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A Study on Fuzzy 2-absorbing Primary Γ-ideals in Γ-rings

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Abstract. In this paper, we initiate the study of a generalization of fuzzy primary Γ-ideals in Γ-rings by introducing fuzzy 2-absorbing primary Γ-ideals and fuzzy strongly 2-absorbing primary Γ-ideals. The notions of a fuzzy 2-absorbing primary Γ-ideal, fuzzy strongly 2-absorbing primary Γ-ideal and fuzzy weakly completely 2-absorbing primary Γ-ideal are defined and their structural characteristics and properties are investigated. The notion of a fuzzy $K-2$-absorbing primary Γ-ideal is introduced and several of its properties are investigated. Finally, the relationships between fuzzy 2-absorbing primary Γ-ideals of a Γ-ring are examined.

1. Introduction

The fuzzy set theory has been proposed in 1965 by Lofti A. Zadeh [33] from the University of Berkeley and since then this concept has been applied to various algebraic structures. Rosenfeld [31] was the first who applied this notion on algebraic structures. After the introduction of the concept of fuzzy sets by Zadeh, a lot of research took place regarding the generalization of the classical notions and results on algebraic structures applying fuzzy sets. (See [13, 14]) The concept of a Γ-ring has a special place among generalizations of rings. One of the most interesting examples of a ring would be the endomorphism ring of an abelian group, i.e., $EndM$ or $Hom(M,M)$ where M is an abelian group. Now if two abelian groups, say A and B instead of one are taken, then $Hom(A,B)$ is no longer a ring in the way as $EndM$ becomes a ring because the composition is no longer defined. However, if one takes an element of $Hom(B,A)$ and put it in between two elements of $Hom(A,B)$, then the composition can be defined. This served as a motivating factor for introducing and studying the notion of a Γ-ring. The notion of a Γ-ring, a generalization of the concept of associative rings, has been introduced and studied by Nobusawa in [28]. Barnes [6] slightly weakened the conditions in the definition of a Γ-ring in the sense of Nobusawa. The structure of Γ-rings was investigated by several authors such as W.E.Barnes in [6], S.Kyuno in [19, 20] and J.Luh in [23] and were obtained various generalizations analogous to corresponding parts in ring theory. The concept of a fuzzy ideal of a ring was introduced by Liu in [22]. Y. B. Jun and C. Y. Lee [15] applied the concept of fuzzy sets to the theory of Γ-rings. They studied some properties of fuzzy ideals of Γ-rings. In fuzzy commutative algebra, primary ideals are the most significant structures. Dutta and Chanda [10], studied the structure of the set of fuzzy ideals of a Γ-ring. Jun [16] defined fuzzy prime ideal of a Γ-ring and obtained several

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characterizations for a fuzzy ideal to be a fuzzy prime ideal. Fuzzy maximal, radical and primary ideal of a ring was studied by Malik and Mordeson in [24] and fuzzy prime ideal in Γ-rings was studied by Dutta and Chanda in [11]. Furthermore, Öztürk et al. [29, 30] characterized the Artinian and Noetherian Γ-rings in terms of fuzzy ideals.

The concept of a 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [2] and which was also studied in [1],[5]. At present, studies on the 2-absorbing ideal theory are progresssing rapidly. It has been studied extensively by many authors (e.g.[3],[7],[17]). Darani [9] investigated and examined the notion of L-fuzzy 2-absorbing ideals and he has obtained interesting results on these concepts. Darani and Hashempoor were focused on the concept of $L-$fuzzy 2-absorbing ideals in semiring [8]. Elkettani and Kasem [12] proposed the notion of 2-absorbing δ-primary Γ-ideal of Γ-rings and obtained interesting results concerning these concepts.

In this paper, we introduce the fuzzy 2-absorbing Γ-ideals, fuzzy 2-absorbing primary Γ-ideals, fuzzy strongly 2-absorbing primary Γ-ideals and fuzzy weakly completely 2-absorbing primary Γ-ideals, some generalizations of 2-absorbing primary fuzzy ideals and describe some of their properties . The notion of a fuzzy $K-2$-absorbing primary Γ-ideal is introduced and several of its properties are investigated. Finally, the relationships between fuzzy 2-absorbing primary Γ-ideals of Γ-rings are examined. We also establish a diagram which transition between definitions of fuzzy 2-absorbing Γ-ideals of a Γ-ring as well as the relationships of these concepts with the concept of 2-absorbing Γ-ideal.

2. Preliminaries

In this section, for the sake of completeness, we first recall some useful definitions and results which are needed in the sequel. Throughout this paper, unless otherwise stated, R is a commutative Γ-ring with $1 \neq 0$ and $L = [0, 1]$ stands for a complete lattice.

Definition 2.1. *[33] A fuzzy subset μ in a set X is a function $\mu : X \to [0, 1]$.*

Definition 2.2. *[25] Let μ and ν be fuzzy subsets of X. We say that μ is a subset of ν, and write $\mu \subseteq \nu$, if and only if $\mu(x) \leq \nu(x)$, for all $x \in X$.*

Definition 2.3. *[25] Let μ be any fuzzy subset of X and $t \in L$. Then the set*

$$\mu_t = \{x \in X \mid \mu(x) \geq t\}$$

is called the $t-$ level subset of X with respect to μ.

Definition 2.4. *[25] Let $x \in X$ and $r \in L - \{0\}$. A fuzzy point, written as x_r, is defined to be a fuzzy subset of X, given by*

$$x_r(y) = \begin{cases} r, & \text{if } y{=}x; \\ 0, & \text{otherwise.} \end{cases}$$

If x_r is a fuzzy point of X and $x_r \subseteq \mu$, where μ is a fuzzy subset of X, then we write $x_r \in \mu$.

Definition 2.5. *[6] Let R and Γ be two abelian groups. R is called a Γ-ring if there exists a mapping*

$$\begin{aligned} R \times \Gamma \times R &\to R \\ (x, \alpha, y) &\mapsto x\alpha y \end{aligned}$$

satisfying the following conditions:

1. $(x + y)\alpha z = x\alpha z + y\alpha z,$
2. $x\alpha(y + z) = x\alpha y + x\alpha z,$
3. $x(\alpha + \beta)y = x\alpha y + x\beta y$
4. $x\alpha(y\beta z) = (x\alpha y)\beta z$ *for all $x, y, z \in R$ and all $\alpha, \beta \in \Gamma$.*

Definition 2.6. *[10] A left (resp. right) Γ-ideal of a Γ-ring R is a subset A of R which is an additive subgroup of R and $R\Gamma A \subseteq A$ (resp,$A\Gamma R \subseteq A$) where,*

$$R\Gamma A = \{x\alpha y \mid x \in R, \alpha \in \Gamma, y \in A\}.$$

If A is both a left and right ideal, then A is called a Γ-ideal of R.

Definition 2.7. *[11] A fuzzy set μ in $\Gamma-$ring R is called a fuzzy Γ-ideal of R, if for all $x, y \in R$ and $\alpha \in \Gamma$, the following requirements are satisfied:*

1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$.

Definition 2.8. *[10] Let R and S be two Γ-rings, and f be a mapping of R into S. Then f is called Γ-homomorphism if*

$$f(a + b) = f(a) + f(b) \text{ and } f(a\alpha b) = f(a)\alpha f(b)$$

for all $a, b \in R$ and $\alpha \in \Gamma$.

