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Connectedness of Ordered Rings of Fractions of $C(X)$ with the m-Topology

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Abstract. An order is presented on the rings of fractions $S^{-1}C(X)$ of $C(X)$, where S is a multiplicatively closed subset of $C(X)$, the ring of all continuous real-valued functions on a Tychonoff space X. Using this, a topology is defined on $S^{-1}C(X)$ and for a family of particular multiplicatively closed subsets of $C(X)$ namely $m.c.$ $\mathfrak{z}$-subsets, it is shown that $S^{-1}C(X)$ endowed with this topology is a Hausdorff topological ring. Finally, the connectedness of $S^{-1}C(X)$ via topological properties of X is investigated.

1. Introduction

In this paper, the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorff space X, is denoted by $C(X)$ ($C^*(X)$). The space X is called pseudocompact if $C(X) = C^*(X)$. For every $f \in C(X)$ the set $Z(f) = \{f \in C(X) : f(x) = 0\}$ is said to be zero-set of f and it's complement which is denoted by coz f, is called cozero-set of f. Moreover, an ideal $I \subseteq C(X)$ is said to be z-ideal if for every $f \in I$ and $g \in C(X)$, the inclusion $Z(f) \subseteq Z(g)$ implies that $g \in I$. $u \in C(X)$ is a unit (i.e., u has multiplicative inverse) if and only if $Z(u) = \emptyset$ and it is not hard to see that an element f of $C(X)$ is zero-divisor if and only if $\mathrm{int}_X Z(f) \neq \emptyset$. The set of all units and the set of all zero-divisors of $C(X)$ are denoted by $U(X)$ and $Zd(X)$ respectively.

Let βX and υX be the Stone-Čech compactification and the Hewitt realcompactification of the space X, respectively. For every $f \in C^*(X)$ the unique extension of f to a continuous function in $C(\beta X)$ is denoted by f^β and for each $p \in \beta X$, $M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ ($M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}$) is a maximal ideal of $C(X)$ ($C^*(X)$) and also, every maximal ideal of $C(X)$ ($C^*(X)$) is precisely of the form M^p (M^{*p}), for some $p \in \beta X$. Moreover, for every $p \in \beta X$, $O^p = \{f \in C(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$ is the intersection of all prime ideals of $C(X)$ which are contained in M^p. In fact, we have;

Lemma 1.1. ([7, Theorem 7.15]) *Every prime ideal P in $C(X)$ contains O^p for some unique $p \in \beta X$, and M^p is the unique maximal ideal containing P.*

Whenever $p \in X$, the ideals M^p and O^p will be the sets $\{f \in C(X) : p \in Z(f)\}$ and $\{f \in C(X) : p \in \mathrm{int}_X Z(f)\}$ respectively and in this case, they are denoted by M_p and O_p. A maximal ideal M of $C(X)$ is called real whenever the residue class field $\frac{C(X)}{M}$ is isomorphic with the real field $\mathbb{R}$. Thus, for every $p \in \upsilon X$, M^p is a

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real maximal ideal, and conversely every real maximal ideal of $C(X)$ is precisely of the form M^p for some $p \in \upsilon X$. Moreover, $M^p \bigcap C^*(X) = M^{*p}$ if and only if $p \in \upsilon X$, see 7.9 (c) in [7].

Let R be a commutative ring with unity and suppose that S is a multiplicatively closed subset or briefly an *m.c.* subset of R. Here $S^{-1}R$ is the ring of all equivalence classes of the formal fractions $\frac{a}{s}$ for $a \in R$ and $s \in S$, where the equivalence relation is the obvious one. Whenever S is the set of all non-zero-divisors of R, then $S^{-1}R$ is called the classical ring of quotients of R.

An *m.c.* subset T of R is called saturated whenever $a, b \in R$ and $ab \in T$ imply that a and b belong to T. For an arbitrary *m.c.* subset S of R, the intersection of all saturated *m.c.* subsets of R which contain S, is called saturation of S and is denoted by $\bar{S}$. Using 5.7 in [11] we have

$$\bar{S} = R \setminus \bigcup_{\substack{P \in \mathrm{Spec}(R) \\ P \cap S = \emptyset}} P.$$

Lemma 1.2. ([11, Exercise 5.12(iv)]) *For an arbitrary m.c. subset S of a commutative ring R with unity, two rings $S^{-1}R$ and $\bar{S}^{-1}R$ are isomorphic.*

In sequel, for every *m.c.* subset S of $C(X)$, the ring of fractions $S^{-1}C(X)$ is often abbreviated as $S^{-1}C$.

2. An Order Relation on $S^{-1}C$

The m-topology on $C(X)$ is defined by taking the sets of the form

$$B(f, u) = \{g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in X\}$$

as a base for the neighborhood system at f, for each $f \in C(X)$, where u runs through the set of all positive units of $C(X)$. This topology on $C(X)$ which is denoted by $C_m(X)$, was first introduced in [9] and studied more in [1–3, 5, 8, 12]. To define a topology on $S^{-1}C$, similar to the m-topology on $C(X)$, we need an ordering to make $S^{-1}C$ a lattice-ordered ring. We define the order relation $\leq$ on $S^{-1}C$ as follows:

Definition 2.1. For $\frac{f}{r} \in S^{-1}C$, we define

$$0 \leq \frac{f}{r} \text{ if there exists } t \in S \text{ such that } 0 \leq (t^2 rf)(x) \text{ for all } x \in X.$$

Clearly $0 \leq \frac{f}{r}$ if and only if $0 \leq (rf)(x)$ for all $x \in \mathrm{coz}\, t$, for some $t \in S$. This definition is similar to the familiar definition of order on $C(X)$. But here we consider restriction of each $\frac{f}{r}$ on a cozero-set of X instead of X itself. To see that the order $\leq$ is well defined, let $\frac{f}{r}, \frac{g}{s} \in S^{-1}C$, $\frac{f}{r} = \frac{g}{s}$ and $0 \leq \frac{f}{r}$. Then there exist $p, q \in S$ such that $qfs = qrg$ and $0 \leq p^2 rf$. Now, the inequality $0 \leq (q^2s^2)(p^2rf)$ implies that $0 \leq (p^2rsq)(qfs) = (p^2rsq)(qrg) = (p^2r^2q^2)(sg)$ and since $prq \in S$, we conclude that $0 \leq \frac{g}{s}$.

