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Hausdorff Dimension of the Nondifferentiability Set of a Convex Function

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Abstract. We find an upper bound for the Hausdorff dimension of the nondifferentiability set of a continuous convex function defined on a Riemannian manifold. As an application, we show that the boundary of a convex open subset of R^n, $n \geq 2$, has Hausdorff dimension at most $n-2$.

1. Introduction

There are many examples of nowhere differentiable continuous functions defined on a differentiable manifold M. In fact, the set of this kind of functions is very big in some points of view. For example, it is proved in [4] that if M is a compact differentiable manifold, then typical elements of the set of continuous functions defined on M are nowhere differentiable. If we impose some important conditions such convexity or Lipschitz condition on a continuous function f, then f is not nowhere differentiable. So, it is natural to ask: how big can be the set of nondifferentiability points of f. The set of nondifferentiability points of a directionally differentiable Lipschitz-function f defined on R^n is σ-porous (see [1]). Thus, it can be included in a countable union of sets E_i with the property that for all $x \in E_i$ and all $0 < r < 1$, there exists $0 < \delta_i(x) < \frac{1}{2}$ such that a ball $B(y, \delta r)$ is included in $B(x, r) - E_i$. This argument also implies the best Hausdorff dimension estimate, $n-1$, for the set of nondifferentiability.

Also, one can find some measure theoric characterizations of the magnitude of the sets of nondifferentiability points of convex functions defined on R^n (see [5]). In the present paper, by a preliminary proof, we give an upper bound estimate for the Hausdorff dimension of the set of nondifferentiability points of convex functions defined on R^n, then we generalize it to the convex functions defined on Riemannian manifolds.

2. Results

We will use the following definitions and facts in the proof of our theorems.

(a) A continuous function $f : R^n \to R$ is called convex if for all $x, y \in R^n$

$$af(x) + (1-a)f(y) \leq f(ax + (1-a)y), \quad 0 \leq a \leq 1.$$

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and it is called concave if for all $x, y \in R^n$

$$af(x) + (1 - a)f(y) \geq f(ax + (1 - a)y), \quad 0 \leq a \leq 1.$$

(b) Let X be a metric space. If $A \subset X$ and $s \in [0, \infty)$, we put

$$H^s_\delta(A) = \inf\left\{ \sum_i r_i^s : \text{ there is a cover of A by balls of radius } 0 < r_i \leq \delta \right\}.$$

The following limit which exits (see [3]), is called the s-dimensional Hausdorff content of A.

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta(A).$$

The Hausdorff dimension of A is defined by

$$\dim_H(A) = \inf\{s : H^s(A) = 0\}.$$

Fact 2.1. Let B be the collection of all line segments in R^2 which have rational coordinates at the end points. If L is a line segment in R^2, it is clear that there is a line segment in B which cuts L. So, it is not hard to show that each line segment in R^3 cuts a triangle in R^3 with vertices having rational coordinates. In more general case, by using induction, we can show that each line segment in R^n cuts an $(n - 1)$-simplex with vertices having rational coordinates.

Fact 2.2. If $f : R \to R$ is a convex or concave continuous function, then the set of points where f is not differentiable is at most a countable set, so its Hausdorff dimension is zero.

The following theorem is a generalization of this fact.

Theorem 2.3. *The Hausdorff dimension of the set of nondifferentiability points of a convex or concave function $f : R^n \to R$ is at most $n - 1$.*

Proof. The ideas of the proof comes from the proof of Theorem 12.3 in [3]. We give the proof for convex functions. The other case is similar. Without lose of generality, we suppose that f is positive function. Consider the graph of f, $\mathcal{G}_f = \{(x, f(x)) : x \in R^n\} \subset R^{n+1}$. For each $x \in R^n$, let $g(x)$ be the point in $\mathcal{G}_f$ which the distance between $(x, 0) \in R^n \times R = R^{n+1}$ and $g(x)$ is least. We get from convexity of f that the map $g : R^n \to \mathcal{G}_f$ is well defined. Given $(y, f(y)) \in \mathcal{G}_f$, let T_y be a hyperplane in R^{n+1} which is tangent to $\mathcal{G}_f$ at $(y, f(y))$, and let L_y be the line in R^{n+1} which is perpendicular to T_y at $(y, f(y))$. Clearly, if $(x, 0) = L_y \cap (R^n \times \{0\})$, then $g(x) = (y, f(y))$. If f is not differentiable at y, then there are infinitely many hyperplanes tangent to $\mathcal{G}_f$ at $(y, f(y))$ and infinitely many Ly such that intersection of these lines L_y with $R^n \times \{0\}$ at least contains a line segment in $R^n \times \{0\} \simeq R^n$. Put
$A = \{(y, f(y)) : f \text{ is not differentiable at } y\}$
and
B= the union of all $(n - 1)$-simplexes in R^n with vertices having rational coordinates.
Since for each point $(y, f(y)) \in (A)$, the set $g^{-1}((y, f(y)))$ contains a line segment, then it intersects at least one $(n - 1)$-simplex in R^n with vertices having rational coordinates. So, $A \subset g(B)$. We can show that $d(g(x), g(y)) \leq d(x, y), x, y \in R^n$. Thus, $\dim_H(A) \leq \dim_H g(B) \leq \dim_H(B)$. Since $\dim_H(B) = n - 1$, then $\dim_H(A) \leq n - 1$. Now, consider the function $F : R^n \to R^{n+1}$, defined by $F(x) = (x, f(x))$. The points of R^n where F is not differentiable is equal to the set of points where f is not differentiable, and this set is mapped by F to A. Since $d(F(x), F(y)) \geq d(x, y)$, then the theorem is proved. □