Proposition 2.9. *[21] If P is an ideal of a Γ-ring R, then the following conditions are equivalent:*

1. *P is a prime ideal of R;*
2. *If $x, y \in R$ and $x\Gamma R\Gamma y \subseteq P$, then $x \in P$ or $y \in P$.*

Definition 2.10. *[16] A non-constant fuzzy Γ-ideal μ of a Γ-ring R is called fuzzy prime Γ-ideal of R if for any two fuzzy Γ-ideals σ and θ of R,*

$$\sigma\Gamma\theta \subseteq \mu \text{ implies that either } \sigma \subseteq \mu \text{ or } \theta \subseteq \mu.$$

Lemma 2.11. *[24] Let R be a commutative Γ-ring with identity and let x_r and y_s be two fuzzy points of R. Then*

1. $x_r\alpha y_s = (x\alpha y)_{r\wedge s}$
2. $\langle x_r\rangle \alpha \langle y_s\rangle = \langle x_r\alpha y_s\rangle$, *where $\langle x_r\rangle$ is fuzzy Γ-ideal of R generated by x_r.*

Theorem 2.12. *[24] Let R be a commutative Γ-ring and μ be a fuzzy Γ-ideal of R. Then the following statements are equivalent:*

1. *$x_r\Gamma y_t \subseteq \mu \Rightarrow x_r \subseteq \mu$ or $y_t \subseteq \mu$ where x_r and y_t are two fuzzy points of R.*
2. *μ is a fuzzy prime Γ-ideal of R.*

Definition 2.13. *[2] A proper ideal I of a commutative ring M is called a 2-absorbing ideal of M if whenever $x, y, z \in M$ and $xyz \in I$, then $xy \in I$ or $xz \in I$ or $yz \in I$.*

Definition 2.14. *[26] A fuzzy ideal μ of R is said to be a fuzzy weakly completely prime ideal if μ is non-constant function and for all $x, y \in R$, $\mu(xy) = \max\{\mu(x), \mu(y)\}$.*

Definition 2.15. *[18] Let μ be a non-constant fuzzy ideal of R. μ is said to be a fuzzy $K-$prime ideal if $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$ for any $x, y \in R$.*

Definition 2.16. *[12] A proper Γ-ideal I of a Γ-ring R is called a 2-absorbing Γ-ideal of R if whenever $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $x\alpha y\beta z \in I$, then $x\alpha y \in I$ or $x\beta z \in I$ or $y\beta z \in I$.*

Definition 2.17. *[27] Let μ be a fuzzy ideal of R. Then $\sqrt{\mu}$, called the radical of μ, is defined by $\sqrt{\mu}(x) = \vee_{n\geq 1} \mu(x^n)$.*

Definition 2.18. *[27] A fuzzy ideal μ of R is called primary fuzzy ideal if for $x, y \in R$, $\mu(xy) > \mu(x)$ implies $\mu(xy) \leq \mu(y^n)$ for some positive integer n.*

Theorem 2.19. *[27] Let μ be fuzzy ideal of a ring R. Then $\sqrt{\mu}$ is a fuzzy ideal of R.*

Theorem 2.20. *[32] If μ and ξ are two fuzzy ideals of R, then $\sqrt{\mu \cap \xi} = \sqrt{\mu} \cap \sqrt{\xi}$.*

Theorem 2.21. *[32] Let $f : R \to S$ be a ring homomorphism and let μ be a fuzzy ideal of R such that μ is constant on Kerf and ξ be a fuzzy ideal of S. Then ,*

$$\sqrt{f(\mu)} = f(\sqrt{\mu}) \quad \text{and} \quad \sqrt{f^{-1}(\xi)} = f^{-1}(\sqrt{\xi}).$$

Definition 2.22. *[4] A proper ideal I of R is called 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$ then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.*

Theorem 2.23. *[4] If I is a 2-absorbing primary ideal of R, then $\sqrt{I}$ is a 2-absorbing ideal of R.*

3. Fuzzy 2-absorbing primary Γ-ideals of a Γ-ring

In this section, we introduce and study fuzzy 2-absorbing primary Γ-ideals of a Γ- ring. Firstly, we will give the structure of fuzzy primary Γ-ideals of a Γ-ring. Throughout this paper we assume that R is a commutative Γ-ring.

Definition 3.1. *Let μ be a non-constant fuzzy Γ-ideal of R. Then μ is said to be a fuzzy primary Γ-ideal of R if*

$$x_r \alpha y_s \text{ implies that either } x_r \in \mu \text{ or } y_s \in \sqrt{\mu}$$

for any fuzzy points x_r, y_s of R and $\alpha \in \Gamma$.

Proposition 3.2. *Let μ be a fuzzy Γ-ideal of R. If μ is a fuzzy primary Γ-ideal of R, then for all $x, y \in R$ and $\alpha \in \Gamma$*

$$\mu(x\alpha y) > \mu(x) \text{ implies that } \mu(x\alpha y) \leq \sqrt{\mu}(y).$$

Proof. Let $\mu(x\alpha y) = r > \mu(x)$. Then $(x\alpha y)_r \in \mu$ and $x_r \notin \mu$. Since μ is a fuzzy primary Γ-ideal of R, then $y_r \in \sqrt{\mu}$. Thus $\mu(x\alpha y) = r \leq \sqrt{\mu}(y)$. □

Example 3.3. *Every fuzzy prime Γ-ideal of R is a fuzzy primary Γ-ideal of R.*

Now, we give the definition of a fuzzy 2-absorbing primary Γ-ideal of a Γ-ring.

Definition 3.4. *Let μ be a non-constant fuzzy Γ-ideal of a Γ-ring R. Then μ is called fuzzy 2-absorbing primary Γ-ideal of R if for any fuzzy points x_r, y_s, z_t of R and $\alpha, \beta \in \Gamma$,*

$$x_r \alpha y_s \beta z_t \in \mu \text{ implies that either } x_r \alpha y_s \in \mu \text{ or } x_r \beta z_t \in \sqrt{\mu} \text{ or } y_s \beta z_t \in \sqrt{\mu}.$$

Proposition 3.5. *Every fuzzy primary Γ-ideal of R is a fuzzy 2-absorbing primary Γ-ideal of R.*

Proof. The proof is straightforward. □

Theorem 3.6. *Let μ be a fuzzy Γ-ideal of R. If μ is a fuzzy 2-absorbing primary Γ-ideal of R, then μ_a is a 2-absorbing primary Γ-ideal of R, for every $a \in [0, \mu(0)]$ with $\mu_a \neq R$.*

Proof. Let μ be fuzzy 2-absorbing primary Γ-ideal of R and suppose that $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ are such that $x\alpha y\beta z \in \mu_a$ for every $a \in [0, \mu(0)]$ with $\mu_a \neq R$. Then

$$\mu(x\alpha y\beta z) \geq a \text{ and } (x\alpha y\beta z)_a (x\alpha y\beta z) = a \leq \mu(x\alpha y\beta z),$$

so we have $(x\alpha y\beta z)_a = x_a\alpha y_a\beta z_a \in \mu$. Since μ is a fuzzy 2-absorbing primary Γ-ideal of R, we have

$$(x\alpha y)_a = x_a\alpha y_a \in \mu \text{ or } (x\beta z)_a = x_a\beta z_a \in \sqrt{\mu} \text{ or } (y\beta z)_a = y_a\beta z_a \in \sqrt{\mu}.$$

Thus $x\alpha y \in \mu_a$ or $x\beta z \in \sqrt{\mu_a}$ or $y\beta z \in \sqrt{\mu_a}$, and μ_a is a 2-absorbing primary Γ-ideal of R. □

The following example shows that the converse of the theorem is not generally true.

Example 3.7. *Let $R = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$, so R is a Γ-ring. Define the fuzzy Γ-ideal μ of R by*

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0; \\ \frac{1}{3}, & \text{if } x \in 15\mathbb{Z} - \{0\}; \\ 0, & \text{if } x \in \mathbb{Z} - 15\mathbb{Z}. \end{cases}$$

Since $\mu_0 = \mathbb{Z}$, $\mu_{\frac{1}{3}} = 15\mathbb{Z}$ and $\mu_1 = 0$, then we get μ_a is a 2-absorbing primary Γ-ideal of R. But, for $\alpha, \beta \in 2\mathbb{Z}$, we get

$$\begin{aligned} 3_{\frac{1}{2}}\alpha 5_{\frac{1}{2}}\beta 1_{\frac{1}{3}} &= (3\alpha 5\beta 1)_{\frac{1}{2}\wedge\frac{1}{2}\wedge\frac{1}{3}} = (3\alpha 5\beta 1)_{\frac{1}{3}} \in \mu, \text{ and} \\ 3_{\frac{1}{2}}\alpha 5_{\frac{1}{2}} &= (3\alpha 5)_{\frac{1}{2}\wedge\frac{1}{2}} = (3\alpha 5)_{\frac{1}{2}} = \frac{1}{2} > \mu(3\alpha 5) = \frac{1}{3}, \\ 3_{\frac{1}{2}}\beta 1_{\frac{1}{3}} &= (3\beta 1)_{\frac{1}{2}\wedge\frac{1}{3}} = (3\beta 1)_{\frac{1}{3}} = \frac{1}{3} > \sqrt{\mu}(3\beta 1) = 0; \\ 5_{\frac{1}{2}}\beta 1_{\frac{1}{3}} &= (5\beta 1)_{\frac{1}{2}\wedge\frac{1}{3}} = (5\beta 1)_{\frac{1}{3}} = \frac{1}{3} > \sqrt{\mu}(5\beta 1) = 0. \end{aligned}$$

Thus μ is not a fuzzy 2-absorbing primary Γ-ideal of R.

