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Modular Best Proximity and Equilibrium Pairs in Free Generalized Games

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Abstract. In this paper, we prove a fixed point theorem for ρ-acyclic factorizable multifunction. Some existence theorems of general best proximity pairs and equilibrium pairs are presented in modular function spaces. Moreover, some equilibrium theorems are established for free generalized n-person game.

1. Introduction

The theory of mappings defined on convex subsets of modular function spaces generalized by Khamsi et al. (see e.g. [4–6]).

We need the following definitions in sequel, from [7, 9]:
Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a σ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$, we denote the linear space of all simple functions with supports in $\mathcal{P}$. By $\mathcal{M}_\infty$, we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, +\infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(w) \to f(w)$ for all $w \in \Omega$. By 1_A, we denote the characteristic function of the set A.

Definition 1.1. *Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*

(i) $\rho(0) = 0$;

(ii) *ρ is monotone, i.e. $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;*

(iii) *ρ is orthogonally subadditive, i.e. $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;*

(iv) *ρ has the Fatou property, i.e. $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;*

(v) *ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(w)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.*

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We say that $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. A property holds ρ-almost everywhere if the exceptional set is ρ-null, we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(w)| < \infty \ \rho - a.e.\}.$$

We will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 1.2. *Let ρ be a regular convex function pseudomodular. We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.*

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\mathfrak{R}$.

Definition 1.3. *Let ρ be a convex function modular. A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by*

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

The formula

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}.$$

defines a norm in L_ρ which is frequently called the Luxemburg norm.

Definition 1.4. *Let $\rho \in \mathfrak{R}$.*

(i) We say $\{f_n\}$ is ρ-convergent to f and write $f_n \to f$ (ρ) if and only if $\rho(f_n - f) \to 0$.

(ii) A subset $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f$ (ρ) implies that f belong to B.

(iii) A nonempty subset K of L_ρ is said to be ρ-compact if for any family $\{A_\alpha; A_\alpha \in 2^{L_\rho}, \alpha \in \Gamma\}$ of ρ-closed subsets with $K \cap A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \emptyset$, for any $\alpha_1, \cdots, \alpha_n \in \Gamma$, we have

$$K \cap (\bigcap_{\alpha \in \Gamma} A_\alpha) \neq \emptyset.$$

Let $\rho \in \mathfrak{R}$. We have $\rho(f) \leq \liminf \rho(f_n)$, whenever $f_n \to f$ $\rho - a.e.$ This property is equivalent to the Fatou property [7, Theorem 2.1].

Definition 1.5. *Let $\rho \in \mathfrak{R}$ and let C be nonempty ρ-closed subset of L_ρ. Let $T : G \to L_\rho$ be a map. T is called ρ-continuous if $\{T(f_n)\}$ ρ-converges to $T(f)$ whenever $\{f_n\}$ ρ-converges to f. Also T will be called strongly ρ-continuous if T is ρ-continuous and*

$$\liminf_{n \to \infty} \rho(g - T(f_n)) = \rho(g - T(f)),$$

for any sequence $\{f_n\} \subset C$ which ρ-converges to f and for any $g \in C$.

Definition 1.6. *Let $X, Y \subseteq L_\rho$. A map $F : X \to 2^Y$ is said to be ρ-upper semi continuous if for each ρ-closed set $B \subseteq Y$, $F^-(B)$ is ρ-closed in X.*

Recal the following definitions of proximity concepts. Let X, and Y be any two nonempty ρ-closed subsets of L_ρ, and $\rho \in \mathfrak{R}$. For $f \in X$, define $d_\rho(f, Y) = \inf\{\|f - g\|_\rho : g \in Y\}$, and

$$d_\rho(X, Y) = \inf\{\|f - g\|_\rho : f \in X, g \in Y\},$$

If $X = \{f\}$ and $Y = \{g\}$, then $\|f - g\|_\rho$ denotes $d_\rho(X, Y)$ which is precisely $\|f - g\|_\rho$.

