}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} \tag{79}$$

$$\begin{aligned}
&= \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} + \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|\le|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \\
&\le \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} + \int\limits_{d(z_1,\, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \\
&\le \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|\,|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}}\,|\zeta - z'|^2} + n^{1+\varepsilon}.
\end{aligned}$$

Lets denote by $I_n(z_1, z')$ the last integral and let $F := \Phi(l^1_{R,2} \cap \{\zeta : |\zeta - z_1| > |\zeta - z'|\})$. For the estimation $I_n(z_1, z')$, first of all, replacing the variable $\tau = \Phi(\zeta)$, we obtain:

$$I^1_{R,2}(z_1, z') := \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|\,|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}}\,|\zeta - z'|^2}$$

$$= \int\limits_{F} \frac{|\Psi(\tau)-\Psi(w_1)|\,|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\frac{1}{\lambda_1}}\,|\Psi(\tau)-\Psi(w')|^2}$$

$$= \int\limits_{F} \left|\frac{\Psi(\tau)-\Psi(w_1)}{\Psi(\tau)-\Psi(w')}\right| \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\frac{1}{\lambda_1}}\,|\Psi(\tau)-\Psi(w')|}$$

Since $z' \in l_2^1$, then $c_1 d_{1,R} \le |\zeta - z'| < |\zeta - z_1| \le \delta_1$. According to Lemma 3.5, in this case we have $|\tau| - 1 \le |\tau - w'| \le |\tau - w_1| \le 1$. Assume that $|\tau - w'| < |\tau - w_1|$ (the inverse is trivial). We set $\varepsilon_0 := |\tau| - 1$. In this case, we take the discs centered at the point w_1, and radius $2^s\varepsilon_0$, $s = 1, 2, ...N$, where we choose a number N such that the circle is $Q_N = \left\{\tau : |\tau - w_1| = 2^N\varepsilon_0\right\}$, that satisfies the conditions $Q_N \cap \{t : |t| = R\} \ne \varnothing$, $Q_{N+1} \cap \{t : |t| = R\} = \varnothing$. Then, setting $F^s := F \cap \left\{t : 2^{s-1}\varepsilon_o \le |t - w_1| \le 2^s\varepsilon_o\right\}$, and applying Lemma 3.5 and Lemma 3.9, we have:

$$I_{R,2}^{1,2}(z_1, z') \tag{80}$$

$$= \int\limits_{F} \left|\frac{\Psi(\tau)-\Psi(w_1)}{\Psi(\tau)-\Psi(w')}\right| \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\frac{1}{\lambda_1}}\,|\Psi(\tau)-\Psi(w')|}$$

$$\preceq \sum_{s=1}^{\infty} \int\limits_{F^s} \left[\frac{|\tau - w_1|}{|\tau| - 1}\right]^{\varepsilon(L)} \frac{|d\tau|}{|\tau - w_1|^{1+\varepsilon}} \frac{\varepsilon_0^{\lambda_1 - 1 - \varepsilon}\,|d\tau|}{|\tau - w'|^{\lambda_1 + \varepsilon}}$$

$$\preceq \sum_{s=1}^{\infty} \left(\frac{2^s\varepsilon_o}{\varepsilon_o}\right)^{\varepsilon(L)} \frac{\varepsilon_0^{\lambda_1 - 1 - \varepsilon}}{(2^{s-1}\varepsilon_o)^{1+\varepsilon}} \int\limits_{F^s} \frac{|d\tau|}{|\tau - w'|^{\lambda_1 + \varepsilon}}$$

$$\preceq 2^{1+\varepsilon}\varepsilon_o^{\lambda_1 - 2 - \varepsilon} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon(L)}}{2^{1+\varepsilon}}\right)^s \int\limits_{F^s_{R_1,1}} \frac{|d\tau|}{|\tau - w'|^{\lambda_1 + \varepsilon}}$$

$$\preceq \varepsilon_o^{\lambda_1 - 2 - \varepsilon} \int\limits_{n^{-1}}^{c_4} \frac{ds}{s^{\lambda_1 + \varepsilon}} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon(L)}}{2^{1+\varepsilon}}\right)^s \preceq \left(\frac{1}{n}\right)^{\lambda_1 - 2 - \varepsilon} \cdot n^{\lambda_1 - 1 + \varepsilon} \le n^{1+\varepsilon},$$

where $\varepsilon(L)$, $0 < \varepsilon(L) < 1$, taken from Lemma 3.5.