Corollary 3.8. *If μ is a fuzzy 2-absorbing primary Γ-ideal of R, then*

$$\mu_* = \{x \in R \mid \mu(x) = \mu(0)\}$$

is a 2-absorbing primary Γ-ideal of R.

Proof. Since μ is a non-constant fuzzy Γ-ideal of R, then $\mu_* \neq R$. Now the result follows from the above theorem. □

In the sequel of the paper, for the sake of simplicity, we denote $x^m = x\gamma_1 x\gamma_2 x....\gamma_{m-1}x$ for some $\gamma_1, \gamma_2, ..., \gamma_{m-1} \in \Gamma$ and for some $m \in Z^+$.

Theorem 3.9. *Let I be a 2-absorbing primary Γ-ideal of R. Then the fuzzy subset of R defined by*

$$\mu(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{otherwise} \end{cases}$$

is a fuzzy 2-absorbing primary Γ-ideal of R.

Proof. We have $I \neq R$ and so μ is non-constant because I is a 2-absorbing primary Γ-ideal of R. Assume that $x_r\alpha y_s\beta z_t \in \mu$, but $x_r\alpha y_s \notin \mu$, $x_r\beta z_t \notin \sqrt{\mu}$ and $y_s\beta z_t \notin \sqrt{\mu}$ where x_r, y_s, z_t are fuzzy points of R and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} \mu(x\alpha y) &< r \wedge s \\ \mu\left((x\beta z)^n\right) &< \sqrt{\mu}(x\beta z) < r \wedge t \\ \mu\left((y\beta z)^n\right) &< \sqrt{\mu}(y\beta z) < s \wedge t \end{aligned}$$

for all $n \geq 1$. Hence

$$\begin{aligned} \mu(x\alpha y) &= 0 \text{ and } x\alpha y \notin I \\ \mu\left((x\beta z)^n\right) &= 0 \text{ and } (x\beta z)^n \notin I \text{ so } x\beta z \notin \sqrt{I} \\ \mu\left((y\beta z)^n\right) &= 0 \text{ and } (y\beta z)^n \notin I \text{ so } y\beta z \notin \sqrt{I}. \end{aligned}$$

Since I is a 2-absorbing Γ-ideal of R, we have $x\alpha y\beta z \notin I$ and so $\mu(x\alpha y\beta z) = 0$ for $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. By our hypothesis, we have $(x\alpha y\beta z)_{(r\wedge s\wedge t)} = x_r\alpha y_s\beta z_t \in \mu$ and $r \wedge s \wedge t \leq \mu(x\alpha y\beta z) = 0$. Hence $r \wedge s = 0$ or $r \wedge t = 0$ or $s \wedge t = 0$, which is a contradiction. Hence $x_r\alpha y_s \in \mu$ or $x_r\beta z_t \in \sqrt{\mu}$ or $y_s\beta z_t \in \sqrt{\mu}$ and μ is a fuzzy 2-absorbing primary Γ-ideal of R. □

Theorem 3.10. *Every fuzzy 2-absorbing Γ-ideal of R is a fuzzy 2-absorbing primary Γ-ideal of R.*

Proof. The proof is straightforward. □

The following example shows that the converse of the above theorem is not true.

Example 3.11. *Let $R = \mathbb{Z}$ and $\Gamma = 5\mathbb{Z}$, so R is a Γ-ring. Define the fuzzy Γ-ideal μ of R by*

$$\mu(x) = \begin{cases} 1, & \text{if } x \in 12\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy 2-absorbing primary Γ-ideal of R, but it is not fuzzy 2-absorbing Γ-ideal of R because for $\alpha, \beta \in 2\mathbb{Z}$ and $r, s, t \in [0,1]$,

$$2_r\alpha 2_s\beta 3_t \in \mu, \text{ but } 2_r\alpha 2_s \notin \mu,\ 2_s\beta 3_t \notin \mu \text{ and } 2_r\beta 3_t \notin \mu.$$

Proposition 3.12. *If μ is a fuzzy 2-absorbing primary Γ-ideal of R, then $\sqrt{\mu}$ is a fuzzy 2-absorbing Γ-ideal of R.*

Proof. Suppose that $x_r\alpha y_s\beta z_t \in \sqrt{\mu}$ and $x_r\alpha y_s \notin \sqrt{\mu}$ where x_r, y_s, z_t are fuzzy points of R and $\alpha, \beta \in \Gamma$. Since $x_r\alpha y_s\beta z_t \in \sqrt{\mu}$, then

$$r \wedge s \wedge t = (x\alpha y\beta z)_{(r\wedge s\wedge t)}(x\alpha y\beta z) = x_r\alpha y_s\beta z_t(x\alpha y\beta z) \leq \sqrt{\mu}(x\alpha y\beta z).$$

From the definition of $\sqrt{\mu}$, we have

$$\sqrt{\mu}(x\alpha y\beta z) = \bigvee_{n\geq 1}\mu\left((x\alpha y\beta z)^n\right) \geq \bigvee_{n\geq 1}\mu\left(x^n\gamma_1 y^n\gamma_2 z^n\right) \geq r \wedge s \wedge t.$$

for some $\gamma_1, \gamma_2 \in \Gamma$. Then there exists $k \in \mathbb{Z}^+$ such that for some $\gamma_1', \gamma_2' \in \Gamma$,

$$r \wedge s \wedge t \leq \mu\left(x^k\gamma_1' y^k\gamma_2' z^k\right) \leq \mu\left((x\alpha y\beta z)^k\right)$$

which implies that $(x_r\alpha y_s\beta z_t)^k \in \mu$. If $x_r\alpha y_s \notin \sqrt{\mu}$, then for all $k \in \mathbb{Z}^+$ and for some $\gamma \in \Gamma$,

$$(x_r\alpha y_s)^k \geq x_r^k\gamma y_s^k \notin \mu.$$

Since μ is a fuzzy 2-absorbing primary Γ-ideal of R, then

$$x_r\beta z_t \in \sqrt{\mu} \text{ or } y_s\beta z_t \in \sqrt{\mu}.$$

Hence $\sqrt{\mu}$ is fuzzy 2-absorbing Γ-ideal of R. □

Definition 3.13. *Let μ be a fuzzy 2-absorbing primary Γ-ideal of R and $\gamma = \sqrt{\mu}$ which is a fuzzy 2-absorbing Γ-ideal of R. Then μ is called a fuzzy γ-2-absorbing primary Γ-ideal of R.*

Theorem 3.14. *Let $\mu_1, \mu_2, ..., \mu_n$ be fuzzy γ-2-absorbing primary Γ-ideals of R for some fuzzy 2-absorbing Γ-ideal γ of R. Then $\mu = \bigcap_{i=1}^{n}\mu_i$ is a fuzzy γ-2-absorbing primary Γ-ideal of R.*

Proof. Assume that $x_r\alpha y_s\beta z_t \in \mu$ and $x_r\alpha y_s \notin \mu$ where x_r, y_s, z_t are fuzzy points of R and $\alpha, \beta \in \Gamma$. Then $x_r\alpha y_s \notin \mu_j$ for some $n \geq j \geq 1$ and $x_r\alpha y_s\beta z_t \in \mu_j$ for all $n \geq j \geq 1$. Since μ_j is a fuzzy γ-2-absorbing primary Γ-ideal of R, we have $y_s\beta z_t \in \sqrt{\mu_j} = \gamma = \bigcap_{i=1}^{n} \sqrt{\mu_i} = \sqrt{\bigcap_{i=1}^{n} \mu_i} = \sqrt{\mu}$ or $x_r\beta z_t \in \sqrt{\mu_j} = \gamma = \bigcap_{i=1}^{n} \sqrt{\mu_i} = \sqrt{\bigcap_{i=1}^{n} \mu_i} = \sqrt{\mu}$. Thus μ is a fuzzy γ-2-absorbing primary Γ-ideal of R. □

In the following example, we show that if μ_1, μ_2 are fuzzy 2-absorbing primary Γ-ideals of a Γ-ring R, then $\mu_1 \cap \mu_2$ need not to be a fuzzy 2-absorbing primary Γ-ideal of R.

