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Sufficient and Necessary Conditions of Stochastic Permanence and Extinction for Stochastic Logistic Model with Markovian Switching and Lévy Noise

Sheng Wang[a], **Linshan Wang**[b], **Tengda Wei**[c]

[a] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454003, PR China*
[b] *School of Mathematics, Ocean University of China, Qingdao, Shandong 266071, PR China*
[c] *College of Oceanic and Atmospheric Sciences, Ocean University of China, Qingdao, Shandong 266071, PR China*

Abstract. In this paper, stochastic permanence and extinction of a stochastic logistic model with Markovian switching and Lévy noise are investigated by combining stochastic analytical techniques with M-matrix analysis. Sufficient and necessary conditions of stochastic permanence and extinction are obtained. In the case of stochastic permanence, both the superior limit and the inferior limit of the average in time of the sample path of the solution are estimated by two constants related to the stationary probability distribution of the Markov chain and the parameters of the subsystems of the logistic model. Finally, our conclusions are illustrated through an example.

1. Introduction

In recent years, stochastic population systems driven by white noise have been received great attention and have been studied extensively (see e.g. [1–7]). Particularly, the stochastic logistic model with white noise can be expressed as follows:

$$\mathrm{d}x(t) = x(t)\left[r - ax(t)\right]\mathrm{d}t + \sigma x(t)\mathrm{d}B(t), \tag{1}$$

with initial condition

$$x(0) = x_0 > 0, \tag{2}$$

where $r > 0$ is the rate of growth, $r/a > 0$ is the carrying capacity [8]. $B(t)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $\sigma \geq 0$ is the noise intensity.

However, in the real world population systems often suffer sudden environmental perturbations, such as earthquakes, hurricanes, planting, harvesting, etc (see e.g. [9–12]). These phenomena cannot be described

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Email addresses: wangpengzhan2012@163.com (Sheng Wang), wangls@ouc.edu.cn (Linshan Wang), tdwei123@163.com (Tengda Wei)

by white noise [13]. Bao et al. (see e.g. [14, 15]) pointed out that introducing Lévy jumps into the underlying population system may be a reasonable way to describe these phenomena. Hence, incorporating Lévy noise into system (1), we obtain

$$\mathrm{d}x(t) = x(t^-)\left\{[r - ax(t^-)]\,\mathrm{d}t + \sigma \mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma(\mu)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \tag{3}$$

where $x(t^-)$ is the left limit of $x(t)$. N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Z}$ of $[0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$ and $\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu) = N(\mathrm{d}t, \mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$. $\gamma(\mu) > -1$ is a bounded function defined on $\mathbb{Z}$.

On the other hand, parameters in some population systems may suffer abrupt changes, for instance, some authors (see e.g. [7, 16]) have claimed that the growth rates of some species in summer will be much different from those in winter, and one can use a continuous-time Markov chain with a finite state space to describe these abrupt changes (see e.g. [9, 10, 17]). Especially, Takeuchi et al. [18] investigated a predator-prey system with regime switching and revealed the significant effect of environmental noise on the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative (see e.g. [8, 18, 19]).

To the best of our knowledge to date, stochastic permanence and extinction of stochastic logistic model with Markovian switching and Lévy noise have not been investigated in the existing literature. So, in this paper we study stochastic permanence and extinction of the following stochastic logistic model with Markovian switching and Lévy noise:

$$\mathrm{d}x(t) = x(t^-)\left\{[r(\rho(t)) - a(\rho(t))x(t^-)]\,\mathrm{d}t + \sigma(\rho(t))\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \tag{4}$$

where $\rho(t)$ is a right-continuous Markov chain on $(\Omega, \mathcal{F}, P)$, taking values in $\mathbb{S} = \{1, 2, ..., S\}$. System (4) is operated as follows: If $\rho(0) = i_0$, then the system obeys

$$\mathrm{d}x(t) = x(t^-)\left\{[r(i) - a(i)x(t^-)]\,\mathrm{d}t + \sigma(i)\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma(\mu, i)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \tag{5}$$

with $i = i_0$ until time τ_1 when the Markov chain jumps to i_1 from i_0; the system will then obey system (5) with $i = i_1$ from τ_1 until τ_2 when the Markov chain jumps to i_2 from i_1. System (4) will go on switching as long as the Markov chain jumps. That is to say, system (4) can be regarded as system (5) switching from one to another in accordance with the law of the Markov chain. The different systems (5) ($1 \le i \le S$) are therefore referred to as the subsystems of system (4). If the switching between environmental regimes disappears, in other words, the Markov chain $\rho(t)$ has only one state, then system (4) degenerates into system (5).