b) If $0 < \lambda_1 < 1$, then

$$J_{n,2}^1(l_{R,2}^1) := \int\limits_{l_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1} - 1}\,|\zeta - z'|^2} \tag{81}$$

$$= \int\limits_{l_{R,2}^1 \cap \{\zeta : |\zeta - z_1| \ge |\zeta - z'|\}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l_{R,2}^1 \cap \{\zeta : |\zeta - z_1| < |\zeta - z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\le \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \preceq \int\limits_{d(z',\,L_R)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\,L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\le d^{-\frac{1}{\lambda_1}}(z',\,L_R) + d^{-\frac{1}{\lambda_1}}(z_1,\,L_R) \le n^{1+\varepsilon}.$$

2.3)

a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,3}) = \int\limits_{l^1_{R,3}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z'|^2} \tag{82}$$

$$= \int\limits_{l^1_{R,3}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z'|^2} + \int\limits_{l^1_{R,3}\cap\{\zeta:|\zeta-z_1|\le|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l^1_{R,3}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z'|^2} + \int\limits_{d(z_1,\, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l^1_{R,3}\cap\{\zeta:|\zeta-z_1|>|\zeta-z'|\}} \frac{|\zeta - z_1|\,|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}}\,|\zeta - z'|^2} + n^{1+\varepsilon}.$$

Denote by $I^{1,2}_{R,3}(z_1, z')$ the last integral. We estimate this integral analogously to integral $I^{1,2}_{R,2}(z_1, z')$. Consequently, in this case for $J^1_{n,2}(l^1_{R,3})$ we will obtain:

$$J^1_{n,2}(l^1_{R,3}) \preceq n^{1+\varepsilon},\ \forall \varepsilon > 0. \tag{83}$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_{R,3}) := \int\limits_{l^1_{R,3}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{84}$$

$$\preceq \int\limits_{l^1_{R,3}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} \preceq \int\limits_{d(z',\, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \preceq \frac{1}{d^{\frac{1}{\lambda_1}}(z',\ L_R)} \preceq n^{1+\varepsilon}.$$

3) Now, lets $z' \in l^1_3$. Note that, this case includes also the case of $z' \in L\backslash L^1$, since the point z' more removed from the z_1.

3.1) a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,1}) \quad := \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{85}$$

$$= \int\limits_{l^1_{R,1}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z'|^2} \preceq \frac{(diam L)^{1-\frac{1}{\lambda_1}}}{(\delta_1 - c_1 d_{1,R})^2} \left|l^1_{R,3}\right| \preceq 1.$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_{R,1}) := \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{86}$$

$$= \int\limits_{l^1_{R,1}} \frac{|\zeta - z_1|\,|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}}\,|\zeta - z'|^2} \preceq \frac{d_{1,R}}{(\delta_1 - c_1 d_{1,R})^2} \int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}}}$$

$$\preceq \frac{d_{1,R}}{(\delta_1 - c_1 d_{1,R})^2} \int\limits_{d_{1,R}}^{c_1 d_{1,R}} \frac{|d\zeta|}{s^{\frac{1}{\lambda_1}}} \preceq \frac{1}{d_{1,R}^{\frac{1}{\lambda_1}-2}} \preceq n^{1+\varepsilon}.$$

3.2) a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,2}) := \int\limits_{l^1_{R,2}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} \tag{87}$$

$$= \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|\ge|\zeta-z'|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} + \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|<|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|\ge|\zeta-z'|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} + \int\limits_{d(z_1,\,L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|\ge|\zeta-z'|\}} \frac{|\zeta - z_1|\,|d\zeta|}{|\zeta - z'|\,|\zeta - z'|^{1+\frac{1}{\lambda_1}}} + n^{1+\varepsilon}.$$