Example 3.15. *Let $R = \mathbb{Z}$ and $\Gamma = p_i\mathbb{Z}$, so R is a Γ-ring. Define the fuzzy Γ-ideals μ_1 and μ_2 of R by*

$$\mu_1(x) = \begin{cases} 1, & \text{if } x \in 50\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \mu_2(x) = \begin{cases} 1, & \text{if } x \in 75\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

such that $p_i \neq 2, 3, 5$ is a prime integer. Hence μ_1 and μ_2 are fuzzy 2-absorbing primary Γ-ideals of a Γ-ring R. Since

$$(\mu_1 \cap \mu_2)(x) = \begin{cases} 1, & \text{if } x \in 150\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \sqrt{\mu_1 \cap \mu_2}(x) = \begin{cases} 1, & \text{if } x \in 30\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

then for $\alpha, \beta \in \Gamma$ and $r, s, t \in [0, 1]$, $25_r\alpha 3_s\beta 2_t \in \mu_1 \cap \mu_2$, but $25_r\alpha 3_s \notin \mu_1 \cap \mu_2$, $3_s\beta 2_t \notin \sqrt{\mu_1 \cap \mu_2}$, $25_r\beta 2_t \notin \sqrt{\mu_1 \cap \mu_2}$. Therefore, $\mu_1 \cap \mu_2$ is not a fuzzy 2-absorbing primary Γ-ideal of R.

Theorem 3.16. *Let μ be a fuzzy Γ-ideal of R. If $\sqrt{\mu}$ is a fuzzy prime Γ-ideal of R, then μ is a fuzzy 2-absorbing primary Γ-ideal of R.*

Proof. Suppose that $x_r\alpha y_s\beta z_t \in \mu$ and $x_r\alpha y_s \notin \mu$ for any $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $r, s, t \in [0, 1]$. Since $x_r\alpha y_s\beta z_t \in \mu$ and R is a commutative Γ-ring, we have

$$x_r\alpha y_s\beta z_t\beta z_t = (x_r\alpha z_t)\beta(y_s\beta z_t) \in \mu \subseteq \sqrt{\mu}.$$

Thus $x_r\alpha z_t \in \sqrt{\mu}$ or $y_s\beta z_t \in \sqrt{\mu}$, since $\sqrt{\mu}$ is a fuzzy prime Γ-ideal of R. Therefore we conclude that μ is a fuzzy 2-absorbing primary Γ-ideal of R. □

Corollary 3.17. *If μ is a fuzzy prime Γ-ideal of R, then μ^n is fuzzy 2-absorbing primary Γ-ideal of R, for any $n \in Z^+$.*

Proof. Let μ be a fuzzy prime Γ-ideal of R and $x_r\alpha y_s\beta z_t \in \mu^n$, but $x_r\alpha y_s \notin \mu^n$ for any $n \in Z^+$, where x_r, y_s, z_t are fuzzy points of R and $\alpha, \beta \in \Gamma$. Since $x_r\alpha y_s\beta z_t \in \mu^n$ and R is a commutative Γ-ring, then

$$x_r\alpha y_s\beta z_t\beta z_t = (x_r\alpha z_t)\beta(y_s\beta z_t) \in \mu^n \subseteq \mu.$$

Since μ is fuzzy prime Γ-ideal of R, then $x_r\alpha z_t \in \mu = \sqrt{\mu^n}$ or $y_s\beta z_t \in \mu = \sqrt{\mu^n}$. Hence μ^n is fuzzy 2-absorbing primary Γ-ideal of R. □

Theorem 3.18. *Let $\{\mu_i \mid i \in I\}$ be a directed collection of fuzzy 2-absorbing primary Γ-ideals of R. Then the fuzzy ideal $\mu = \bigcup_{i \in I} \mu_i$ is a fuzzy 2-absorbing primary Γ-ideal of R.*

Proof. Assume that $x_r\alpha y_s\beta z_t \in \mu$ and $x_r\alpha y_s \notin \mu$ for some x_r, y_s, z_t fuzzy points of R and $\alpha, \beta \in \Gamma$. Then there exists $j \in I$ such that $x_r\alpha y_s\beta z_t \in \mu_j$ and $x_r\alpha y_s \notin \mu_j$ for all $j \in I$. Since μ_j is a fuzzy 2-absorbing Γ-ideal of R, then

$$y_s\beta z_t \in \sqrt{\mu_j} \text{ or } x_r\beta z_t \in \sqrt{\mu_j}.$$

Thus

$$y_s\beta z_t \in \sqrt{\mu_j} \subseteq \bigcup_{i \in I} \sqrt{\mu_i} = \sqrt{\bigcup_{i \in I} \mu_i} = \mu \text{ or } x_r\beta z_t \in \sqrt{\mu_j} \subseteq \bigcup_{i \in I} \sqrt{\mu_i} = \sqrt{\bigcup_{i \in I} \mu_i} = \mu.$$

Hence $\mu = \bigcup_{i \in I} \mu_i$ is a fuzzy 2-absorbing primary Γ-ideal of R. □

Theorem 3.19. *Let $f : R \to S$ be a surjective Γ-ring homomorphism. If μ is a fuzzy 2-absorbing primary Γ-ideal of R which is constant on Kerf, then $f(\mu)$ is a fuzzy 2-absorbing primary Γ-ideal of S.*

Proof. Suppose that $x_r\alpha y_s\beta z_t \in f(\mu)$, where x_r, y_s, z_t are fuzzy points of S and $\alpha, \beta \in \Gamma$. Since f is a surjective Γ-ring homomorphism, then there exist $a, b, c \in R$ such that $f(a) = x$, $f(b) = y$, $f(c) = z$. Thus

$$\begin{aligned} x_r\alpha y_s\beta z_t(x\alpha y\beta z) &= r \wedge s \wedge t \\ &\leq f(\mu)(x\alpha y\beta z) \\ &= f(\mu)(f(a)\alpha f(b)\beta f(c)) \\ &= f(\mu)(f(a\alpha b\beta c)) \\ &= \mu(a\alpha b\beta c) \end{aligned}$$

because μ is constant on Ker$f$. Then we get $a_r\alpha b_s\beta c_t \in \mu$. Since μ is a fuzzy 2-absorbing primary Γ-ideal of R, then

$$a_r\alpha b_s \in \mu \text{ or } a_r\beta c_t \in \sqrt{\mu} \text{ or } b_s\beta c_t \in \sqrt{\mu}.$$

Thus,

$$\begin{aligned} r \wedge s &\leq \mu(a\alpha b) = f(\mu)(f(a\alpha b)) \\ &= f(\mu)(f(a)\alpha f(b)) \\ &= f(\mu)(x\alpha y) \end{aligned}$$

and so, $x_r\alpha y_s \in f(\mu)$ or

$$\begin{aligned} r \wedge t &\leq \sqrt{\mu}(a\beta c) = f\left(\sqrt{\mu}\right)(f(a\beta c)) \\ &= f\left(\sqrt{\mu}\right)(f(a)\beta f(c)) \\ &= f\left(\sqrt{\mu}\right)(x\beta z), \end{aligned}$$

so $x_r\beta z_t \in f\left(\sqrt{\mu}\right)$ or

$$\begin{aligned} s \wedge t &\leq \sqrt{\mu}(b\beta c) = f\left(\sqrt{\mu}\right)(f(b\beta c)) \\ &= f\left(\sqrt{\mu}\right)(f(b)\beta f(c)) \\ &= f\left(\sqrt{\mu}\right)(y\beta z), \end{aligned}$$

so $y_s\beta z_t \in f\left(\sqrt{\mu}\right)$. Hence $f(\mu)$ is a fuzzy 2-absorbing primary Γ-ideal of S. □

Theorem 3.20. *Let $f : R \to S$ be a Γ-ring homomorphism. If ν is a fuzzy 2-absorbing primary $\Gamma-$ideal of S, then $f^{-1}(\nu)$ is a fuzzy 2-absorbing primary Γ-ideal of R.*

Proof. Suppose that $x_r\alpha y_s\beta z_t \in f^{-1}(\nu)$, where x_r, y_s, z_t are any fuzzy points of R and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} r \wedge s \wedge t &\leq f^{-1}(\nu)((x\alpha y\beta z)) \\ &= v(f(x\alpha y\beta z)) \\ &= v(f(x)\alpha f(y)\beta f(z)). \end{aligned}$$