2. Global Positive Solutions

Throughout this paper, the generator $\Gamma = (\gamma_{ij})_{S\times S}$ of $\rho(t)$ is given by

$$P\{\rho(t+\varsigma) = j | \rho(t) = i\} = \begin{cases} \gamma_{ij}\varsigma + o(\varsigma), & i \neq j, \\ 1 + \gamma_{ij}\varsigma + o(\varsigma), & i = j, \end{cases} \tag{6}$$

where $\varsigma > 0$. Here γ_{ij} represents the transition rate from i to j and $\gamma_{ij} \geq 0$ if $i \neq j$, while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}. \tag{7}$$

We assume that $\rho(t)$, $B(t)$ and N are mutually independent. As a standing hypothesis we also assume that the Markov chain $\rho(t)$ is *irreducible*. Under this condition, system (4) can switch from any regime to any other regime and the Markov chain $\rho(\cdot)$ has a unique stationary probability distribution $\pi = (\pi_1, \pi_2, ..., \pi_S) \in \mathbb{R}^{1\times S}$ which can be determined by solving the following linear equation:

$$\pi\Gamma = 0, \tag{8}$$

subject to

$$\sum_{i=1}^{S} \pi_i = 1 \textit{ and } \pi_i > 0, \ \forall i \in \mathbb{S}. \tag{9}$$

In this paper, we impose the following assumptions:
Assumption 1 : $r(i) > 0, a(i) > 0, \sigma(i) \geq 0, \forall i \in \mathbb{S}$.
Assumption 2 : There exist $\gamma^*(i) \geq \gamma_*(i) > -1$ such that $\gamma_*(i) \leq \gamma(\mu, i) \leq \gamma^*(i), \forall i \in \mathbb{S}, \mu \in \mathbb{Z}$.
Assumption 3 : For some $j \in \mathbb{S}, \gamma_{ij} > 0, \forall i \neq j$.
Assumption 4 : $\sum_{i=1}^{S} \pi_i\alpha(i) > 0$, where

$$\alpha(i) = r(i) - \frac{1}{2}\sigma^2(i) - \int_{\mathbb{Z}} [\gamma(\mu, i) - \ln(1 + \gamma(\mu, i))]\, \lambda(\mathrm{d}\mu). \tag{10}$$

For convenience, denote $\sigma = \max_{1\leq i\leq S} \sigma(i)$ and $\gamma^\star = \max_{1\leq i\leq S} \{|\gamma_*(i)|, |\gamma^*(i)|\}$.

Theorem 2.1. *Under assumptions 1 and 2, for any initial value $x_0 \in \mathbb{R}_+$, system (4) has a unique positive global solution $x(t)$ on $t \geq 0$ a.s.*

Proof. Consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}u(t) = \left[\alpha(\rho(t)) - a(\rho(t))\mathrm{e}^{u(t)}\right]\mathrm{d}t + \sigma(\rho(t))\mathrm{d}B(t) + \displaystyle\int_{\mathbb{Z}} \ln[1 + \gamma(\mu, \rho(t))]\, \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \\ u(0) = \ln x_0. \end{cases} \tag{11}$$

Since the coefficients of system (11) are locally Lipschitz continuous, from [20] and [21] we observe that system (11) admits a unique local solution $u(t)$ on $t \in [0, \tau_e)$ a.s., where τ_e is the explosion time. By the generalized Itô's formula, $x(t) = \mathrm{e}^{u(t)}$ is the unique local solution to system (4) with initial value $x_0 \in \mathbb{R}_+$. The proof of its global solution is almost identical to that for systems with regime switching driven by white noise (see e.g. [17, 22, 23]), and here is omitted. □

3. Extinction

Theorem 3.1. *Under assumptions 1 and 2, let $x(t)$ be the solution to system (4) with initial value $x_0 \in \mathbb{R}_+$. Then*

$$\sum_{i=1}^{S} \pi_i\alpha(i) < 0 \textit{ a.s.} \implies \lim_{t\to+\infty} x(t) = 0 \textit{ a.s.} \tag{12}$$

Proof. From system (11) we have

$$\mathrm{d}\ln x(t) = [\alpha(\rho(t)) - a(\rho(t))x(t)]\,\mathrm{d}t + \sigma(\rho(t))\mathrm{d}B(t) + \int_{\mathbb{Z}} \ln[1 + \gamma(\mu, \rho(t))]\, \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu). \tag{13}$$