Example 2.4. An easy example of a convex function with infinite set of nondifferentiability points is the function $h : [0, 1] \to R$ defined by $h(x) = \sum_{n=1}^{\infty} 2^{-n}|x - \frac{1}{n}|$. h can be extended to a convex function $g : R \to R$ in such a way that g be differentiable on $R - [0, 1]$. The set of nondifferentiability points of g, which we denote it by $\mathcal{A}_g$, is countable and $\dim_H(\mathcal{A}_g) = 0$. Put $f : R^n \to R$, $f(x_1, ..., x_n) = g(x_1)$. Clearly, f is convex and $\mathcal{A}_f$, the set of nondifferentiability points of f, is equal to $\mathcal{A}_g \times R^{n-1}$. Thus, $\dim_H \mathcal{A}_f = n - 1$.

Theorem 2.5. *If M is a submanifold of R^{n+1} contained in the boundary of a convex open subset of R^{n+1}, then the Hausdorff dimension of nondifferentiable points of M is at most $n-1$.*

Proof. We show that M is locally isometric to the graph of a convex function. Then, by Theorem 2.3, we get the result. Let D be an open convex subset of R^{n+1} such that $M = \partial D$ and let $a \in M$. Consider an open subset W of a hyperplane in R^{n+1} with the following properties:

(1) $W \subset D$;

(2) There is a unit vector V perpendicular to W at a point y_0, such that the half line $y_0 + tV$, $t \geq 0$, contains a, and for all $y \in W$, the half line $y + tV$, $t \geq 0$, intersects M.

Let M_1 be the set of points of M belonging to the mentioned half lines. Clearly M_1 is open set in M containing a.

Given $x \in W$, let $\tau(x)$ be the intersection point of the half line $x + tV$, $t \geq 0$, and M_1. Consider the function $f : W \to R$, $f(x) = |\tau(x) - x|$. It is sufficient to prove the following assertions:

(1) f is well defined;

(2) f is convex;

(3) graph$(f) \subset W \times R$ is isometric to $M_1 \subset R^{n+1}$.

(1): Consider a point $x \in W$. We show that the intersection point of the half line $L = \{x + tV, t \geq 0\}$ and M_1 is unique. Then, f will be well defined. Let y_1 and y_2 be two different points belonging to $L \cap M_1$. Let one of the points y_1, y_2, say y_1, is contained between the points x and y_2 on the half line L. Since y_2 belongs to the boundary of D and D is open, by an small rotation of the line segment xy_2 around the point y_1, we get a line segment $x'y_2'$ with $x', y_2' \in D$. D is convex, so $y_1 \in D$ and we have a contradiction.

(2): Let $0 \leq \lambda \leq 1$ and $x, y \in D$. Note that if $x + sV = \tau(x)$ then the half open line segment $\{x + tV : 0 \leq t < s\}$ is included in D.

We show that

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \qquad (1)$$

Let $\tau(x) = x + s_1 V, \tau(y) = y + s_2 V$. Then, for any positive number $\epsilon < min\{s_1, s_2\}$, $x + (s_1 - \epsilon)V, y + (s_2 - \epsilon)V \in D$. Thus, by convexity of D,

$$A_\epsilon = \lambda(x + (s_1 - \epsilon)V) + (1-\lambda)(y + (s_2 - \epsilon)V) \in D \qquad (2)$$

Put $B = \lambda\tau(x) + (1-\lambda)\tau(y)$, $C_1 = \lambda x + (1-\lambda)y$ and $C = \tau(C_1) = C_1 + s_3 V$. It is an easy computation to show that B (as like as C) belongs to the half line $\{C_1 + tV : t \geq 0\}$. Let $B = C_1 + s_4 V$. Since $\lim_{\epsilon\to 0} A_\epsilon = B$, then $B \in \overline{D}$, so $s_4 \leq s_3$.