Let $f(x) = a$, $f(y) = b$, $f(z) = c \in S$. Hence we have that $r \wedge s \wedge t \leq v(a\alpha b\beta c)$ and $a_r\alpha b_s\beta c_t \in v$. Since v is a fuzzy 2-absorbing primary $\Gamma-$ideal of R, then $a_r\alpha b_s \in v$ or $a_r\beta c_t \in \sqrt{v}$ or $b_s\beta c_t \in \sqrt{v}$. If $a_r\alpha b_s \in v$, then

$$\begin{aligned} r \wedge s &\leq v(a\alpha b) = v(f(x)\alpha f(y)) \\ &= v(f(x\alpha y)) \\ &= f^{-1}(v(x\alpha y)). \end{aligned}$$

Thus we get $x_r \alpha y_s \in f^{-1}(\nu)$. If $a_r \beta c_t \in \sqrt{v}$, then

$$\begin{aligned} r \wedge t &\leq \sqrt{v}(a\alpha c) = \sqrt{v}(f(x)\alpha f(z)) \\ &= \sqrt{v}(f(x\alpha z)) \\ &= f^{-1}\left(\sqrt{v}(x\alpha z)\right) \end{aligned}$$

so we have $x_r \beta z_t \in f^{-1}\left(\sqrt{\nu}\right)$ or if $b_s \beta c_t \in \sqrt{v}$, then

$$\begin{aligned} s \wedge t &\leq \sqrt{v}(b\alpha c) = \sqrt{v}(f(y)\alpha f(z)) \\ &= \sqrt{v}(f(y\alpha z)) \\ &= f^{-1}\left(\sqrt{v}(y\alpha z)\right) \end{aligned}$$

and we get $y_s \beta z_t \in f^{-1}\left(\sqrt{\nu}\right)$. Therefore, we see that $f^{-1}(v)$ is a fuzzy 2-absorbing primary Γ-ideal of R. □

Definition 3.21. *Let μ be a fuzzy Γ-ideal of R. μ is called a fuzzy strongly 2-absorbing primary Γ-ideal of R if it is non-constant and whenever λ, η, ν are fuzzy Γ-ideals of R with $\lambda \Gamma \eta \Gamma \nu \subseteq \mu$, then $\lambda \Gamma \eta \subseteq \mu$ or $\lambda \Gamma \nu \subseteq \sqrt{\mu}$ or $\eta \Gamma \nu \subseteq \sqrt{\mu}$.*

Theorem 3.22. *Every fuzzy primary Γ-ideal of R is a fuzzy strongly 2-absorbing primary Γ-ideal of R.*

Proof. The proof is straightforward. □

Theorem 3.23. *Every fuzzy strongly 2-absorbing primary Γ-ideal of R is a fuzzy 2-absorbing primary Γ-ideal of R.*

Proof. Assume that μ is a fuzzy strongly 2-absorbing primary Γ-ideal of R. Suppose that $x_r, y_s, z_t \in \mu$ for some fuzzy points x_r, y_s, z_t of R. We get $\langle x_r \rangle \Gamma \langle y_s \rangle \Gamma \langle z_t \rangle = \langle x_r \Gamma y_s \Gamma z_t \rangle \subseteq \mu$. Since μ is a fuzzy strongly 2-absorbing primary Γ-ideal of R, then we get $\langle x_r \Gamma y_s \rangle = \langle x_r \rangle \Gamma \langle y_s \rangle \subseteq \mu$ or $\langle x_r \Gamma z_t \rangle = \langle x_r \rangle \Gamma \langle z_t \rangle \subseteq \sqrt{\mu}$ or $\langle y_s \Gamma z_t \rangle = \langle y_s \rangle \Gamma \langle z_t \rangle \subseteq \sqrt{\mu}$. Hence $x_r \Gamma y_s \subseteq \mu$ or $x_r \Gamma z_t \subseteq \sqrt{\mu}$ or $y_s \Gamma z_t \subseteq \sqrt{\mu}$ and then, for $\alpha, \beta \in \Gamma$, we get $x_r \alpha y_s \in \mu$ or $x_r \beta z_t \in \sqrt{\mu}$ or $y_s \beta z_t \in \sqrt{\mu}$ which implies that μ is a fuzzy 2-absorbing primary Γ-ideal of R. □

4. Fuzzy Weakly Completely 2-absorbing Primary Γ-ideals

In this section, we study fuzzy weakly completely 2-absorbing primary Γ-ideals of a Γ-ring. Firstly, we give the definitions of fuzzy weakly completely 2-absorbing Γ-ideal and fuzzy weakly completely primary Γ-ideal of a Γ-ring.

Definition 4.1. *Let μ be a fuzzy Γ-ideal of R. μ is called a fuzzy weakly completely 2-absorbing Γ-ideal of R if for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,*

$$\mu(x\alpha y\beta z) \leq \mu(x\alpha y) \text{ or } \mu(x\alpha y\beta z) \leq \mu(x\beta z) \text{ or } \mu(x\alpha y\beta z) \leq \mu(y\beta z).$$

Definition 4.2. *Let μ be a fuzzy Γ-ideal of R. μ is said to be a fuzzy weakly completely primary Γ-ideal of R if μ is non-constant fuzzy Γ-ideal of R and for all $x, y \in R$ and $\alpha \in \Gamma$,*

$$\mu(x\alpha y) \leq \mu(x) \text{ or } \mu(x\alpha y) \leq \sqrt{\mu}(y).$$

Proposition 4.3. *Let μ be a non-constant fuzzy Γ-ideal of R. μ is a fuzzy weakly completely primary Γ-ideal of R if and only if for every $x, y \in R$ and $\alpha \in \Gamma$,*

$$\mu(x\alpha y) = \max\left\{\mu(x), \sqrt{\mu}(y)\right\}.$$

Now, we give the definition of a fuzzy weakly completely 2-absorbing primary Γ-ideal of a Γ-ring.

Definition 4.4. *Let μ be a fuzzy Γ-ideal of R. μ is called a fuzzy weakly completely 2-absorbing primary Γ-ideal of R if for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,*

$$\mu(x\alpha y\beta z) \leq \mu(x\alpha y) \text{ or } \mu(x\alpha y\beta z) \leq \sqrt{\mu}(x\beta z) \text{ or } \mu(x\alpha y\beta z) \leq \sqrt{\mu}(y\beta z).$$

Proposition 4.5. *Let μ be a non-constant fuzzy Γ-ideal of R. μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R if and only if for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,*

$$\mu(x\alpha y\beta z) = \max\left\{\mu(x\alpha y), \sqrt{\mu}(x\beta z), \sqrt{\mu}(y\beta z)\right\}.$$

Theorem 4.6. *Every fuzzy weakly completely 2-absorbing Γ-ideal of R is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R.*

Proof. The proof is straightforward. □

Theorem 4.7. *Every fuzzy primary Γ-ideal of R is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R.*

Proof. Let μ be a fuzzy primary Γ-ideal of R. Suppose that $\mu(x\alpha y\beta z) > \mu(x\alpha y)$ for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. From the definition of a fuzzy primary Γ-ideal of R, we get $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(z)$. Since $\sqrt{\mu}$ is a fuzzy Γ-ideal, then

$$\begin{aligned} \sqrt{\mu}(x\beta z) &\geq \sqrt{\mu}(z) \geq \mu(x\alpha y\beta z) \text{ or} \\ \sqrt{\mu}(y\beta z) &\geq \sqrt{\mu}(z) \geq \mu(x\alpha y\beta z). \end{aligned}$$

Hence μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R. □

Theorem 4.8. *Every fuzzy weakly completely primary Γ-ideal of R is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R.*

Proof. Let μ be a fuzzy weakly completely primary Γ-ideal of R. Then for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$, $\mu(x\alpha y\beta z) \leq \mu(x)$ or $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(y)$ or $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(z)$. Suppose that $\mu(x\alpha y\beta z) \leq \mu(x)$. Since μ is a fuzzy $\Gamma-$ideal of R, then $\mu(x\alpha y\beta z) \leq \mu(x) \leq \mu(x\alpha y)$, and we get $\mu(x\alpha y\beta z) \leq \mu(x\alpha y)$.

If $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(y)$, then since $\sqrt{\mu}$ is a fuzzy Γ-ideal of R, we have $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(y) \leq \sqrt{\mu}(y\beta z)$, and we get $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(y\beta z)$, or if $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(z)$,then $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(z) \leq \sqrt{\mu}(x\beta z)$, and we get $\mu(x\alpha y\beta z) \leq \sqrt{\mu}(x\beta z)$.