Proposition 2.2. *let S be an m.c. subset of $C(X)$, then $(S^{-1}C, \leq)$ is a lattice-ordered ring.*

Proof. Clearly for every $\frac{f}{r} \in S^{-1}C$ if $0 \leq \frac{f}{r}$ and $0 \leq -\frac{f}{r}$, then $\frac{f}{r} = 0$. Now, suppose that $\frac{f}{r}, \frac{g}{s} \in S^{-1}C$, $0 \leq \frac{f}{r}$ and $0 \leq \frac{g}{s}$. There exist $t_1, t_2 \in S$ such that $0 \leq rf$ on $\mathrm{coz}\, t_1$ and $0 \leq sg$ on $\mathrm{coz}\, t_2$. Therefore, $0 \leq r^2s^2(s^2rf + r^2sg)$ and $0 \leq r^2s^2(rfsg)$ on $\mathrm{coz}\, t_1t_2$ and thus, $0 \leq \frac{s^2rf + r^2sg}{r^2s^2} = \frac{f}{r} + \frac{g}{s}$ and $0 \leq \frac{rfsg}{r^2s^2} = \frac{f}{r} \cdot \frac{g}{s}$ on $\mathrm{coz}\, t_1t_2$. To prove that $S^{-1}C$ is lattice, it can be shown that

$$\frac{f}{r} \wedge \frac{g}{s} = \frac{rf}{r^2} \wedge \frac{sg}{s^2} = \frac{s^2rf}{r^2s^2} \wedge \frac{r^2sg}{r^2s^2} = \frac{s^2rf \wedge r^2sg}{s^2r^2}.$$

□

If S is an *m.c.* subset of a commutative ring R, then for every $n \in \mathbb{N}$, the set $S^n = \{s^n : s \in S\}$ is an *m.c.* subset of R and clearly two rings $(S^n)^{-1}R$ and $S^{-1}R$ are isomorphic. In fact, the map $i_n(\frac{f}{r}) = \frac{r^{n-1}f}{r^n}$ is an isomorphism from $S^{-1}R$ onto $(S^n)^{-1}R$. Now we define an ordering $\leq^*$ on $(S^2)^{-1}C$ as follows;

Definition 2.3. For every $\frac{f}{r} \in (S^2)^{-1}C$, we define

$$0 \leq^* \frac{f}{r} \text{ if there exists } t \in S^2 \text{ such that } 0 \leq t(x)f(x) \text{ for all } x \in X.$$

If S is an *m.c.* subset of $C(X)$ then $S^2 \subseteq \{f \in S : 0 \leq f\}$. Therefore $0 \leq^* \frac{f}{r}$ if and only if $0 \leq f$ on $\operatorname{coz} t$ for some $t \in S$. Similar to Definition 2.1, it can be shown that $((S^2)^{-1}C, \leq^*)$ is a lattice-ordered ring. Moreover, we have the following result whose proof is left to the readers.

Proposition 2.4. *Let S be an m.c. subset of $C(X)$. Two rings $(S^{-1}C, \leq)$ and $((S^2)^{-1}C, \leq^*)$ are lattice isomorphic. In fact, the map $i_2(\frac{f}{r}) = \frac{rf}{r^2}$ from $S^{-1}C$ onto $(S^2)^{-1}C$ is an isomorphism and also order-preserving, i.e., $\frac{f}{r} \leq \frac{g}{s}$ if and only if $\frac{rf}{r^2} \leq^* \frac{sg}{s^2}$.*

Now using the above proposition, without loss of generality, for every lattice-ordered ring $(S^{-1}C, \leq)$ we can assume that each member of S is non-negative. In addition, we can consider $0 \leq \frac{f}{r}$ whenever $0 \leq f$ on $\operatorname{coz} t$ for some $t \in S$.

Definition 2.5. A subset S of $C(X)$ is called $\mathfrak{z}$-subset whenever $f, g \in C(X)$ and $f \in S$, then $Z(f) = Z(g)$ implies that $g \in S$.

Example 2.6. The set $C(X)\backslash Zd(X) = \{f \in C(X) : int_X Z(f) = \emptyset\}$ of all non-zero-divisor elements of $C(X)$, is a multiplicatively closed $\mathfrak{z}$-subset (or briefly an *m.c.* $\mathfrak{z}$-subset) of $C(X)$. Another example of *m.c.* $\mathfrak{z}$-subset is $U(X) = \{f \in C(X) : Z(f) = \emptyset\}$, the set of all units of $C(X)$. If $\{P_\lambda\}_{\lambda \in \Lambda}$ is a family of prime z-ideals of $C(X)$, then $S = C(X)\backslash \bigcup_{\lambda \in \Lambda} P_\lambda$ is also an *m.c.* $\mathfrak{z}$-subset of $C(X)$. Note that whenever P is a prime ideal of $C(X)$ which is not z-ideal, then $S = C(X)\backslash P$ is a saturated *m.c.* subset of $C(X)$ which is not a $\mathfrak{z}$-subset.

Proposition 2.7. *If S is an m.c. $\mathfrak{z}$-subset of $C(X)$, then the set $T := \{f \in C(X) : Z(f) \subseteq Z(s) \text{ for some } s \in S\}$ is the saturation of S.*

Proof. We show that T is the smallest saturated *m.c.* subset containing S. First, note that T is a saturated *m.c.* subset of $C(X)$ containing S. In fact, if $f, g \in T$ then there exist s_1, s_2 in S such that $Z(f) \subseteq Z(s_1)$ and $Z(g) \subseteq Z(s_2)$. Therefore, $Z(fg) = Z(f) \cup Z(g) \subseteq Z(s_1 s_2)$ which implies $fg \in T$. Moreover, if $fg \in T$ then $Z(fg) \subseteq Z(s)$, for some $s \in S$. Thus $Z(f) \subseteq Z(s)$ and also $Z(g) \subseteq Z(s)$ which imply that $f, g \in T$. Next, let T' be a saturated *m.c.* subset of $C(X)$ containing S and suppose that $f \in T$. Hence $Z(f) \subseteq Z(s)$, for some $s \in S$ and thus $Z(fs) = Z(f) \cup Z(s) = Z(s)$. Since S is a $\mathfrak{z}$-subset, $fs \in S \subseteq T'$ and so $f \in T'$, i.e., $T \subseteq T'$ which complete the proof. □