Integrating both sides of system (13) from 0 to t leads to

$$\ln x(t) - \ln x_0 = \int_0^t \alpha(\rho(s))\mathrm{d}s - \int_0^t a(\rho(s))x(s)\mathrm{d}s + \sum_{j=1}^{2} M_j(t), \tag{14}$$

where

$$M_1(t) = \int_0^t \sigma(\rho(s))\mathrm{d}B(s), M_2(t) = \int_0^t \int_{\mathbb{Z}} \ln\left[1 + \gamma(\mu, \rho(s))\right] \widetilde{N}(\mathrm{d}s, \mathrm{d}\mu). \tag{15}$$

In view of (15), we compute

$$\begin{cases} \langle M_1(t)\rangle = \int_0^t \sigma^2(\rho(s))\mathrm{d}s \le \sigma^2 t, \\ \langle M_2(t)\rangle = \int_0^t \int_{\mathbb{Z}} \{\ln\left[1 + \gamma(\mu, \rho(s))\right]\}^2 \lambda(\mathrm{d}\mu)\mathrm{d}s \le \max\limits_{1\le i\le S} \left\{[\ln(1 + \gamma_*(i))]^2, [\ln(1 + \gamma^*(i))]^2\right\} \lambda(\mathbb{Z})t. \end{cases} \tag{16}$$

By Lemma 3.1 in [14] and the strong law of large numbers, we have

$$\lim_{t\to+\infty} \frac{M_j(t)}{t} = 0 \ a.s., \ \ 1 \le j \le 2. \tag{17}$$

According to system (14), we get

$$\frac{\ln x(t)}{t} - \frac{\ln x_0}{t} = t^{-1}\int_0^t \alpha(\rho(s))\mathrm{d}s - t^{-1}\int_0^t a(\rho(s))x(s)\mathrm{d}s + t^{-1}\sum_{j=1}^{2} M_j(t). \tag{18}$$

Based on (17) and (18), we obtain

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{t} \le \limsup_{t\to+\infty} t^{-1}\int_0^t \alpha(\rho(s))\mathrm{d}s = \sum_{i=1}^{S} \pi_i \alpha(i). \tag{19}$$

Hence, the conclusion follows from (19). □

Corollary 3.2. *Assume that for some $i \in \mathbb{S}$, $\alpha(i) < 0$. Then the solutions of system (5) tend to zero a.s.*

Remark 3.3. *Corollary 3.2 implies that Theorem 3.1 contains Lemma 1 (ii) in [10] as a special case.*

If $\gamma(\mu, i) = 0$ ($\forall i \in \mathbb{S}$), then system (4) becomes the following stochastic logistic model with Markovian switching:

$$\mathrm{d}x(t) = x(t)\left\{\left[r(\rho(t)) - a(\rho(t))x(t)\right]\mathrm{d}t + \sigma(\rho(t))\mathrm{d}B(t)\right\}. \tag{20}$$

Corollary 3.4. *Under assumption 1, let $x(t)$ be the solution to system (20) with initial value $x_0 \in \mathbb{R}_+$. Then*

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{t} \le \sum_{i=1}^{S} \pi_i \left[r(i) - \frac{1}{2}\sigma^2(i)\right]. \tag{21}$$

Particularly, if $\sum_{i=1}^{S} \pi_i \left[r(i) - \frac{1}{2}\sigma^2(i)\right] < 0$, then

$$\lim_{t\to+\infty} x(t) = 0 \ a.s. \tag{22}$$

Remark 3.5. *Corollary 3.4 implies that Theorem 3.1 contains Theorem 4.1 in [8] as a special case.*

4. Stochastic Permanence

Definition 4.1. *(see e.g. [8, 14]) System (4) is said to be stochastically permanent, if, for any $\epsilon \in (0,1)$, there exist $\delta_* = \delta_*(\epsilon) > 0$ and $\delta^* = \delta^*(\epsilon) > 0$ such that*

$$\liminf_{t\to+\infty} P\{x(t) \geq \delta_*\} \geq 1-\epsilon, \ \liminf_{t\to+\infty} P\{x(t) \leq \delta^*\} \geq 1-\epsilon. \tag{23}$$

Lemma 4.2. *Under assumptions 1 and 2, let $x(t)$ be the solution to system (4) with initial value $x_0 \in \mathbb{R}_+$, then for any constant $\theta > 0$, there exists a constant $K(\theta) > 0$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}[x^\theta(t)] \leq K(\theta). \tag{24}$$