Hence, μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R. □

Lemma 4.9. *Let μ be a fuzzy Γ-ideal of R and $a \in [0, \mu(0)]$. Then μ is a fuzzy weakly completely 2-absorbing Γ-ideal of R if and only if μ_a is a 2-absorbing Γ- ideal of R.*

Theorem 4.10. *Let μ be a fuzzy $\Gamma-$ideal of R. The following statements are equivalent:*

1. *μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R.*
2. *For every $a \in [0, \mu(0)]$, the $a-$level subset μ_a of μ is a 2-absorbing primary Γ-ideal of R.*

Proof. $(1) \Rightarrow (2)$. Suppose that μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R. Let $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $x\alpha y\beta z \in \mu_a$ for some $a \in [0, \mu(0)]$. Then

$$\max\left\{\mu(x\alpha y), \sqrt{\mu}(x\beta z), \sqrt{\mu}(y\beta z)\right\} = \mu(x\alpha y\beta z) \geq a.$$

Hence $\mu(x\alpha y) \geq a$ or $\sqrt{\mu}(x\beta z) \geq a$ or $\sqrt{\mu}(y\beta z) \geq a$, which implies that

$$x\alpha y \in \mu_a \text{ or } x\beta z \in \sqrt{\mu}_a = \sqrt{\mu_a} \text{ or } y\beta z \in \sqrt{\mu}_a = \sqrt{\mu_a}.$$

Thus, μ_a is a 2-absorbing primary Γ-ideal of R.

(2) ⇒ (1). Assume that μ_a is a 2-absorbing primary Γ-ideal of R for every $a \in [0,1]$. Let $\mu(x\alpha y\beta z) = a$ for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then $x\alpha y\beta z \in \mu_a$ and μ_a is a 2-absorbing primary Γ-ideal. Thus it gives

$$x\alpha y \in \mu_a \text{ or } x\beta z \in \sqrt{\mu_a} \text{ or } y\beta z \in \sqrt{\mu_a}.$$

Hence $\mu(x\alpha y) \geq a$ or $\sqrt{\mu}(x\beta z) \geq a$ or $\sqrt{\mu}(y\beta z) \geq a$, which implies that

$$\mu(x\alpha y) \geq a = \mu(x\alpha y\beta z) \text{ or } \sqrt{\mu}(x\beta z) \geq a = \mu(x\alpha y\beta z) \text{ or } \sqrt{\mu}(y\beta z) \geq a = \mu(x\alpha y\beta z).$$

Therefore, μ is a fuzzy weakly completely 2-absorbing primary Γ- ideal of R. □

Theorem 4.11. *If μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R, then $\sqrt{\mu}$ is a fuzzy weakly completely 2-absorbing Γ- ideal of R.*

Proof. If μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R, then by the previous theorem, we get that μ_a is a 2-absorbing primary Γ-ideal of R for any $a \in [0, \mu(0)]$. Since μ_a is 2-absorbing primary Γ−ideal of R, then $\sqrt{\mu_a} = \sqrt{\mu}_a$ is a 2-absorbing Γ-ideal of R. From the previous lemma, since $\sqrt{\mu}_a$ is a 2-absorbing Γ-ideal of R, we get that $\sqrt{\mu}$ is a fuzzy weakly completely 2-absorbing Γ-ideal of R. Hence we see that $\sqrt{\mu}$ is a fuzzy weakly completely 2-absorbing Γ-ideal of R. □

Theorem 4.12. *Let $f : R \to S$ be a surjective Γ-ring homomorphism. If μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R which is constant on Ker$f$, then $f(\mu)$ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of S.*

Proof. Suppose that $f(\mu)(x\alpha y\beta z) > f(\mu)(x\alpha y)$ for any $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Since f is a surjective Γ-ring homomorphism, then

$$f(a) = x,\ f(b) = y,\ f(c) = z \text{ for some } a, b, c \in R.$$

Hence

$$\begin{aligned} f(\mu)(x\alpha y\beta z) &= f(\mu)(f(a)\alpha f(b)\beta f(c)) = f(\mu)(f(a\alpha b\beta c)) \\ &\neq f(\mu)(x\alpha y) = f(\mu)(f(a)\alpha f(b)) = f(\mu)(f(a\alpha b)). \end{aligned}$$

Since μ is constant on Ker$f$,

$$\begin{aligned} f(\mu)(f(a\alpha b\beta c)) &= \mu(a\alpha b\beta c) \text{ and} \\ f(\mu)(f(a\alpha b)) &= \mu(a\alpha b). \end{aligned}$$

It means that

$$f(\mu)(f(a\alpha b\beta c)) = \mu(a\alpha b\beta c) > \mu(a\alpha b) = f(\mu)(f(a\alpha b)).$$

Since μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R, we have that

$$\begin{aligned} \mu(a\alpha b\beta c) &= f(\mu)(f(a)\alpha f(b)\beta f(c)) = f(\mu)(x\alpha y\beta z) \\ &\leq \sqrt{\mu}(a\beta c) = f\left(\sqrt{\mu}\right)(f(a\beta c)) = f\left(\sqrt{\mu}\right)(f(a)\beta f(c)) = f\left(\sqrt{\mu}\right)(x\beta z) \end{aligned}$$

so, we get $f(\mu)(x\alpha y\beta z) \leq f\left(\sqrt{\mu}\right)(x\beta z) = \sqrt{f(\mu)}(x\beta z)$ or

$$\begin{aligned} \mu(a\alpha b\beta c) &= f(\mu)(f(a)\alpha f(b)\beta f(c)) = f(\mu)(x\alpha y\beta z) \\ &= \sqrt{\mu}(b\beta c) = f\left(\sqrt{\mu}\right)(f(b\beta c)) = f\left(\sqrt{\mu}\right)(f(b)\beta f(c)) = f\left(\sqrt{\mu}\right)(y\beta z) \end{aligned}$$

and we have $f(\mu)(x\alpha y\beta z) \leq f\left(\sqrt{\mu}\right)(y\beta z) = \sqrt{f(\mu)}(y\beta z)$. Thus, $f(\mu)$ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of S. □

Theorem 4.13. *Let $f : R \to S$ be a Γ-ring homomorphism. If v is a fuzzy weakly completely 2-absorbing primary Γ-ideal of S, then $f^{-1}(v)$ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R.*

Proof. Suppose that $f^{-1}(v)(x\alpha y\beta z) > f^{-1}(v)(x\alpha y)$ for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} f^{-1}(v)(x\alpha y\beta z) &= v(f(x\alpha y\beta z)) = v(f(x)\alpha f(y)\beta f(z)) \\ &> f^{-1}(v)(x\alpha y) = v(f(x\alpha y)) = v(f(x)\alpha f(y)). \end{aligned}$$

Since v is a fuzzy weakly completely 2-absorbing primary Γ-ideal of S, we have that

$$\begin{aligned} f^{-1}(v)(x\alpha y\beta z) &= v(f(x)\alpha f(y)\beta f(z)) \\ &\leq \sqrt{v}(f(x)\beta f(z)) = \sqrt{v}(f(x\beta z)) \\ &= f^{-1}\left(\sqrt{v}\right)(x\beta z) \\ &= \sqrt{f^{-1}(v)}(x\beta z) \end{aligned}$$

or

$$\begin{aligned} f^{-1}(v)(x\alpha y\beta z) &= v(f(x)\alpha f(y)\beta f(z)) \\ &\leq \sqrt{v}(f(y)\beta f(z)) = \sqrt{v}(f(y\beta z)) \\ &= f^{-1}\left(\sqrt{v}\right)(y\beta z) \\ &= \sqrt{f^{-1}(v)}(y\beta z). \end{aligned}$$

Thus $f^{-1}(v)$ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R. □

Corollary 4.14. *Let f be a Γ-ring homomorphism from R onto S. f induces a one-to-one inclusion preserving correspondence between fuzzy weakly completely 2-absorbing primary Γ-ideals of S in such a way that if μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R constant on $Ker f$, then $f(\mu)$ is the corresponding fuzzy weakly completely 2-absorbing primary Γ-ideal of S, and if v is a fuzzy weakly completely 2-absorbing primary Γ-ideal of S, then $f^{-1}(v)$ is the corresponding fuzzy weakly completely 2-absorbing primary Γ-ideal of R.*

5. Fuzzy K-2-absorbing primary Γ-ideals

Let μ be a fuzzy Γ-ideal of R. μ is said to be a *fuzzy K-Γ-ideal* of R if for $x, y \in R$ and $\alpha, \beta \in \Gamma$

$$\mu(x\alpha y) = \mu(0) \text{ implies that } \mu(x) = \mu(0) \text{ or } \mu(y) = \mu(0)$$

and μ is called a *fuzzy K-primary Γ-ideal* of R if

$$\mu(x\alpha y) = \mu(0) \text{ implies that } \mu(x) = \mu(0) \text{ or } \sqrt{\mu}(y) = \mu(0).$$

Also, μ is called a *fuzzy K-2-absorbing Γ-ideal* of R if

$$\mu(x\alpha y\beta z) = \mu(0) \text{ implies that } \mu(x\alpha y) = \mu(0) \text{ or } \mu(x\beta z) = \mu(0) \text{ or } \mu(y\beta z) = \mu(0)$$

for $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Now, we give the definition of fuzzy K-2-absorbing primary Γ-ideal of R.