Denote by $I^{1,3}_{R,3}(z_1, z')$ the last integral. We estimate this integral analogously to integral $I^{1,2}_{R,2}(z_1, z')$. Consequently, in this case for $J^1_{n,2}(l^1_{R,3})$ we will obtain:

$$J^1_{n,2}(l^1_{R,3}) \preceq n^{1+\varepsilon},\ \forall \varepsilon > 0. \tag{88}$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_{R,2}) := \int\limits_{l^1_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{89}$$

$$= \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|\ge|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^1_{R,2}\cap\{\zeta:|\zeta-z_1|<|\zeta-z'|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\le \int\limits_{L^1_R} \frac{|d\zeta|}{|\zeta - z'|^{\frac{1}{\lambda_1}+1}} + \int\limits_{L^1_R} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \preceq \int\limits_{d(z',\,L_R)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\,L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq d^{-\frac{1}{\lambda_1}}(z',\,L_R) + d^{-\frac{1}{\lambda_1}}(z_1,\,L_R) \preceq n^{1+\varepsilon}.$$

3.3) a) If $1 \le \lambda_1 \le 2$, then

$$J^1_{n,2}(l^1_{R,3}) = \int\limits_{l^1_{R,3}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}}\,|d\zeta|}{|\zeta - z'|^2} \tag{90}$$

$$\preceq (diamL)^{1-\frac{1}{\lambda_1}} \int\limits_{l^1_{R,3}} \frac{|d\zeta|}{|\zeta - z'|^2} \preceq \int\limits_{d(z',\, L_R)}^{c_4} \frac{ds}{s^2} \preceq \frac{1}{d(z',\, L_R)} \preceq n^{1+\varepsilon}.$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_{R,3}) := \int\limits_{l^1_{R,3}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1}\,|\zeta - z'|^2} \tag{91}$$

$$\preceq \frac{1}{\delta_1^{1-\frac{1}{\lambda_1}}} \int\limits_{d(z',\, L_R)}^{c_4} \frac{ds}{s^2} \preceq \frac{1}{d(z',\, L_R)} \preceq n^{1+\varepsilon}.$$

Combining estimations (68)-(91), we get the proof of (19). □

4.4. Proof of Remark 2.20.

Proof. Lets $P^*_n(z) = \sum\limits_{j=0}^{n-1} z^j$ and $L := \{z : |z| = 1\}$. Then, $L \in \widetilde{Q}_1$.

a) $h^*(z) \equiv 1$; b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$.

Obviously,

$$\left|P^*_n(z)\right| \le \sum_{j=0}^{n-1} \left|z^j\right| = n, \;\; |z| = 1; \;\; \left|P^*_n(1)\right| = n.$$

So,

$$\left\|P^*_n\right\|_{\mathcal{L}_\infty} = n;$$

On the other hand, according to [38, p. 236], we have:

$$\left\|P^*_n\right\|_{\mathcal{L}_p(h^*,L)} \asymp \begin{cases} n^{1-\frac{\gamma+1}{p}}, & p > 1, \\ \ln n, & p = 1, \\ 1, & 0 < p < 1. \end{cases}$$

and

$$\left\|P^*_n\right\|_{\mathcal{L}_p(h^{**},L)} \asymp n^{1-\frac{\gamma+1}{p}}, \quad p > \gamma + 1.$$

Therefore,

$$a) \;\; \left\|P^*_n\right\|_{\mathcal{L}_\infty} = n \asymp n^{\frac{1}{p}} \left\|P^*_n\right\|_{\mathcal{L}_p(h^*,L)}, \; p > 1;$$

$$b) \;\; \left\|P^*_n\right\|_{\mathcal{L}_\infty} = n = n^{1-\frac{\gamma+1}{p}} \cdot n^{\frac{\gamma+1}{p}} \asymp n^{\frac{\gamma+1}{p}} \left\|P^*_n\right\|_{\mathcal{L}_p(h^{**},L)}, \; p > \gamma + 1.$$

□

4.5. Proof of Corollary 2.11

Proof. If $L \in C(1, \alpha, \lambda_1)$, then the curve $L = \partial G$ has a interior (with respect to $\overline{G}$) $(2 - \lambda_1)-$ angle at the z_1. Then, according to [22], $\psi \in Lip\frac{1}{2-\lambda_1}$, and so, by [22], $\Phi \in Lip\frac{1}{\lambda_1}$. Therefore, $L \in \widetilde{Q}_{\alpha,\beta_1}$ for $\alpha = 1$ (2.3) and $\beta_1 = \frac{1}{\lambda_1}$. In this case, for $p = 2$ from (14) and (15), we have the proof. □

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