Corollary 2.8. *Let S be an m.c. subset of $C(X)$. S is a saturated m.c. $\mathfrak{z}$-subset if and only if for every $f \in C(X)$ and $s \in S$, the inclusion $Z(f) \subseteq Z(s)$ implies that $f \in S$.*

Corollary 2.9. *The saturation of every m.c. $\mathfrak{z}$-subset of $C(X)$ is a $\mathfrak{z}$-subset.*

Example 2.10. Let $f(x) = |x| - 1$ be a function of $C(\mathbb{R})$. Then $S_1 = \{1, f, f^2, ...\}$ is an *m.c.* subset of X which is not $\mathfrak{z}$-subset nor saturated. In fact,

$$S_2 = \{g \in C(\mathbb{R}) : Z(g) = \emptyset \text{ or } Z(g) = \{1, -1\}\}$$

is the smallest *m.c.* $\mathfrak{z}$-subset of $C(\mathbb{R})$ containing S_1 and for saturation of S_2 we have

$$\bar{S}_2 = \{\, g \in C(\mathbb{R}) : \; Z(g) \subseteq \{1, -1\}\}.$$

Moreover, it is easy to see that $S_1 \subsetneq S_2 \subsetneq \bar{S}_2$.

Similarly to the order relation $\leq$, for every $\frac{f}{r} \in S^{-1}C$ we define $0 < \frac{f}{r}$ if $0 < f$ on $\operatorname{coz} t$ for some $t \in S$.

Proposition 2.11. *The set $U^+ = \{\frac{f}{r} \in S^{-1}C : 0 < \frac{f}{r}\}$ is closed with respect to the operations $\vee$ and $\wedge$. Moreover, if S is an m.c. $\mathfrak{z}$-subset, then every member of U^+ is a unit of $S^{-1}C$.*

Proof. If $\frac{f}{r}, \frac{g}{s} \in U^+$, then there exist $t_1, t_2 \in S$ such that $0 < f$ on $\operatorname{coz} t_1$ and $0 < g$ on $\operatorname{coz} t_2$. Since $0 \leq r, s$ we have $0 < sf \wedge rg$ on $\operatorname{coz} t_1 t_2 rs$ which implies that $0 < \frac{sf \wedge rg}{rs} = \frac{f}{r} \wedge \frac{g}{s}$. To prove the second part of the proposition, let $0 < \frac{f}{r}$. We have $0 < f$ on $\operatorname{coz} t$ for some $t \in S$ and so $\operatorname{coz} t \subseteq \operatorname{coz} f$. Therefore, $\operatorname{coz} t = \operatorname{coz} tf$ and since S is an *m.c.* $\mathfrak{z}$-subset, then $tf \in S$. Now, $\frac{f}{r} = \frac{tf}{tr} \in S^{-1}C$ implies that $\frac{f}{r}$ is a unit. □

3. The *m*-Topology on $S^{-1}C$

Before defining the *m*-topology on $S^{-1}C$, we note that $|\frac{f}{r}| = \frac{f}{r} \vee (\frac{-f}{r}) = \frac{f \vee (-f)}{r} = \frac{|f|}{r}$. Now, for each $\frac{f}{r} \in S^{-1}C$ and each $\frac{u}{t} \in U^+$ if we consider the set $B(\frac{f}{r}, \frac{u}{t}) := \{\frac{g}{s} : |\frac{f}{r} - \frac{g}{s}| < \frac{u}{t}\}$, then clearly we have:

$$B(\frac{f}{r}, \frac{u}{t}) = \{\frac{g}{s} : |\frac{f}{r}(x) - \frac{g}{s}(x)| < \frac{u}{t}(x) \text{ for all } x \in \operatorname{coz} q \subseteq \operatorname{coz} rstu \text{ for some } q \in S\}.$$

The collection $\mathcal{B} = \{B(\frac{f}{r}, \frac{u}{t}) : \frac{f}{r} \in S^{-1}C \text{ and } \frac{u}{t} \in U^+\}$ is a base for a topology on $S^{-1}C$. In fact, $\frac{f}{r} \in B(\frac{f}{r}, \frac{u}{t})$ and $B(\frac{f}{r}, \frac{u}{t} \wedge \frac{v}{s}) \subseteq B(\frac{f}{r}, \frac{u}{t}) \cap B(\frac{f}{r}, \frac{v}{s})$ for every $\frac{u}{t}, \frac{v}{s} \in U^+$. Moreover, if $\frac{g}{s} \in B(\frac{f}{r}, \frac{u}{t})$, then $\frac{p}{q} := \frac{u}{t} - |\frac{f}{r} - \frac{g}{s}| \in U^+$ and we have $B(\frac{g}{s}, \frac{p}{q}) \subseteq B(\frac{f}{r}, \frac{u}{t})$. As the *m*-topology on $C(X)$, this topology on $S^{-1}C$ is called the *m*-topology and $S^{-1}C$ endowed with this topology is denoted by $S_m^{-1}C$. This topology is in fact a generalization of the *m*-topology on $C(X)$. Note that whenever $S = U(X)$ then $S_m^{-1}C = C_m(X)$.

Recall that a topological ring is simply a ring furnished with a topology for which its algebraic operations are continuous, see [13]. We also notice that a Hausdorff topological ring is completely regular, see 8.1.17 in [6]. To prove that $S_m^{-1}C$ is a Hausdorff topological ring we need the following lemmas.