Definition 5.1. *Let μ be a fuzzy Γ-ideal of R. μ is called a fuzzy K-2-absorbing primary Γ-ideal of R if for $x, y, z \in R$ and $\alpha, \beta \in \Gamma$*

$$\mu(x\alpha y\beta z) = \mu(0) \textit{ implies that } \mu(x\alpha y) = \mu(0) \textit{ or } \sqrt{\mu}(x\beta z) = \mu(0) \textit{ or } \sqrt{\mu}(y\beta z) = \mu(0).$$

Theorem 5.2. *Every fuzzy weakly completely 2-absorbing primary Γ-ideal of R is a fuzzy K-2-absorbing primary Γ-ideal of R.*

Proof. Suppose that μ is a fuzzy weakly completely 2-absorbing primary Γ-ideal of R. If $\mu(x\alpha y\beta z) = \mu(0)$ for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$, then since μ is a fuzzy weakly completely 2-absorbing primary $\Gamma-$ideal of R, we have

$$\begin{aligned}\mu(0) &\geq \mu(x\alpha y) \geq \mu(x\alpha y\beta z) = \mu(0) \text{ or}\\ \mu(0) &\geq \sqrt{\mu}(x\beta z) \geq \mu(x\alpha y\beta z) = \mu(0) \text{ or}\\ \mu(0) &\geq \sqrt{\mu}(y\beta z) \geq \mu(x\alpha y\beta z) = \mu(0).\end{aligned}$$

Then,

$$\mu(x\alpha y) = \mu(0) \text{ or } \sqrt{\mu}(x\beta z) = \mu(0) \text{ or } \sqrt{\mu}(y\beta z) = \mu(0).$$

Therefore μ is a fuzzy K-2-absorbing primary Γ-ideal of R. □

Theorem 5.3. *Every fuzzy K-primary Γ-ideal of R is a fuzzy K-2-absorbing primary Γ-ideal of R.*

Proof. Let μ be a fuzzy K-primary Γ-ideal of R. Then for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,

$$\mu(x\alpha y\beta z) = \mu(0) \text{ implies that } \mu(x) = \mu(0) \text{ or } \sqrt{\mu}(y) = \mu(0) \text{ or } \sqrt{\mu}(z) = \mu(0).$$

Suppose that $\mu(x) = \mu(0)$. Then from

$$\mu(0) = \mu(x\alpha y\beta z) \geq \mu(x\alpha y) \geq \mu(x) = \mu(0),$$

we get $\mu(x\alpha y) = \mu(0)$. If $\sqrt{\mu}(y) = \mu(0)$, then since μ is a fuzzy K-primary Γ-ideal of R, we have

$$\mu(0) = \mu(x\alpha y\beta z) \geq \sqrt{\mu}(y\beta z) \geq \sqrt{\mu}(y) = \mu(0).$$

Thus, $\sqrt{\mu}(y\beta z) = \mu(0)$ or if $\sqrt{\mu}(z) = \mu(0)$. Then

$$\mu(0) = \mu(x\alpha y\beta z) \geq \sqrt{\mu}(x\beta z) \geq \sqrt{\mu}(z) = \mu(0)$$

and we get $\sqrt{\mu}(x\beta z) = \mu(0)$. We conclude that μ is a fuzzy K-2-absorbing primary Γ-ideal of R. □

Theorem 5.4. *Every fuzzy K-2-absorbing Γ-ideal of R is a fuzzy K-2-absorbing primary Γ-ideal of R.*

Proof. The proof is obvious. □

Theorem 5.5. *Let $f : R \to S$ be a surjective Γ-ring homomorphism. If μ is a fuzzy K-2-absorbing primary Γ-ideal of R which is constant on $\mathrm{Ker}f$, then $f(\mu)$ is a fuzzy K-2-absorbing primary Γ-ideal of S.*

Proof. Suppose that $f(\mu)(a\alpha b\beta c) = f(\mu)(0_S)$ for any $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Since f is a surjective Γ-ring homomorphism, then $f(x) = a$, $f(y) = b$, $f(z) = c$ for some $x, y, z \in R$. Hence

$$\begin{aligned}f(\mu)(a\alpha b\beta c) &= f(\mu)(f(x)\alpha f(y)\beta f(z))\\ &= f(\mu)(f(x\alpha y\beta z))\end{aligned}$$

and

$$f(\mu)(0_S) = \vee\{\mu(x) : f(x) = 0_S\}.$$

Thus we have $x \in \mathrm{Ker}f$ and so μ is constant on $\mathrm{Ker}f$, $\mu(x) = \mu(0)$

$$f(\mu)(0_S) = \vee\{\mu(x) : \mu(x) = \mu(0)\}.$$

Therefore we get

$$f(\mu)(f(x\alpha y\beta z)) = \mu(x\alpha y\beta z) = \mu(0).$$

Since μ is a fuzzy K-2-absorbing primary Γ-ideal of R,

$$\mu(x\alpha y\beta z) = \mu(0) \text{ implies that } \mu(x\alpha y) = \mu(0) \text{ or } \sqrt{\mu}(x\beta z) = \mu(0) \text{ or } \sqrt{\mu}(y\beta z) = \mu(0).$$

Then the rest of the proof can easily be made similar to the proof of the previous theorems and we can see that $f(\mu)$ is a fuzzy K-2-absorbing primary Γ-ideal of S. □

Theorem 5.6. *Let $f : R \to S$ be a Γ-ring homomorphism. If v is a fuzzy K-2-absorbing primary Γ-ideal of S, then $f^{-1}(v)$ is a fuzzy K-2-absorbing primary Γ-ideal of R.*

Proof. Assume that $f^{-1}(v)(x\alpha y\beta z) = f^{-1}(v)(0)$ for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then from

$$\begin{aligned} f^{-1}(v)(x\alpha y\beta z) &= v(f(x\alpha y\beta z)) = v(f(x)\alpha f(y)\beta f(z)) \\ &= f^{-1}(v)(0) = v(f(0)) = v(0), \end{aligned}$$

we have $v(f(x)\alpha f(y)\beta f(z)) = v(0)$. Since v is a fuzzy K-2-absorbing primary Γ-ideal of S, then we get

$$\begin{aligned} v(f(x)\alpha f(y)\beta f(z)) &= v(0) \text{ implies that} \\ v(f(x)\alpha f(y)) &= v(0) \text{ or } \sqrt{v}(f(x)\beta f(z)) = v(0) \text{ or } \sqrt{v}(f(y)\beta f(z)) = v(0). \end{aligned}$$

From this, we get

$$\begin{aligned} v(f(x)\alpha f(y)) &= v(f(x\alpha y)) = f^{-1}(v)(x\alpha y) \\ &= v(0) = v(f(0)) = f^{-1}(v)(0) \\ f^{-1}(v)(x\alpha y) &= f^{-1}(v)(0) \end{aligned}$$

or

$$\begin{aligned} \sqrt{v}(f(x)\beta f(z)) &= \sqrt{v}(f(x\beta z)) = f^{-1}\left(\sqrt{v}\right)(x\beta z) \\ v(0) &= v(f(0)) = f^{-1}(v)(0) \\ f^{-1}\left(\sqrt{v}\right)(x\beta z) &= f^{-1}(v)(0) \end{aligned}$$

or

$$\begin{aligned} \sqrt{v}(f(y)\beta f(z)) &= \sqrt{v}(f(y\beta z)) = f^{-1}\left(\sqrt{v}\right)(y\beta z) \\ v(0) &= v(f(0)) = f^{-1}(v)(0) \\ f^{-1}\left(\sqrt{v}\right)(y\beta z) &= f^{-1}(v)(0). \end{aligned}$$