Let I be a finite or an infinite index set. For each $i \in I$, let X and Y_i be non-empty ρ-closed subsets of L_ρ. Then we can use the following notations: for each $i \in I$,

$$d_\rho(f, Y_i) = \inf\{\|f - g\|_\rho : g \in Y_i\},$$

$$X^o := \{f \in X \,|\, \text{for each } i \in I, \exists g_i \in Y_i \text{ such that } \|f - g_i\|_\rho = d_\rho(X, Y_i)\}$$

$$Y_i^o := \{g \in Y_i \,|\, \exists f \in X \text{ such that } \|f - g\|_\rho = d_\rho(X, Y_i)\}$$

Let X, and Y be any two nonempty ρ-closed subsets of L_ρ and $T : X \to 2^Y$ be a multifunction. Then the pair $(\bar{f}, T(\bar{f}))$ is called the best proximity pair for T if $d_\rho(\bar{f}, T(\bar{f})) = \|\bar{f} - \bar{g}\|_\rho = d_\rho(X, Y)$, for some $g \in T(\bar{f})$.

If $X \in L_\rho$ is a nonempty ρ-closed, convex and ρ-compact, then the set

$$P_X(f) = \{g \in X : \|f - g\|_\rho = d_\rho(f, X)\},$$

of all ρ-best approximations in X to any element $f \in X$ is a nonempty ρ-closed, convex and ρ-compact subset of X and every point in $P_X(f)$ is called a best proximity point of f in X. Also, any point $f \in X$ for which $d_\rho(f, Y) = d_\rho(X, Y)$ is called a best proximity point of Y in X and the points $f \in X$, $g \in Y$ satisfying $\|f - g\|_\rho = d_\rho(X, Y)$ are called best proximity points of the pair (X, Y).

Definition 1.7. *Let C be a nonempty, convex subsets of L_ρ. A single value function $h : C \to L_\rho$ is said to be quasi ρ-affine if for each real number $r \geqslant 0$ and $f \in L_\rho$ the set $\{g \in C \,|\, \|h(g) - f\|_\rho \leqslant r\}$ to be convex.*

A nonempty topological space is called acyclic if all its reduced Čach homology groups over rationals vanish.

Definition 1.8. *Let $X, Y \subset L_\rho$. A multifunction $T : X \to 2^Y$ is said to be ρ-acyclic multifunction if is ρ-upper semi continuous and $T(x)$ is a nonempty ρ-compact and acyclic subset of Y.*

The collection of all ρ-acyclic multifunctions from X to Y is denoted by $\mathbb{V}(X, Y)$. A multifunction $T : X \to 2^Y$ is said to be a ρ-acyclic factorizable multifunction if it can be expressed as a composition of finitely many acyclic multifunction. The collection of all ρ-acyclic factorizable multifunctions from X to Y is denoted by $\mathbb{V}_C(X, Y)$.

Now we recall the following equilibrium pair concept of [8]. Let I be a finite or an infinite set of locations or agents. For each $i \in I$, let X_i nonempty set of manufacturing commodities and Y_i be a nonempty set of selling commodities. A free generalized game or free abstract economy $\Gamma = (X_i, Y_i, A_i, P_i)_{i\in I}$ is defined as a family of ordered quadruples, where X_i and Y_i are nonempty subsets of L_ρ, $A_i : X = \prod_{j\in I} X_j \to 2^{Y_i}$ is a constraint correspondence and $P_i : Y = \prod_{j\in I} Y_j \to 2^{Y_i}$ is a preference correspondence. An equilibrium pair for Γ is a pair of points $(\bar{f}, \bar{g}) = ((\bar{f}_i)_{i\in I}, (\bar{g}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$

$$\bar{g}_i \in A_i(\bar{f}) \text{ with } \|\bar{f}_i - \bar{g}_i\|_\rho = d_\rho(X_i, Y_i) \text{ and } A_i(\bar{f}) \cap P_i(\bar{g}) = \emptyset.$$

In particular, when $I = \{1, \cdots, n\}$, we may call Γ a free n-person game.
When $X_i = Y_i$ for each $i \in I$, then the previous definitions can be reduced to the standard definitions of equilibrium theory in mathematical economics.

Section 2 is devoted to fixed point theorem and some existence theorems for best proximity pairs in modular function spaces. In the last section, some equilibrium theorems are proved for free n-person game.

2. General Best Proximity Pairs

Here, first we established the following existence theorem of general best proximity.