Lemma 3.1. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$. For every $0 \leq \frac{f}{r} \in S^{-1}C$ there exists $\frac{g}{s} \in S^{-1}C$ such that $0 \leq g, s \leq 1$ and $\frac{f}{r} = \frac{g}{s}$.*

Proof. Consider $s = \frac{r}{1+r+|f|}$ and $g = \frac{|f|}{1+r+|f|}$. Clearly $Z(s) = Z(r)$ implies $s \in S$ and we have $\frac{g}{s} = \frac{|f|}{r} = |\frac{f}{r}| = \frac{f}{r}$. □

Lemma 3.2. *If S is an m.c. $\mathfrak{z}$-subset of $C(X)$, then the set $\{B(\frac{f}{r}, \frac{v}{1}) : f \in C(X), r, v \in S \text{ and } 0 \leq v \leq 1\}$ is a base for the m-topology on $S^{-1}C$.*

Proof. By Lemma 3.1, for each $B(\frac{f}{r}, \frac{u}{t})$ there exist $v, s \in S$ such that $0 \leq v, s \leq 1$, and $\frac{u}{t} = \frac{v}{s}$. But $s(x)v(x) \leq v(x)$ for all $x \in \operatorname{coz} sv$, then $\frac{v}{1} \leq \frac{v}{s}$ and so $\frac{f}{r} \in B(\frac{f}{r}, \frac{v}{1}) \subseteq B(\frac{f}{r}, \frac{v}{s}) = B(\frac{f}{r}, \frac{u}{t})$. □

Proposition 3.3. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$. Then $S_m^{-1}C$ is a Hausdorff topological ring.*

Proof. To prove the continuity of addition and multiplication, let $\frac{f}{r}, \frac{g}{s} \in S^{-1}C$ and $\frac{u}{1} \in U^+$. Then

$$+\left(B\left(\frac{f}{r}, \frac{\frac{u}{2}}{1}\right) \times B\left(\frac{g}{s}, \frac{\frac{u}{2}}{1}\right)\right) \subseteq B\left(\frac{f}{r} + \frac{g}{s}, \frac{u}{1}\right)$$

$$\cdot\left(B\left(\frac{f}{r}, \frac{v}{1}\right) \times B\left(\frac{g}{s}, \frac{v}{1}\right)\right) \subseteq B\left(\frac{f}{r} \cdot \frac{g}{s}, \frac{u}{1}\right)$$

where $\frac{v}{1} \in U^+$ such that $\left(\frac{1}{1} + \frac{u}{1} + |\frac{f}{r}| + |\frac{g}{s}|\right)\frac{v}{1} < \frac{u}{1}$. In fact, if we consider $w := \left(\frac{1}{1} + \frac{u}{1} + |\frac{f}{r}| + |\frac{g}{s}|\right)$, then $\frac{1}{1} < w \in U^+$, $w^{-1} < \frac{1}{1}$ and $w^{-1} \in U^+$. Now, it is enough to take $\frac{v}{1} = w^{-1}\frac{u}{2}$. To show that $S_m^{-1}C$ is Hausdorff, let $\frac{f}{r}, \frac{g}{s} \in S^{-1}C$ and $\frac{f}{r} \neq \frac{g}{s}$. Thus, $fs \neq rg$ on $\operatorname{coz} rs$ and so $\operatorname{coz} rs \subseteq \operatorname{coz}(fs - rg)$. Therefore, $\operatorname{coz} rs = \operatorname{coz} rs(fs - rg)$ and since S is an *m.c.* $\mathfrak{z}$-subset and $rs \in S$, we have $t := |rs(fs - rg)| \in S$. Now, it is not hard to see that $B(\frac{f}{r}, \frac{t}{2r^2s^2})$ and $B(\frac{g}{s}, \frac{t}{2r^2s^2})$ are disjoint. □

Corollary 3.4. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$. Then $S^{-1}C$ with the m-topology is a completely regular Hausdorff space.*

4. Connectedness of $S_m^{-1}C$

In this section, in imitate of [2], we first find the component of zero in $S_m^{-1}C$, where S is an *m.c.* $\mathfrak{z}$-subset. Next using this, we give a necessary and sufficient condition for connectedness of $S_m^{-1}C$.

Definition 4.1. A member $\frac{f}{r} \in S^{-1}C$ is called bounded if there exists $k \in \mathbb{N}$ such that $|\frac{f}{r}| \leq \frac{k}{1}$, i.e., $|f(x)| \leq k|r(x)|$ for all $x \in \operatorname{coz} t$ for some $t \in S$.

Clearly the set $(S^{-1}C)^*$ of all bounded elements of $S^{-1}C$ is a subring of $S^{-1}C$.

Lemma 4.2. *$(S^{-1}C)^*$ is a clopen subset of $S_m^{-1}C$.*

Proof. If $\frac{f}{r} \in (S^{-1}C)^*$, then $B(\frac{f}{r}, \frac{1}{1}) \subseteq (S^{-1}C)^*$. In fact, $|\frac{f}{r} - \frac{g}{s}| < \frac{1}{1}$ implies that $|\frac{g}{s}| < |\frac{f}{r}| + \frac{1}{1} \leq \frac{k}{1} + \frac{1}{1}$ for some $k \in \mathbb{N}$ and hence $\frac{g}{s}$ is bounded. On the other hand, if $\frac{f}{r} \notin (S^{-1}C)^*$, then $B(\frac{f}{r}, \frac{1}{1}) \cap (S^{-1}C)^* = \emptyset$. □

Lemma 4.3. *$J_\psi = \{\frac{f}{r} \in S^{-1}C : \frac{f}{r} \cdot \frac{s}{t}$ is bounded for each $\frac{s}{t} \in U^+\}$ is an ideal of $S^{-1}C$.*

Proof. It is not hard to see that J_ψ is closed with respect to addition. Let $\frac{f}{r} \in J_\psi$, $\frac{g}{s} \in S^{-1}C$ and $\frac{p}{q} \in U^+$. We claim that $\frac{fgp}{rsq}$ is bounded and so $\frac{fg}{rs} \in J_\psi$. Since $\frac{p}{q} \in U^+$, $0 < p$ on $\operatorname{coz} t$ for some $t \in S$ and so $0 < (1 + |g|)p$ on $\operatorname{coz} t$. Therefore, $\frac{(1+|g|)p}{sq} \in U^+$. Now by our hypothesis, $\frac{f}{r} \cdot \frac{(1+|g|)p}{sq}$ is bounded which implies that $\frac{f|g|p}{rsq} = \left(\frac{f}{r} \cdot \frac{(1+|g|)p}{sq}\right)\left(\frac{|g|}{1+|g|}\right)$ is bounded and $\frac{fgp}{rsq}$ is bounded as well. □