Proof. For any constant $\theta > 0$, define $W(x) = x^\theta$. By the generalized Itô's formula, we compute

$$\begin{aligned}\mathcal{L}[W(x)] =&\theta x^\theta \left[r(\rho(t)) - a(\rho(t))x\right] + \frac{1}{2}\theta(\theta-1)\sigma^2(\rho(t))x^\theta + x^\theta \int_{\mathbb{Z}} \left\{[1+\gamma(\mu,\rho(t))]^\theta - 1 - \theta\gamma(\mu,\rho(t))\right\}\lambda(\mathrm{d}\mu)\\ \leq&\theta x^\theta \left[\max_{1\leq i\leq S} r(i) - \min_{1\leq i\leq S} a(i)x\right] + \frac{1}{2}\theta^2\sigma^2 x^\theta + x^\theta \int_{\mathbb{Z}} \max_{1\leq i\leq S}\left\{[1+\gamma^*(i)]^\theta - 1 - \theta\gamma_*(i)\right\}\lambda(\mathrm{d}\mu).\end{aligned} \tag{25}$$

From (25) we deduce that there exists a constant $K(\theta) > 0$ such that

$$\mathcal{L}[W(x)] + W(x) \leq K(\theta). \tag{26}$$

So in view of the generalized Itô's formula and (26), we obtain

$$\mathcal{L}[\mathrm{e}^t W(x)] = \mathrm{e}^t W(x) + \mathrm{e}^t \mathcal{L}[W(x)] \leq \mathrm{e}^t K(\theta). \tag{27}$$

Based on (27), integrating $\mathrm{d}[\mathrm{e}^t W(x(t))]$ from 0 to t and then taking the expectations of both sides lead to

$$\mathrm{e}^t \mathbb{E}[x^\theta(t)] \leq [x_0]^\theta + K(\theta)\left(\mathrm{e}^t - 1\right), \tag{28}$$

which implies the required assertion (24). □

Now, by combining stochastic analytical techniques with M-matrix analysis, we are in the position to prove stochastic permanence of system (4). For convenience, let C be a vector or matrix. Denote by $C \gg 0$ all elements of C are positive. Let

$$Y^{S\times S} = \left\{C = (c_{ij})_{S\times S} : c_{ij} \leq 0, i \neq j\right\}. \tag{29}$$

We shall also need two classical results.

Lemma 4.3. *(see Lemma 5.3 in [21]) If $C = (c_{ij})_{S\times S} \in Y^{S\times S}$ has all of its row sums positive, that is*

$$\sum_{j=1}^{S} c_{ij} > 0 \ for\ all\ 1 \leq i \leq S, \tag{30}$$

then $det(C) > 0$.

Lemma 4.4. *(See Theorem 2.10 in [21]) If $C = (c_{ij})_{S\times S} \in Y^{S\times S}$, then the following statements are equivalent:*
(1) C is a nonsingular M-matrix.
(2) All of the principal minors of C are positive; that is,

$$\det\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k}\\ c_{21} & c_{22} & \dots & c_{2k}\\ \vdots & \vdots & & \vdots\\ c_{k1} & c_{k2} & \dots & c_{kk}\end{pmatrix} > 0 \ for\ every\ k = 1,2,...,S. \tag{31}$$

(3) C is semi-positive; that is, there exists $x \gg 0$ in $\mathbb{R}^S$ such that $Cx \gg 0$.

Lemma 4.5. *Assumptions 3 and 4 indicate that there exists a constant $\theta_0 > 0$ such that for $0 < \theta < \theta_0$, the matrix*

$$G(\theta) = diag\,(\nu_1(\theta), ..., \nu_S(\theta)) - \Gamma \tag{32}$$

is a nonsingular M-matrix, where

$$\nu_i(\theta) = \alpha(i)\theta - \frac{1}{2}\sigma^2(i)\theta^2 - \int_{\mathbb{Z}} \left\{[1+\gamma(\mu,i)]^{-\theta} - 1 + \theta \ln[1+\gamma(\mu,i)]\right\} \lambda(\mathrm{d}\mu). \tag{33}$$

Proof. It is known that switching the ith row with the jth row and then switching the ith column with the jth column do not change the value of a determinant. It is also known that for a nonsingular M-matrix, if we switch the ith row with the jth row and then switch the ith column with the jth column, then the new matrix is still a nonsingular M-matrix. For assumption 3, without loss of generality, let $j = S$, that is

$$\gamma_{iS} > 0, \ \forall\ 1 \le i \le S-1. \tag{34}$$

By Appendix A in [24], under assumption 3, assumption 4 is equivalent to

$$\det\begin{pmatrix} \alpha(1) & -\gamma_{12} & \dots & -\gamma_{1S} \\ \alpha(2) & -\gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ \alpha(S) & -\gamma_{S2} & \dots & -\gamma_{SS} \end{pmatrix} > 0. \tag{35}$$