Theorem 2.1. *For each $I = \{1, \cdots, n\}$, let X and Y_i be nonempty ρ-compact and convex subsets of L_ρ, and let $T_i : X \to 2^{Y_i}$ be a ρ-upper semi continuous multifunction in X^o such that $T_i(f)$ is a nonempty ρ-closed and convex subset of Y_i for each $f \in X$. Assume that $T_i(f) \cap Y_i^o \neq \emptyset$ for each $f \in X^o$. Then there exists a system of best proximity pairs $\{\bar{f}_i\} \times T_i(f_i) \subseteq X \times Y_i$, i.e., for each $i \in I$, $d_\rho(\bar{f}_i, T(\bar{f}_i)) = d_\rho(X, Y_i)$.*

Proof. Let $f_1, f_2 \in X^o$ be arbitrary. Then, for each $i \in I$, there exist $g_i^1, g_i^2 \in Y_i$ such that $\|f_i - g_i^j\|_\rho = d_\rho(X, Y_i)$ for each $j = 1, 2$. For any $\lambda \in (0, 1)$, we let $\hat{f} = \lambda f_1 + (1-\lambda) f_2$ and $\hat{g}_i = \lambda g_i^1 + (1-\lambda) g_i^2$. Since Y_i is convex, $\hat{g}_i \in Y_i$. Then we have

$$\begin{aligned}
\|\hat{f} - \hat{g}_i\|_\rho &= \|(\lambda f_1 + (1-\lambda) f_2) - (\lambda g_i^1 + (1-\lambda) g_i^2)\|_\rho \\
&= \|\lambda(f_1 - g_i^1) + (1-\lambda)(f_2 - g_i^2)\|_\rho \\
&\leq \lambda \|f_1 - g_i^1\|_\rho + (1-\lambda)\|f_2 - g_i^2\|_\rho \\
&= \lambda d_\rho(X, Y_i) + (1-\lambda) d_\rho(X, Y_i) \\
&= d_\rho(X, Y_i),
\end{aligned}$$

so $\hat{f} \in X^o$. Hence X^o is convex. Similarly, the convexity for Y_i^o can be proved. Now we show that X^o is a ρ-closed subset of X. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in X^o, which converges to $\tilde{f} \in X$. If $i \in I$ be fixed and $k_i = d_\rho(X, Y_i) = \inf\{\|f - g_i\|_\rho : f \in X, g_i \in Y_i\}$ then, for each $n \in \mathbb{N}$ there exists $g_n^i \in Y_i$ such that $\|f_n - g_n^i\|_\rho = k_i$. Since Y_i is compact, there exists a convergent subsequence $(g_{n_k}^i)$ of (g_n^i) which converges to $\tilde{g}_i \in Y_i$. Also

$$\begin{aligned}
k_i \leq \|\tilde{f} - \tilde{g}_i\|_\rho &= \|\tilde{f} - f_{n_k} + f_{n_k} - g_{n_k}^i + g_{n_k}^i - \tilde{g}_i\|_\rho \\
&\leq \|\tilde{f} - f_{n_k}\|_\rho + \|f_{n_k} - g_{n_k}^i\|_\rho + \|g_{n_k}^i - \tilde{g}_i\|_\rho.
\end{aligned}$$

Since $\|\tilde{f} - f_{n_k}\|_\rho \to 0$, $\|g_{n_k}^i - \tilde{g}_i\|_\rho \to 0$ and $\|f_{n_k} - g_{n_k}^i\|_\rho = k_i$ we have $k_i = \|\tilde{f} - \tilde{g}_i\|_\rho$ so that $\tilde{f} \in X^o$. Therefore X^o is ρ-closed. Similarly the closedness of Y_i^o can be shown. Also, the metric projection map $P_X : l_\rho \to 2^X$ is upper semicontinuous in L_ρ such that $P_X(z)$ is a nonempty ρ-compact convex subset of X for each $h \in L_\rho$. For each $i \in I$, we define a multifunction $T_i' : X^o \to 2^{Y^o}$ by

$$T_i'(f) := T_i(f) \cap Y_i^o \qquad \text{for each} \quad f \in X^o.$$

Then, by assumptions, each $T_i'(f)$ is a nonempty and ρ-compact convex in Y_i^o. Also, T_i' is ρ-upper semi continuous in X^o. Next, we claim that if $g \in Y_i^o$, then $P_X(g)$ is a nonempty subset of X^o. In fact, if $g \in Y_i^o$, then there exists $f_i \in X$ such that $\|f_i - g\|_\rho = d_\rho(X, Y_i)$. Let $f \in P_X(g)$ be arbitrary. Thus $\|g - f\|_\rho = d_\rho(g, X) \leq \|f_i - g\|_\rho = d_\rho(X, Y_i)$ so that $\|g - f\|_\rho = d_\rho(X, Y_i)$ for each $i \in I$ and hence $f \in X^o$. That is, $P_X(Y_i^o) \subseteq X^o$. Now we define the following multifunctions $T' : \prod_{i \in I} X^o \to 2^{\prod_{i \in I} Y_i^o}$ by