Using Lemmas 3.1 and 3.2 we have $J_\psi = \{\frac{f}{r} \in S^{-1}C : \frac{f}{t}$ is bounded, $\forall t \in U^+,\ 0 \leq t \leq 1\}$

Lemma 4.4. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$ and consider $\frac{f}{r} \in J_\psi$. The function $\varphi_{\frac{f}{r}} : \mathbb{R} \longrightarrow S^{-1}C$ defined by $\varphi_{\frac{f}{r}}(a) = \frac{af}{r}$ is continuous.*

Proof. Using Lemma 3.2, for every $a \in \mathbb{R}$ and $\frac{v}{1} \in U^+$, we must show that $\varphi_{\frac{f}{r}}^{-1}(B(\frac{af}{r}, \frac{v}{1}))$ contains a neighborhood of a in $\mathbb{R}$. Since $\frac{f}{r} \in J_\psi$, there exists $k \in \mathbb{N}$ such that $|\frac{f}{r}\frac{1}{v}| \leq \frac{k}{1}$. Now, we show that the interval $(a - \frac{1}{k}, a + \frac{1}{k})$ is contained in $\varphi_{\frac{f}{r}}^{-1}(B(\frac{af}{1}, \frac{v}{1}))$. In fact, $b \in (a - \frac{1}{k}, a + \frac{1}{k})$ implies that $\frac{|b-a|}{1}|\frac{f}{r}|\frac{1}{v} \leq \frac{1}{k} \cdot \frac{k}{1} = \frac{1}{1}$ and hence $|\frac{bf}{r} - \frac{af}{r}| < \frac{v}{1}$, i.e. $b \in \varphi_{\frac{f}{r}}^{-1}(B(\frac{af}{r}, \frac{v}{1}))$. □

The following theorem is in fact a generalization of Corollary 3.3 in [2].

Theorem 4.5. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$. The ideal J_ψ is the component of zero in $S_m^{-1}C$.*

Proof. First, since $\mathbb{R}$ is connected, using Lemma 4.4, $\varphi_{\frac{f}{r}}(\mathbb{R})$ is a connected set containing 0 for every $\frac{f}{r} \in J_\psi$. Therefore, $J_\psi = \cup_{\frac{f}{r}\in J_\psi} \varphi_{\frac{f}{r}}(\mathbb{R})$ is a connected set containing 0. Next, If I is the component of 0 in $S_m^{-1}C$, then $J_\psi \subseteq I$. Moreover, since $S_m^{-1}C$ is topological ring, I is an ideal of $S_m^{-1}C$. To complete the proof, it is enough to show that $I \subseteq J_\psi$. On the contrary, let $\frac{f}{r} \in I \backslash J_\psi$. By Lemma 4.3, there exists $\frac{s}{t} \in U^+$ such that $\frac{f}{r}\frac{s}{t} \notin (S^{-1}C)^*$. Consider the sets $I \cap (S^{-1}C)^*$ and $I \backslash (S^{-1}C)^*$. By Lemma 4.2, these two sets are open in I and since $0 \in I \cap (S^{-1}C)^*$ and $\frac{f}{r}\frac{s}{t} \in I \backslash (S^{-1}C)^*$, they are non-empty disjoint open subsets of the connected set I, a contradiction. □

Corollary 4.6. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$. $S_m^{-1}C$ is connected if and only if $S_m^{-1}C = J_\psi$, i.e., for every $f \in C(X)$ and each $r \in S$, there exist $k \in \mathbb{N}$ and $t \in S$ such that $|f(x)| \leq kr(x)$ for all $x \in \operatorname{coz} t$.*

Motivated by the previous corollary, we are going to investigate the connectedness of $S_m^{-1}C$ via topological properties of X for some particular *m.c.* $\mathfrak{z}$-subsets of X. For example, let $p \in \beta X$ and put $S_p = C(X) \backslash M^p$ or more generally, suppose that $A \subseteq \beta X$ and $S_A := C(X) \backslash \bigcup_{p\in A} M^p$. Clearly S_A is an *m.c.* $\mathfrak{z}$-subset of $C(X)$ and $S_A = \{f \in C(X) : p \notin cl_{\beta X}Z(f) \text{ for each } p \in A\} = \{f \in C(X) : A \cap cl_{\beta X}Z(f) = \emptyset\}$. Now, it is natural to ask the following questions.

When is the topological ring $(S_A)_m^{-1}C$ connected? what can we say about the connectedness of $(S_A)_m^{-1}C$ if we replace $\bigcup_{p\in A} M^p$ in S_A by an arbitrary union of family of particular prime ideals of C(X)? We will address such questions in the next section.

5. Connectedness of $S_A^{-1}C$ with the *m*-Topology

In this section, we study the connectedness of $S_m^{-1}C$, where $S = C(X) \backslash \bigcup_{\lambda\in\Lambda} P_\lambda$, and $\{P_\lambda\}_{\lambda\in\Lambda}$ is a family of prime z-ideals of $C(X)$. Using this, we conclude that $C(X)$ with the m-topology is connected if and only if X is pseudocompact. Also, It is shown that the classical ring of quotients of $C(X)$ endowed with the m-topology, is connected if and only if every dense cozero-set of $C(X)$ is pseudocompact.