Now, we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f(C_1) = |\tau(C_1) - C_1| = s_3|V| \geq s_4|V| \\ &= |B - C_1| = |\lambda\tau(x) + (1-\lambda)\tau(y) - (\lambda x + (1-\lambda)y)| \\ &= |\lambda(\tau(x) - x) + (1-\lambda)(\tau(y) - y)|. \end{aligned}$$

Since the vectors $\tau(x) - x$ and $\tau(y) - y$ in R^{n+1} are both perpendicular to the hyperplane W, then

$$\begin{aligned} |\lambda(\tau(x) - x) + (1-\lambda)(\tau(y) - y)| &= |\lambda(\tau(x) - x)| + |(1-\lambda)(\tau(y) - y)| \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

Thus,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y).$$

(3) Define the map $\psi : \text{graph}(f) \subset W \times R \to M_1 \subset R^{n+1}$ by

$$\psi(x, f(x)) = x + |\tau(x) - x|V.$$

Clearly, ψ is one to one and onto, and we have:

$$d^2(\psi(x, f(x)), \psi(y, f(y))) = |\psi(x, f(x)) - \psi(y, f(y))|^2$$

$$= |(x + |\tau(x) - x|V) - (y + |\tau(y) - y|V)|^2 \quad (\star)$$

Since x, y belong to the hyperplane W, and V is perpendicular to W, then $(\star)$ will be equal to

$$|(x - y)|^2 + |(x - \tau(x)) - (y - \tau(y))|^2 = |x - y|^2 + |f(x) - f(y)|^2$$

$$= d^2((x, f(x)), (y, f(y))).$$

Thus, ψ is an isometry. □

Example 2.6. Let Δ be a triangle in R^2 with vertices A, B, C, and Δ^o be its interior. Put $M = \Delta \times R^{n-1}$. M is boundary of the convex set $\Delta^o \times R^{n-1} \subset R^{n+1}$. The set of nondifferentiability points of M is equal to $\{A, B, C\} \times R^{n-1}$, which is of Hausdorff dimension $n - 1$.

Remark 2.7. If M is a Riemannian manifold, a function $f : M \to R$ is called convex if for each geodesic $\gamma : I \to M$, the function $f \circ \gamma : I \to R$ is convex.

Theorem 2.8. *If M is a complete Riemannian manifold and $f : M \to R$ is a convex function, then the Hausdorff dimension of nondifferentiability set of f is at most* $\dim M - 1$.

Proof. It is sufficient to show that each point $a \in M$ has an open neighborhood W such that the theorem is true for the function $f : W \to R$. By Nash's embedding theorem, M can be considered as a Riemannian submanifold of R^n for some $n > \dim M$. Given $a \in M$, consider an open set W in M around a with compact closure $\overline{W}$. There exists a tube $U = U(W, r) = \{x \in R^n : d(x, W) < r\}$ of radius r around W in R^n, with the property that for each $x \in U$, there exists only one point $x_w \in W$ such that

$$d(x, W) = d(x, x_w) \quad (\star)$$

Now, consider the following function which is an extension of f to U

$$F : U \subset R^n \to R, \;\; F(x) = f(x_w).$$

We show that F is a convex function.

Let $x, y \in U$ and $0 \leq \lambda \leq 1$. Put $z = \lambda x + (1 - \lambda)y$. Consider the points x_M, y_M, z_W in W with the property $(\star)$. Let α, β be geodesics in W such that $\alpha(0) = x_w$, $\alpha(1) = z_w = \beta(1)$ and $\beta(0) = y_w$.

Consider the points $x_s = \alpha(s)$ and $y_s = \beta(s)$, $0 < s < 1$, on α and β, close to z_w such that there is a minimizing geodesic $\gamma_s : [0, 1] \to W$ joining x_s to y_s. Since f is convex on W, then

$$f(\gamma_s(\lambda)) \leq \lambda f(\alpha(s)) + (1 - \lambda) f(\beta(s)) \quad (1)$$

$$f(\alpha(s)) \leq s f(x_W) + (1 - s) f(z_W) \quad (2)$$

$$f(\beta(s)) \leq (1 - s) f(z_M) + s f(y_W) \quad (3)$$

If $s \to 1$ then $\gamma_s(\lambda) \to z_w$, so if we let $s \to 1$ in (1), (2) and (3), then we get

$$f(z_w) \leq \lambda f(x_w) + (1 - \lambda) f(y_w).$$

Therefore,

$$F(z) \leq \lambda F(x) + (1 - \lambda) F(y).$$

This means that F is convex. Since U is open in R^n, we get from Theorem 2.3, that the dimension of the set of nondifferentiability points of F is at most $n-1$. Consider a point $x \in U$. If f is not differentiable at x_w then F is nondifferentiable along the line segment xx_w. So, the Hausdorff dimension of the set of nondifferentiability points of f is less than or equal to $\dim M - 1$(because, if the dimension of nondifferntiability set of f is bigger than $\dim M - 1$, then the Hausdorff dimension of the set of nondifferentiability points of F must be bigger than $\dim M - 1 + (n - \dim M) = n - 1$, which is contradiction). □

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