$$T'(f_1, \cdots f_n) = \prod_{i \in I} T_i'(f_i) \qquad \text{for each} \quad (f_1, \cdots f_n) \in \prod_{i \in I} X^o;$$

and $P_X' : \prod_{i \in I} Y_i^o :\to 2^{\prod_{i \in I} X^o}$ by

$$P_X'(g_1, \cdots g_n) = \prod_{i \in I} P_X(g_i) \qquad \text{for each} \quad (g_1, \cdots g_n) \in \prod_{i \in I} Y_i^o.$$

Then T' and P_X' are both ρ-upper semi continuous such that each $T'(f_1, \cdots f_n)$ is nonempty ρ-compact and convex in $\prod_{i \in I} Y_i^o$, and each $P'(g_1, \cdots g_n)$ is nonempty ρ-compact and convex in $\prod_{i \in I} X^o$. Hence, T' and P_X' are Kakutani multifunctions so that the composition map $P_X' \circ T' : \prod_{i \in I} X^o \to 2^{\prod_{i \in I} X^o}$ is a Kakutani factorizable multifunction. Therefore there exists a fixed point $\bar{f} = (\bar{f}_1 \cdots, \bar{f}_n) \in \prod_{i\in I} X^o$ such that $\bar{f} \in (P'_X \circ T')(\bar{f})$. Then $(\bar{f}_1 \cdots, \bar{f}_n) \in P'_X(T'(\bar{f}_1 \cdots, \bar{f}_n))$ so that there exists a $(\bar{g}_1 \cdots, \bar{g}_n) \in \prod_{i\in I} Y_i^o$ such that $(\bar{g}_1 \cdots, \bar{g}_n) \in T'(\bar{f}_1 \cdots, \bar{f}_n) = \prod_{i\in I}(T_i(\bar{f}_i) \cap Y_i^o)$ and $\bar{f}_1 \in P_X(\bar{g}_1), \cdots \bar{f}_n \in P_X(\bar{g}_n)$. Since each $\bar{g}_i$ is an element in Y_i^o, there exists an $f'_i \in X$ such that $\|f'_i - \bar{g}_i\|_\rho = d_\rho(X, Y_i)$ for each $i \in I$. Therefore, for each $i \in I$, we have

$$d_\rho(\bar{f}_i, T_i(\bar{f}_i)) \leqslant \|\bar{f}_i - \bar{g}_i\|_\rho = d_\rho(\bar{g}_i, X) \leqslant \|\bar{g}_i - f'_i\|_\rho = d_\rho(X, Y_i),$$

so that $d_\rho(\bar{f}_i, T_i(\bar{f}_i)) = d_\rho(X, Y_i)$, which is completes the proof. □

Theorem 2.2. *Let X be a nonempty ρ-compact convex subset of L_ρ, then any ρ-acyclic factorizable multifunction $T : X \to 2^X$ has a fixed point,i.e., if $T \in \mathbb{V}_C(X, X)$, then there exists a point $\hat{f} \in X$ such that $\hat{f} \in T(\hat{f})$.*

Proof. Since T is ρ-upper semi continuous for each neighborhood V of 0 in L_ρ, there exist $f_V, g_V \in X$ such that $g_V \in T(f_V)$ and $f_V, g_V \in X$ and $g_V - f_V \in V$. But $T(X)$ is relatively ρ-compact, we may assume that g_V converges to some $\hat{f}$. Since the graph of T is ρ-closed in $X \times \overline{T(X)}$, we have $\hat{f} \in T(\hat{f})$. □

Using Theorem 2.2, we obtain the following existence theorem for general best proximity pairs.

Theorem 2.3. *For each $I = \{1, \cdots, n\}$, let X and Y_i be nonempty ρ-compact and convex subsets of L_ρ and X^o is a nonempty subset of X. Let $T_i : X \to 2^{Y_i}$ be a ρ-upper semi continuous multifunction in X^o such that $T_i(f)$ is a nonempty ρ-compact and $T_i(f) \cap Y_i^o$ is a ρ-acyclic subset of Y_i and let $g : X^o \to X^o$ be a ρ-continuous, proper, quasi ρ-affine, and surjective mapping on X^o. Assume that for each $f \in X^o$, there exists $(g_1, \cdots g_n) \in \prod_{i\in I} T_i(f)$ such that*