We use the following lemma frequently. But, before that, we review some results which are needed in sequel. First, notice that for every $f \in C(X)$ we have

$$\operatorname{coz} f \subseteq \beta X \backslash \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} \operatorname{coz} f. \qquad (1)$$

The proof of the first inclusion is clear. To prove the second, let $x \notin \mathrm{cl}_{\beta X} Z(f)$. There exists an open neighborhood G of x in βX such that $G \subseteq \beta X \backslash Z(f)$. Now, for an arbitrary open subset H of βX containing x, we have

$$\emptyset \neq X \cap (G \cap H) \subseteq (X \cap H) \cap (\beta X \backslash Z(f)) = H \cap \operatorname{coz} f$$

which implies that $x \in \mathrm{cl}_{\beta X} \operatorname{coz} f$. Next, by part (1), we conclude that

$$\mathrm{cl}_{\beta X} \operatorname{coz} f = \mathrm{cl}_{\beta X}(\beta X \backslash \mathrm{cl}_{\beta X} Z(f)) = \beta X \backslash \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(f). \qquad (2)$$

Finally, if $f \in C^*(X)$, then $\operatorname{coz} f = X \cap \operatorname{coz} f^\beta$ and in this case, we have

$$\mathrm{cl}_{\beta X} \operatorname{coz} f = \mathrm{cl}_{\beta X} \operatorname{coz} f^\beta = \beta X \backslash \mathrm{int}_{\beta X} Z(f^\beta).$$

Using part (2), the next lemma is now evident.

Lemma 5.1. *Let $f \in C(X)$ and $p \in \beta X$. $f \notin O^p$ if and only if $p \in cl_{\beta X} \operatorname{coz} f$.*

Proposition 5.2. *Let S be an m.c. $\mathfrak{z}$-subset of $C(X)$ and consider $S_p := C(X) \backslash M^p$, for some $p \in \beta X \setminus \upsilon X$. If $S_p \subseteq S$ then $S_m^{-1}C$ is disconnected.*

Proof. Let $p \in \beta X \backslash \upsilon X$. By 8.7.(b) in [7], there exists $r \in C^*(X)$ such that $Z(r) = \emptyset$, while $r^\beta(p) = 0$. Since S is a $\mathfrak{z}$-subset and $Z(r) = Z(1)$, then $r \in S$ and so $\frac{1}{r} \in S^{-1}C$. To complete the proof, we claim that $\frac{1}{r}$ is unbounded on $\operatorname{coz} t$ for every $t \in S$. Let $\bar{S}$ be the saturation of S. Recall that $\bar{S} = C(X) \backslash \bigcup_{P \cap S = \emptyset} P$ where each P is a prime ideal of $C(X)$. Furthermore, for every prime ideal P which doesn't intersect S, we have $P \subseteq C(X) \backslash S \subseteq C(X) \backslash S_p = M^p$. Now, for every $t \in S$, $tr \in S \subseteq \bar{S}$ and so, there exists a prime ideal $P_\circ \subseteq M^p$ such that $tr \notin P_\circ$ and consequently $tr \notin O^p$. Thus by Lemma 5.1, $p \in \operatorname{cl}_{\beta X} \operatorname{coz} tr$ and hence there exists a net $\{x_\lambda\}$ contained in $\operatorname{coz} tr = \operatorname{coz} t \cap \operatorname{coz} r$ which converges to p. Since r^β is continuous, $r^\beta(x_\lambda) \to r^\beta(p) = 0$ and this implies that the function r converges to zero on $\operatorname{coz} tr \subseteq \operatorname{coz} r$ and so the fraction $\frac{1}{r} \in S^{-1}C$ is not bounded on $\operatorname{coz} tr \subseteq \operatorname{coz} r$. Therefore, the claim is true and so $S_m^{-1}C$ is disconnected. □

Proposition 5.3. *Let P be a prime z-ideal of $C(X)$ and suppose that $S = C(X) \backslash P$. The topological ring $S_m^{-1}C$ is connected if and only if P is a real maximal ideal.*

Proof. We first prove the necessity. By contrary, assume that P is not real maximal ideal. Now, using 7.15 in [7], let $p \in \beta X$ and M^p be the unique maximal ideal of $C(X)$ containing P. We consider two cases:

Case 1. $p \in \beta X \backslash \upsilon X$. In this case, Proposition 5.2 implies that $S_m^{-1}C$ is disconnected, a contradiction.

Case 2. let $p \in \upsilon X$. In this case, using 7.9.(c) in [7], we have $M^p \cap C^*(X) = M^{*p}$. On the other hand, $P \subsetneq M^p$ by our assumption. Then there exists a function $r \in C^*(X)$ such that $r \in M^p \backslash P$ and so $\frac{1}{r} \in S^{-1}C$. Moreover, $r \in M^p \cap C^*(X)$ implies $r^\beta(p) = 0$. Now, for every $t \in S$, $tr \notin P$ which shows that $tr \notin O^p$ and hence $p \in \operatorname{cl}_{\beta X} \operatorname{coz} tr$, by Lemma 5.1. Finally, similar to the proof of the Proposition 5.2, we conclude that $\frac{1}{r}$ is unbounded on $\operatorname{coz} tr \subseteq \operatorname{coz} t$ and consequently using the Corollary 4.6, $S_m^{-1}C$ is not connected, a contradiction.

Next, to prove the sufficiency, let $p \in \beta X$, M^p be a real maximal ideal of $C(X)$ and $S = C(X) \backslash M^p$. suppose that $\frac{f}{r} \in S^{-1}C$. By Lemma 3.1, we can assume that $f, r \in C^*(X)$. Since $r \notin M^p$ and M^p is real, we have $r \notin M^{*P}$, by 7.9.(c) in [7], and hence $r^\beta(p) \neq 0$. Moreover, for every $f \in C(X)$, $f^\beta(p)$ does not approach to infinity. Now, consider the open subset $H = \{x \in \operatorname{coz} r^\beta : |\frac{f^\beta}{r^\beta}(x) - \frac{f^\beta}{r^\beta}(p)| < 1\}$ of $\operatorname{coz} r^\beta \subseteq \beta X$. We observe that H is an open neighborhood of p in βX and since $\{\operatorname{coz} t^\beta : t \in C^*(X)\}$ is a base for the space βX, there exists $t \in C^*(X)$ such that $p \in \operatorname{coz} t^\beta \subseteq H$. Thus, for every $x \in \operatorname{coz} t^\beta$, $|\frac{f^\beta}{r^\beta}(x)| < |\frac{f^\beta}{r^\beta}(p)| + 1$ and hence for every $x \in X \cap \operatorname{coz} t^\beta = \operatorname{coz} t$, we have $|\frac{f}{r}(x)| < |\frac{f^\beta}{r^\beta}(p)| + 1$ which implies that $\frac{f}{r}$ is bounded on $\operatorname{coz} t$. Therefore, by Corollary 4.6, $S^{-1}C$ with the m-topology, i.e., $S_m^{-1}C$ is connected. □

The following result is an immediate consequence of the previous proposition.