Hence $f^{-1}(v)$ is fuzzy K-2-absorbing primary Γ-ideal of R. □

Corollary 5.7. *Let f be a Γ-ring homomorphism from R onto S. f induces a one-to-one inclusion preserving correspondence between fuzzy K-2-absorbing primary Γ-ideals of S in such a way that if μ is a fuzzy $K-2$-absorbing primary Γ-ideal of R constant on $Ker f$, then $f(\mu)$ is the corresponding fuzzy K-2-absorbing primary Γ-ideal of S, and if v is a fuzzy K-2-absorbing primary Γ-ideal of S, then $f^{-1}(v)$ is the corresponding fuzzy K-2-absorbing primary Γ-ideal of R.*

Remark 5.8. *The following table summarizes findings of fuzzy 2-absorbing primary Γ-ideals.*

$$\begin{array}{ccccc} & & \text{f. primary } \Gamma-\text{i.} & \rightarrow & \text{f. str. } 2-\text{abs. primary } \Gamma-\text{i.} \\ & \nearrow & & & \downarrow \\ \text{f. p. } \Gamma-\text{i.} & \rightarrow & \text{f. } 2-\text{abs. } \Gamma-\text{i.} & \rightarrow & \text{f. } 2-\text{abs. primary } \Gamma-\text{i.} \\ \downarrow & & \downarrow & & \downarrow \\ \text{f. w.c. p. } \Gamma-\text{i.} & \rightarrow & \text{f. w.c. } 2-\text{abs. } \Gamma-\text{i.} & \rightarrow & \text{f. w.c. } 2-\text{abs. primary } \Gamma-\text{i.} \\ \downarrow & & \downarrow & & \downarrow \\ \text{f. K}-\Gamma-\text{i.} & \rightarrow & \text{f. K}-2-\text{abs. } \Gamma-\text{i.} & \rightarrow & \text{f. K}-2-\text{abs. primary } \Gamma-\text{i.} \\ & \searrow & & \nearrow & \\ & & \text{f. K}-\text{primary } \Gamma-\text{i.} & & \end{array}$$

6. Conclusion

In this paper, the theoretical point of view of fuzzy 2-absorbing primary Γ-ideals in a Γ-ring was discussed. The work was focused on fuzzy 2-absorbing primary Γ-ideals, fuzzy weakly completely 2-absorbing primary Γ-ideals and fuzzy K-2-absorbing primary Γ-ideals of a Γ-ring and their properties were investigated. Finally, we have given a diagram in which transition between definitions of fuzzy 2-absorbing Γ-ideals of a Γ-ring are presented. These concepts are basic structures for improvement of fuzzy primary Γ-ideals in a Γ-ring. In the future, one could investigate intuitionistic fuzzy 2-absorbing primary Γ-ideals and intuitionistic fuzzy weakly completely 2-absorbing primary Γ-ideals in a Γ-ring.

References

[1] D.F. Anderson and A. Badawi, On n-absorbing ideals of commutavie rings, Comm. Algebra **39** (2011), 1646-1672.
[2] A. Badawi, On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc. **75** (2007), 417-429.
[3] A. Badawi and A.Y. Darani, On weakly 2-absorbing ideals of commutative rings. Houston J. Math. **39** (2013), 441-452.
[4] A. Badawi, U. Tekir, E. Yetkin, On 2-absorbing primary ideals in commutative rings, Bull. Austral. Math. Soc. **51** (2014), no. 4, 1163-1173.
[5] A. Badawi, U. Tekir, E. Yetkin, On 2-absorbing primary ideals in commutative rings, Bull. Austral. Math. Soc. **51** (2014), no. 4, 1163-1173.
[6] W.E. Barnes, On the Γ-rings of Nobusawa, Pacific J.Math. **18** (1966), 411-422.
[7] A.Y. Darani and E.R. Puczylowski, On 2-absorbing commutative semigroups and their applications to rings, Semigroup Forum **86**(2013), 83-91.
[8] A.Y. Darani and A. Hashempoor, $L-$fuzzy 0-(1- or 2- or 3-) 2-absorbing ideals in semiring, Annals of Fuzzy Math. and Inform. **7** (2014), no. 2, 303-311.
[9] A.Y. Darani, On $L-$fuzzy 2-absorbing ideals, Ital. J. Pure Appl. Math. 36 (2016), 147-154.
[10] T.K. Dutta and T. Chanda, Structures of fuzzy ideals of Γ-ring, Bull. Malays. Math. Sci. Soc. (2) **28**(2005), no. 1, 9-18.
[11] T.K. Dutta and T. Chanda, Fuzzy prime ideals in Γ-rings, Bull. Malays. Math. Sci. Soc. **30** (2007), 65-73.
[12] M.Y. Elkettani and A. Kasem, On 2-absorbing $\delta-$primary gamma ideal of gamma ring, Int. J. Pure Appl. Math. **106**(2016), n0. 2, 543-550.
[13] Hoskova-Mayerova, S., Maturo, A., Fuzzy Sets and Algebraic Hyperoperations to Model Interpersonal Relations. Switzerland: Springer, Recent Trends in Social Systems: Quantitative Theories and Quantitative Models. Studies in Systems, Decision and Control **66** (2006), 211-223.
[14] Hoskova, S., Chvalina J. A survey of investigations of the Brno reseach group in the hyperstructure theory since the last AHA Congress, AHA 2008, (2009), 71-84.
[15] Y.B. Jun and C.Y. Lee, Fuzzy Γ-rings, Pusan Kyongnam Math. J. (presently, East Asian Math. J.) **8**(1992), no. 2, 163-170.
[16] Y.B. Jun, On fuzzy prime ideals of Γ-rings, Soochow J. Math. **21**(1995), no. 1, 41-48.
[17] P. Kumar, M.K. Dubey and P. Sarohe, Some results on 2-absorbing ideals in commutative semirings, J. Math. Appl. **38** (2015), 77-84.
[18] T. Kuraoka and T. Kuroki, On fuzzy quotient rings induced by fuzzy ideals, Fuzzy Sets Syst. **47** (1992), 381-386.
[19] S. Kyuno, On prime gamma rings, Pacific J. Math. **75**(1978), no. 1, 185-190.
[20] S. Kyuno, A gamma ring with the right and left unities, Math. Jopanica **24**(1979), n0. 2, 191-193.
[21] S. Kyuno, Prime ideals in gamma rings, Pacific J. Math. **98** (1982), no. 2, 375-379.
[22] W.J. Liu, Operation on fuzzy ideals, Fuzzy Sets Syst. **11** (1983), 31-41.
[23] J. Luh, On the theory of simple $\Gamma-$rings, Michigan Math. J. **16** (1969), 65-75.
[24] D.S. Malik and J. N. Mordeson, Fuzzy maximal, radical and primary ideals of a ring, Inf. Sci. **53**(1991), no. 3, 237-250.
[25] D.S. Malik and J.N. Mordeson, Fuzzy Commutative Algebra, World Scientific Publishing, (1998), 424 pages; ISBN 9789814495592.
[26] T.K. Mukherjee and M.K.Sen, Prime fuzzy ideals in rings, Fuzzy Sets Syst. **32** (1989),345-350.
[27] T.K. Mukherjee and M.K. Sen, Primary fuzzy ideals and radical of fuzzy ideals, Fuzzy Sets Syst. **56** (1993), 97-101.
[28] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math. **1** (1964), 81-89.
[29] M.A. Öztürk, M. Uckun and Y. B. Jun, Fuzzy ideals in gamma-rings. Turk. J. Math. **27** (2003), No. 3, 369-374.
[30] M.A. Öztürk, M. Uckun and Y. B. Jun, Characterizations of Artinian and Noetherian gamma-rings in terms of fuzzy ideals. Turk. J. Math. **26** (2002), no. 2, 199-205.
[31] A. Rosenfeld, Fuzzy groups, J. Math Anal. Appl. **35** (1971), 512-517.
[32] L.I. Sidky , S.A. Khatab, Nil radical of fuzzy ideal, Fuzzy Sets Syst. **47**(1992), no. 1, 117-120.
[33] L.A. Zadeh, Fuzzy sets, Informat. Control **8** (1965), 338-353.