$$\exists f_0 \in X \quad with \quad \|f_0 - g_i\|_\rho = d_\rho(X, Y_i) \quad for\ each \quad i \in I, \qquad (*)$$

and $\cap_{i\in I} P_X(g_i)$ is nonempty for each $(g_1, \cdots g_n) \in \prod_{i\in I} Y_i^o$. Then there exists a point $\bar{f} \in X$ satisfying the system of best proximity pairs, i.e., for each $i \in I$, $\{g(\bar{f})\} \times T_i(\bar{f}) \subseteq X \times Y_i$ such that $d_\rho(g(\bar{f}), T_i(\bar{f}) = d_\rho(X, Y_i)$.

Proof. As shown in the proof of Theorem 2.1, we can see that X^o and Y_i^o are nonempty ρ-compact and convex. Also since X is nonempty ρ-compact and convex, it is known that the metric projection map $P_X : L_\rho \to 2^X$ is ρ-upper semi continuous in L_ρ such that $P_X(h)$ is a nonempty ρ-compact and convex subset of X for each $h \in L_\rho$. Now we define a multifunction $T'_i : X^o \to 2^{Y^o}$ by

$$T'_i(f) := T_i(f) \cap Y_i^o \qquad \text{for each} \quad f \in X^o.$$

Then, by assumption, T'_i is ρ-upper semi continuous in X^o such that each $T'_i(f)$ is a nonempty ρ-compact and ρ-acyclic subset in Y_i^o. Also $P_X(Y_i^o) \subseteq X^o$ as in the proof of Theorem 2.1. Now we introduce the multifunctions $T' : X^o \to 2^{\prod_{i\in I} Y_i^o}$ by

$$T'(f) := \prod_{i\in I} T'_i(f) \qquad \text{for each} \quad f \in X^o,$$

and $P'_X : \prod_{i\in I} Y_i^o :\to 2^{X^o}$ by

$$P'_X(g_1, \cdots g_n) = \cap_{i\in I} P_X(g_i) \quad \text{for each} \quad (g_1, \cdots g_n) \in \prod_{i\in I} Y_i^o.$$

Then T' is ρ-upper semi continuous in X^o such that each $T'(f)$ is a nonempty ρ-compact and ρ-acyclic subset in $\prod_{i\in I} Y_i^o$. By assumption (∗) each $P'_X(g_1, \cdots, g_n)$ is a nonempty ρ-closed convex subset in X^o and we can see that the multifunction $g^{-1} \circ P'_X : \prod_{i\in I} Y_i^o \to 2^{X^o}$ is a ρ-acyclic multifunction. Therefor the composition map $(g^{-1} \circ P'_X) \circ T' : X^o \to X^o$ is a ρ-acyclic factorizable multifunction in X^o. Therefore by Theorem 2.2 there exists a fixed point $\bar{f} \in X^o$ such that $\bar{f} \in ((g^{-1} \circ P'_X) \circ T')(\bar{f})$, that is, $g(\bar{x}) \in (P'_X \circ T')(\bar{f})$. Then

$g(\bar{f}) \in P'_X(T'_1(\bar{f}), \cdots, T'_n(\bar{f}))$ so that there exists $(\bar{g}_1, \cdots, \bar{g}_n) \in \prod_{i\in I}(T_i(\bar{f}) \cap Y_i^o)$ such that $g(\bar{f}) \in P'_X(\bar{g}_1, \cdots, \bar{g}_n) = \cap_{i\in I} P_X(\bar{g}_i)$. Since each $\bar{g}_i$ is an element in Y_i^o, there exists an $f'_i \in X$ such that $\|f'_i - \bar{g}_i\|_\rho = d_\rho(X, Y_i)$ for each $i \in I$. Therefore, for each $i \in I$,

$$d_\rho(g(\bar{f}), T_i(\bar{f})) \leq d_\rho(g(\bar{f}), \bar{g}_i) = d_\rho(\bar{g}_i, X) \leq d_\rho(\bar{g}_i, f'_i) = d_\rho(X, Y_i)$$

so that $d_\rho(g(\bar{f}), T_i(\bar{f})) = d_\rho(X, Y_i)$, which completes the proof. □

3. Equilibrium Pair for the Free n-person Game

Let X be a topological space, Y be a nonempty subset of L_ρ, $\theta : X \to L_\rho$ be a map, $\phi : X \to 2^Y$ be a correspondence and *con* A denoted the convex hull of A. Then