Corollary 5.4. *Let $p \in \beta X$. $S_p^{-1}C$ with the m-topology is connected if and only if $p \in \upsilon X$.*

By 8A.4 in [7], $\upsilon X = \beta X$ if and only if X is pseudocompact. Using this and corollary 5.4 the following result is now evident.

Corollary 5.5. *$S_p^{-1}C$ with the m-topology is connected for every $p \in \beta X$ if and only if X is pseudocompact.*

Recall that whenever $\bar{S}$ is the saturation of an *m.c.* subset S of $C(X)$, then two rings $S^{-1}C$ and $(\bar{S})^{-1}C$ are isomorphic. By Corollary 2.9, the saturation of every *m.c.* $\mathfrak{z}$-subset S of $C(X)$ is a $\mathfrak{z}$-subset. If we consider $S = C(X) \backslash \bigcup_{\lambda \in \Lambda} P_\lambda$ where $\{P_\lambda\}_{\lambda \in \Lambda}$ is a family of prime ideals of $C(X)$, then for every $\lambda \in \Lambda$, we have $P_\lambda \cap S = \emptyset$ and conversely, for each prime ideal P disjoint from S there exists $\lambda \in \Lambda$ such that $P = P_\lambda$.

Definition 5.6. An ideal I of $C(X)$ is called real whenever every maximal ideal containing I, is real.

As 7O in [7], for an ideal I in $C(X)$ if we define $\theta(I) = \{p \in \beta X : I \subseteq M^P\}$, then $\theta(I) = \bigcap_{f \in I} \operatorname{cl}_{\beta X} Z(f)$. Thus, an ideal of $C(X)$ is real ideal if and only if $\theta(I) \subseteq \upsilon X$ or equivalently $\bigcap_{f \in I} \operatorname{cl}_{\beta X} Z(f) \subseteq \upsilon X$.

Proposition 5.7. *Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of prime z-ideals of $C(X)$ and take $S := C(X) \backslash \bigcup_{\lambda \in \Lambda} P_\lambda$. Then S is an m.c. $\mathfrak{z}$-subset of $C(X)$ and if $S_m^{-1}C$ is connected, then for every $\lambda \in \Lambda$ the ideal P_λ is real. Moreover, $\bigcup_{\lambda \in \Lambda} P_\lambda = \bigcup_{p \in A} M^p$ where $A = \bigcup_{\lambda \in \Lambda} \theta(P_\lambda)$.*

Proof. By contrary, suppose that $S_m^{-1}C$ is connected but at least one of the prime ideals is not real. Thus, there exists $\lambda_\circ \in \Lambda$ and $p \in \beta X \backslash \upsilon X$ such that $P_{\lambda_\circ} \subseteq M^p$. Since $p \notin \upsilon X$, there is a function $r \in C^*(X)$ such that $Z(r) = \emptyset$ and $r^\beta(p) = 0$. Now, similar to the proof of Proposition 5.2, we conclude that $\frac{1}{r} \in S^{-1}C$ and for every $t \in S$ we have $tr \notin O^p$, since $tr \notin P_{\lambda_\circ}$. So by Lemma 5.1, $p \in \mathrm{cl}_{\beta X}\mathrm{coz}\, tr$. Therefore, $\frac{1}{r}$ is not bounded on $\mathrm{coz}\, t$ and thus $S_m^{-1}C$ is not connected, by Corollary 4.6, a contradiction.

To prove the last part of the proposition, by contrary, let $r \in \bigcup_{p\in A} M^p \setminus \bigcup_{\lambda\in\Lambda} P_\lambda$. As above, for every $t \in S$ it can be shown that $\frac{1}{r}$ is unbounded on $\mathrm{coz}\, t$ and so $S_m^{-1}C$ is disconnected, a contradiction. □

Corollary 5.8. *Let $p \in \beta X$ and $\{P_\lambda^p\}_{\lambda\in\Lambda}$ be a family of prime z-ideals of $C(X)$ contained in the maximal ideal M^p and suppose that $S = C(X) \setminus \bigcup_{\lambda\in\Lambda} P_\lambda^p$. Then $S_m^{-1}C$ is connected if and only if $p \in \upsilon X$ and $M^p = \bigcup_{\lambda\in\Lambda} P_\lambda^p$.*

Corollary 5.9. *Let $A \subseteq \beta X$ and suppose that $S_A = C(X) \setminus \bigcup_{p\in A} M^p$. If $S^{-1}C$ with the m-topology is connected, then $A \subseteq \upsilon X$.*

The following theorem which is in fact a generalization of Corollary 5.4, shows that whenever A is a compact subset of βX, the converse of the previous corollary is also true. But, we were unable to answer the converse of the corollary.

Theorem 5.10. *Let A be a compact subset of βX and consider $S_A = C(X) \setminus \bigcup_{p\in A} M^p$. Then $S_A^{-1}C$ with the m-topology is connected if and only if $A \subseteq \upsilon X$.*