(1) ϕ is said to be of class $\rho - L^*_\theta$, if for every $f \in X$, $\phi(f) \subset Y$ and $\theta(f) \notin \phi(f)$ and for each $g \in Y$, $\phi^{-1}(g) = \{f \in X : g \in \phi(g)\}$ is ρ-open in X;

(2) A correspondence $\phi_f : X \to 2^Y$ is said to be an $\rho - L^*_\theta$ majorant of ϕ at f if there exists an ρ-open neighborhood N_f of f in X such that (a) for each $h \in N_f$, $\phi(h) \subset \phi_f(h)$ and $\theta(h) \notin \phi_f(h)$, (b) for each $h \in X$, con $\phi_f(h) \subset Y$ and (c) for each $g \in Y$, $\phi_f^{-1}(g)$ is ρ-open in X;

(3) ϕ is $\rho - L^*_\theta$ majorized if for each $f \in X$ with $\phi(f) \neq \emptyset$, there exists an $\rho - L^*_\theta$ majorant of ϕ at f.

Theorem 3.1. *Let Y be a nonempty ρ-compact and convex subset of L_ρ. If $\Phi : Y \to 2^Y$ be $\rho - \mathcal{L}^*$-majorized then there exists a point $\bar{g} \in con\, Y \subset Y$ such that $\Phi(\bar{g}) = \emptyset$.*

Proof. suppose the contrary, then the set $\{g \in \text{con}\, Y : \Phi(g) \neq \emptyset\} = \text{con}\, Y$ is ρ-compact and there exists a correspondence $\phi : \text{con}\, Y \to 2^Y$ of class $\rho - \mathcal{L}^*$ such that $\Phi(g) \subset \phi(g)$ for each $g \in \text{con}\, Y$. It is easy to see that ϕ satisfies all hypothesis of [1, Theorem5.3] and hence there exists a point $\bar{g} \in \text{con}\, Y \subset Y$ such that $\phi(\bar{g}) = \emptyset$; it is follows that $\Phi(\bar{g}) = \emptyset$ which contradicts our assumption. Hence the conclusion must hold. □

Before starting the existence of equilibrium pair for the free n-person game, we shall need the following lemma.

Lemma 3.2. *Let $\Gamma = (Y_i, \Phi_i)_{i\in I}$ be a qualitative game where I is a (possibly infinite) set of agents such that for each $i \in I$,*

(1) Y_i is a nonempty ρ-compact and convex subset of L_ρ,

(2) the correspondence $\Phi_i : Y = \prod_{j\in I} Y_j \to 2^{Y_i}$ is $\rho - \mathcal{L}^*$-majorized in Y,

(3) the set $W_i := \{g \in Y : \Phi_i(g) \neq \emptyset\}$ is (possibly empty) ρ-open in Y.

Then there exists $\bar{g} \in Y$ such that for each $i \in I$, $\Phi_i(\bar{g}) = \emptyset$.

Proof. For each $f \in Y$, let $I(g) = \{i \in I : \Phi_i(g) \neq \emptyset\}$. Define a correspondence $\Phi : Y \to 2^Y$ by

$$\Phi(g) := \begin{cases} \cap_{i\in I(g)} \Phi'_i(g) & \text{if} \quad I(g) \neq \emptyset, \\ \emptyset & \text{if} \quad I(g) = \emptyset. \end{cases}$$

where $\Phi'_i(g) = \prod_{i\neq j, j\in I} Y_j \otimes \Phi_i(g)$ for each $g \in Y$. Then for each $g \in Y$ with $I(g) \neq \emptyset$, $\Phi(g) \neq \emptyset$. Let $g \in Y$ be such that $\Phi(g) \neq \emptyset$. Then $\Phi'_i(g) \neq \emptyset$ for all $i \in I(g)$. Fix one $i \in I(g)$. By assumption (2), there exists a ρ-open neighborhood $N_\rho(g)$ of g in Y and $\rho - \mathcal{L}^*$-majorant ϕ_i of Φ at g such that

(i) for each $h \in N_\rho(g)$, $\Phi_i(h) \subset \phi_i(h)$ and $h \notin con\, \phi_i(h)$,

(ii) for each $h \in Y$, $con\, \phi_i(h) \subset Y_i$,

(iii) for each $g \in Y_i, \phi_i^{-1}(h)$ is ρ-open in Y.

Now, by assumption (3), we may assume $N_\rho(g) \subset Y_i$, so that $\Phi_i(h) \neq \emptyset$ for all $h \in N_\rho(g)$. we define $\Psi_g : Y \to 2^Y$ by

$$\Psi_g(h) = \prod_{i\neq j, j\in I} Y_j \otimes \phi_i(h) \qquad \text{for all} \qquad h \in Y.$$

We claim that Ψ_g is an $\rho - \mathcal{L}^*$-majorant of Φ at g. Indeed, for each $h \in N_\rho(g)$, by (i),

$$\Phi(h) = \cap_{j\in I(h)} \Phi'_j(h) \subset \Phi'_i(h) \subset \Psi_g(h),$$

and

$$h \notin con\, \Psi_g(h).$$

By (ii), for each $h \in Y$

$$con\Psi_g(h) \subset \prod_{i\neq j, j\in I} con\, Y_j \otimes con\, \phi_i(h) \subset Y.$$

Since for each $f \in Y$,

$$\Phi_g^{-1}(f) = \begin{cases} \phi_i^{-1}(f_i) & \text{if} \quad f_i \in Y_j \quad \text{for all} \quad j \neq i, \\ \emptyset & \text{if} \quad f_i \notin Y_j \quad \text{for some} \quad j \neq i, \end{cases}$$

and $\phi_i^{-1}(f_i)$ is ρ-open in Y, $\Phi_g^{-1}(f)$ is also ρ-open in Y. Therefore, Φ_g^{-1} is a $\rho - \mathcal{L}^*$-majorant of Φ at g. This shows that Φ is $\rho - \mathcal{L}^*$-majorized. By Theorem 3.1 there exists a point $\bar{g} \in Y$ so that $I(g) = \emptyset$ and hence for each $i \in I$, $\Phi_i(\bar{g}) = \emptyset$. □

Next, using Lemma 3.2, we shall prove the existence of equilibrium pairs for free n-person game as follows:

Theorem 3.3. *Let $\Gamma = (X, Y_i, A_i, P_i)_{i\in I}$ be a free n-person game such that for each $i \in I = \{1, \cdots, n\}$*

(1) Let X, Y be nonempty ρ-compact and convex subsets of L_ρ, X^o a nonempty subset of X, Y_i and $Y := \prod_{j\in I} Y_j$;

(2) $A_i : X \to 2^{Y_i}$ is a ρ-upper semi continuous correspondence such that each $A_i(f)$ is a nonempty ρ-closed and convex subsets of Y_i and satisfies in condition $(*)$ in Theorem 2.3;

(3) $P_i : Y \to 2^{Y_i}$ is a preference correspondence which is $\rho - \mathcal{L}^*$-majorized in Y;

(4) $P_i(g)$ is nonempty for each $g = (g_i)_{i\in I} \in Y$ with $g_i \in Y \setminus \mathcal{A}_{if}$, whenever $\mathcal{A}_{if} = \{h \in Y_i : h \in A_i(f) \text{ and } \|f - h\|_\rho = d_\rho(X, Y_i)\}$ is nonempty;

(5) the set $W_i = \{g \in Y : A_i(f) \cap P_i(g) \neq \emptyset\}$ is ρ-open in Y whenever $\mathcal{A}_{if}$ is nonempty.

Then there exists an equilibrium pair $(\bar{f}, \bar{g}) = (\bar{f}, (\bar{g}_i)_{i\in I}) \in X \times Y$ for Γ, i.e., for each $i \in I$, $\bar{g}_i \in A_i(\bar{f})$ and $\|\bar{f} - \bar{g}_i\|_\rho = d_\rho(X, Y_i)$ such that $A_i(\bar{f}) \cap P_i(\bar{g}) \neq \emptyset$.

Proof. For each $i \in I$, since A_i satisfies the whole assumption of Theorem 2.3 in case $g = id_{X^o}$, there exists a point $\bar{f} \in X$ satisfying the system of best proximity pairs, i.e., for each $i \in I$, $\{\bar{f}\} \times A_i(\bar{f}) \subseteq X \times Y_i$ such that $d_\rho(\bar{f}, A_i(\bar{f})) = d_\rho(X, Y_i)$.
Now, we may denote the nonempty best proximity set of the correspondence A_i at $\bar{f}$ simply by

$$\mathcal{A}_i = \{h \in Y_i : h \in A_i(\bar{f}) \text{ and } \|\bar{f} - h\|_\rho = d_\rho(X, Y_i)\}.$$