Proof. Necessity is clear by Corollary 5.9. To prove the sufficiency, let A be a compact subset of υX. Using Corollary 4.6, it is enough to show that for every $\frac{f}{r} \in S_A^{-1}C$, there exists $t \in S_A$ such that $\frac{f}{r}$ is bounded on $\mathrm{coz}\, t$. Since $r \in S_A$, then for every $p \in A \subseteq \upsilon X, r \notin M^p \cap C^*(X) = M^{*p}$ and so $r^\beta(p) \neq 0$. Moreover, $p \in \upsilon X$ implies that $f^\beta(p) \neq \infty$ and thus for each $p \in A$, $\frac{f^\beta}{r^\beta}(p)$ is a real number. As in the proof of Proposition 5.3, the subset $H = \{x \in \mathrm{coz}\, r^\beta : |\frac{f^\beta}{r^\beta}(x) - \frac{f^\beta}{r^\beta}(p) < 1\}$ is an open neighborhood of p in $\mathrm{coz}\, r^\beta$ and hence in βX as well. Thus, there exists $t \in C^*(X)$ such that $p \in \mathrm{coz}\, t_p^\beta \subseteq H \subseteq \mathrm{coz}\, r^\beta$ and so we conclude that $\frac{f}{r}$ is bounded on $\mathrm{coz}\, t_p$. In fact, for every $x \in \mathrm{coz}\, t_p$ we have $|\frac{f}{r}(x)| < |\frac{f^\beta}{r^\beta}(p)| + 1$. Now, since A is compact and $A \subseteq \bigcup_{p\in A} \mathrm{coz}\, t_p^\beta$, there are functions $t_{p_1}, ..., t_{p_n}$ in $C^*(X)$ such that $A \subseteq \bigcup_{i=1}^n \mathrm{coz}\, t_{p_i}^\beta$. We claim that $t = t_{p_1}^2 + ... + t_{p_n}^2$ is the function which we look for.

First, note that for every $p \in A$ we have $t \notin M^p$. Otherwise, if for some $q \in A$ we have $t \in M^q$, then $Z(t) \subseteq Z(t_{p_i})$ $(1 \leq i \leq n)$ implies that $t_{p_i} \in M^q$ for every $1 \leq i \leq n$, (since M^q is a z-ideal) which contradicts $q \in A \subseteq \bigcup_{i=1}^n \mathrm{coz}\, t_{p_i}^\beta$.

Next, because $\frac{f}{r}$ is bounded on every $\mathrm{coz}\, t_{p_i}$ $(1 \leq i \leq n)$, it is bounded on $\mathrm{coz}\, t = \mathrm{coz}\,(t_{p_1}^2 + ... + t_{p_n}^2) = \bigcup_{i=1}^n \mathrm{coz}\, t_{p_i}$ too, which completes the proof. □

Whenever a subset A of X is completely separated from every zero-set disjoint from it, in particular, if A is a zero-set or a C-embedded subset of X, then for every $f \in C(X), \mathrm{cl}_{\beta X}A \cap \mathrm{cl}_{\beta X}Z(f) = \emptyset$ if and only if $A \cap \mathrm{cl}_{\beta X}Z(f) = \emptyset$, see Theorems 1.18 and 6.5 in [7]. Therefore, $S_A = S_{\mathrm{cl}_{\beta X}A}$ and since $\mathrm{cl}_{\beta X}A$ is a compact subset of βX, the following result is now evident by Theorem 5.10.

Corollary 5.11. *Let a subset $A \subseteq X$ be completely separated from every zero-set disjoint from it. Then $S_A^{-1}C$ with the m-topology is connected if and only if $\mathrm{cl}_{\beta X}A \subseteq \upsilon X$.*

If we consider $S = C(X) \setminus \bigcup_{p\in\beta X} M^p$, then S is the set of all units of $C(X)$ and so $S_m^{-1}C = C_m(X)$. Therefore, by Theorem 5.10, $C_m(X)$ is connected if and only if $\beta X \subseteq \upsilon X$. Now, using 8A.4 in [7], the following results is evident.

Corollary 5.12. ([2, Proposition 3.12]) *$C(X)$ with the m-topology is connected if and only if X is pseudocompact.*

Using Proposition 5.7 and Theorem 5.10, we conclude the paper by another proof for Corollary 3.11 in [3]. First, we recall that a point $p \in X$ is called an almost P-point, if every G_δ-set (zero-set) containing p has nonempty interior and a space X is called an almost P-space if each point of X is an almost P-point. Thus, X is an almost P-space if and only if every non-zero-divisor of $C(X)$ is unit, i.e., $U(X) = C(X)\setminus Zd(X) = C(X)\setminus \bigcup P$, where P is a prime ideal of $C(X)$ contained in $Zd(X)$. It is proved that $p \in X$ is an almost P-point if and only if whenever $f \in C(X)$ and $p \in Z(f)$ imply that $p \in \mathrm{cl}_X\mathrm{int}_X Z(f)$. In fact, if p is an almost P-point, then $M_p \subseteq Zd(X)$ and thus, for every $f \in C(X)$ if $p \in Z(f)$, then the ideal (O_p, f) generated by $O_p \cup \{f\}$, is contained in M_p. Now, using Corollary 3.3 in [4] we conclude that $p \in \mathrm{cl}_X\mathrm{int}_X Z(f)$. See [10] for more information about almost P-spaces.

Corollary 5.13. ([3, Corollary 3.11]) *The classical ring of quotients of* $C(X)$ *with the m-topology is connected if and only if X is a pseudocompact almost P-space.*

Proof. Let $S^{-1}C$ be the classical ring of quotients of $C(X)$ and for every $p \in \beta X$, suppose that $\{P^p_\lambda\}_{\lambda\in\Lambda_p}$ is the family of all prime ideals of $C(X)$ contained in $M^p \bigcap Zd(X)$. It is not hard to see that $Zd(X) = \bigcup_{\substack{\lambda\in\Lambda_p \\ p\in\beta X}} P^p_\lambda$ and so, $S = C(X)\setminus \bigcup_{\substack{\lambda\in\Lambda_p \\ p\in\beta X}} P^p_\lambda$. Now, Using Proposition 5.7, if $S^{-1}_m C$ is connected, then every P^p_λ is real ideal which implies that $\beta X \subseteq \upsilon X$, i.e., X is pseudocompact. On the other hand, since for every $\lambda \in \Lambda_p$ we have $\theta(P^p_\lambda) = \{p\}$, using the same proposition, we conclude that $Zd(X) = \bigcup_{p\in\beta X} M^p$. Thus, each non-unit of $C(X)$ is zero-divisor and this means that X is almost p-space.

Conversely, let X be pseudocompact almost P-space. Since X is an almost P-space, $Zd(X) = \bigcup_{p\in\beta X} M^p$ and by pseudocompactness of X we conclude that $\beta X = \upsilon X$. Now, by Theorem 5.10, $S^{-1}_m C$ is connected for $S = C(X)\setminus Zd(X)$. □

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