Then, it is easy to see that each $\mathcal{A}_i$ is ρ-closed and convex subset of a ρ-compact convex set $A_i(\bar{f})$. It remains to show that there exists a point $g = (g_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{g}_i \in A_i(\bar{f})$ and $A_i(\bar{f}) \cap P_i(\bar{g}) \neq \emptyset$.
For each $i \in I$, we now defie a multifunction $\phi : Y \to 2^{Y_i}$ by

$$\phi_i(g) := \begin{cases} P_i(g) & \text{if} \quad g_i \notin \mathcal{A}_i, \\ A_i(\bar{f}) \cap P_i(g) & \text{if} \quad g_i \in \mathcal{A}_i, \end{cases}$$

for each $g = (g_1, \cdots, g_n) \in Y$. In order to apply Lemma 3.2 to ϕ_i for each $i \in I$, we should chek the assumption (2) and (3) of Lemma 3.2. We first show that the set $\{g \in Y : \phi_i(g) \neq \emptyset\}$ is ρ-open in $Y$for each $i \in I$. By assumption (5) the set $W_i = \{g \in Y : A_i(\bar{f}) \cap P_i(g) \neq \emptyset\}$ is ρ-open in Y. Note that the projection map $\pi_i : Y \to Y_i$ defined by $\pi_i(g_1, \cdots, g_n) = g_i$ is ρ-open in Y. Then we have

$$\begin{aligned}\{g \in Y : \phi_i(g) \neq \emptyset\} &= \{g \in Y \setminus \pi_i^{-1}(\mathcal{A}_i) : \phi_i(g) \neq \emptyset\} \cup \{g \in \pi_i^{-1}(\mathcal{A}_i) : \phi_i(g) \neq \emptyset\} \\ &= \{g \in Y \setminus \pi_i^{-1}(\mathcal{A}_i) : P_i(g) \neq \emptyset\} \cup \{g \in \pi_i^{-1}(\mathcal{A}_i) : A_i(\bar{f}) \cap P_i(g) \neq \emptyset\} \\ &= (Y \setminus \pi_i^{-1}(\mathcal{A}_i)) \cup (W_i \cap \pi_i^{-1}(\mathcal{A}_i)) = (Y \setminus \pi_i^{-1}(\mathcal{A}_i)) \cup W_i.\end{aligned}$$

Since the projection mapping π_i is ρ-open and $\mathcal{A}_i$ is ρ-compact, we have $\pi_i^{-1}(\mathcal{A}_i)$ is ρ-closed so that the set $\{g \in Y : \phi_i(g) \neq \emptyset\}$ is ρ-open in Y by the assumption (5).

Next we shal show that ϕ_i is $\rho - \mathcal{L}^*$-majorized in Y. By assumption (4), for each $g \in Y$ with $g_i \notin \mathcal{A}_i$, $\phi_i(g) = P_i(g)$ is nonempty so that there exists a $\rho - \mathcal{L}^*$-majorant ψ_i of ϕ_i in Y by the assumption (3). For each $g \in Y$ with $g_i \in \mathcal{A}_i$, $\phi_i(g) = A_i(\bar{f}) \cap P_i(g)$. If $A_i(\bar{f}) \cap P_i(g) \neq \emptyset$ then $P_i(g) \neq \emptyset$. Again by the assumption (3), there exist a $\rho - \mathcal{L}^*$-majorant ψ_i of P_i in Y. Since $\phi_i(g) \subset P_i(g)$ for each $g \in Y$ with $g_i \in \mathcal{A}_i$, ψ_i is also $\rho - \mathcal{L}^*$-majorant ϕ_i in Y. Therefore ϕ_i is $\rho - \mathcal{L}^*$-majorized in Y for each $i \in I$ and hence the whole hypotheseses of Lemma 3.2 are stisfied so that there exists a point $\bar{g} = (\bar{g}_i)_{i \in I} \in Y$ such that for each $i \in I$, $\Phi_i(\bar{g}) = \emptyset$ for each $i \in I$. If $\bar{g}_i \notin \mathcal{A}_i$ for some $i \in I$ then by assumption (4), $\phi_i(\bar{g}) = P_i(\bar{g})$ is a nonempty subset of Y_i, which is a contradiction. Therefore for each $i \in I$, we must have $\bar{g}_i \in \mathcal{A}_i$ and $\phi_i(\bar{g}) = A_i(\bar{f}) \cap P_i(\bar{g}) \neq \emptyset$. This completed the proof. □

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