From (7) we compute

$$\det G(\theta) = \det\begin{pmatrix} \nu_1(\theta) & -\gamma_{12} & \dots & -\gamma_{1S} \\ \nu_2(\theta) & \nu_2(\theta)-\gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ \nu_S(\theta) & -\gamma_{S2} & \dots & \nu_S(\theta)-\gamma_{SS} \end{pmatrix} = \sum_{i=1}^{S} \nu_i(\theta) M_i(\theta), \tag{36}$$

where $M_i(\theta)$ represents the corresponding minor of $\nu_i(\theta)$ in the first column. Based on (33) and (10), we have

$$\nu_i(0) = 0, \ \frac{\mathrm{d}\nu_i(\theta)}{\mathrm{d}\theta}|_{\theta=0} = \alpha(i). \tag{37}$$

In view of (36) and (37), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta}[\det G(\theta)]\,|_{\theta=0} = \sum_{i=1}^{S} \alpha(i) M_i(0) = \det\begin{pmatrix} \alpha(1) & -\gamma_{12} & \dots & -\gamma_{1S} \\ \alpha(2) & -\gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ \alpha(S) & -\gamma_{S2} & \dots & -\gamma_{SS} \end{pmatrix}. \tag{38}$$

Combining (35) with (38), we get

$$\frac{\mathrm{d}}{\mathrm{d}\theta}[\det G(\theta)]\,|_{\theta=0} > 0. \tag{39}$$

By (36) and (37), we observe that $\det G(0) = 0$. So there exists $0 < \theta_0 \ll 1$ such that for all $0 < \theta < \theta_0$, $\det G(\theta) > 0$ and

$$\nu_i(\theta) > -\gamma_{iS}, \ 1 \le i \le S-1. \tag{40}$$

For each $k = 1, 2, ..., S-1$, consider the leading principal sub-matrix

$$G_k(\theta) := \begin{pmatrix} \nu_1(\theta)-\gamma_{11} & -\gamma_{12} & \dots & -\gamma_{1k} \\ -\gamma_{21} & \nu_2(\theta)-\gamma_{22} & \dots & -\gamma_{2k} \\ \vdots & \vdots & & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \dots & \nu_k(\theta)-\gamma_{kk} \end{pmatrix} \tag{41}$$

of $G(\theta)$. According to (29) and the fact that $\gamma_{ij} \geq 0\,(i \neq j)$, we obtain that $G_k(\theta) \in Y^{k\times k}$. Moreover, in view of (7) and (40), we compute

$$\nu_i(\theta) - \sum_{j=1}^{k} \gamma_{ij} = \nu_i(\theta) + \sum_{j=k+1}^{S} \gamma_{ij} \geq \nu_i(\theta) + \gamma_{iS} > 0,\ \ i = 1, 2, ..., k. \tag{42}$$

By Lemma 4.3, we obtain that $\det G_k(\theta) > 0$. That is to say, all the leading principal minors of $G(\theta)$ are positive. Hence, the required assertion follows from Lemma 4.4. □

Lemma 4.6. *If there exists a constant $\theta > 0$ such that $G(\theta)$ is a nonsingular M-matrix, then the solution $x(t)$ of system (4) with initial value $x_0 \in \mathbb{R}_+$ has the property that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[\frac{1}{x^{\theta}(t)}\right] \leq H(\theta), \tag{43}$$

where $H(\theta) > 0$ is a constant (defined by (55) in the proof).

Proof. Define

$$U(t) = \frac{1}{x(t)}. \tag{44}$$

By the generalized Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}\,[U(t)] =& U(t)\left\{-r(\rho(t)) + \sigma^2(\rho(t)) + \int_{\mathbb{Z}} \tfrac{\gamma^2(\mu,\rho(t))}{1+\gamma(\mu,\rho(t))}\lambda(\mathrm{d}\mu) + \tfrac{a(\rho(t))}{U(t)}\right\}\mathrm{d}t \\ &- \sigma(\rho(t))U(t)\mathrm{d}B(t) - \int_{\mathbb{Z}} \tfrac{\gamma(\mu,\rho(t))}{1+\gamma(\mu,\rho(t))} U(t)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu). \end{aligned} \tag{45}$$

Noting that $G(\theta)$ is a nonsingular M-matrix, then on the basis of Lemma 4.4, there exists $\overrightarrow{p} = (p_1, p_2, ..., p_S)^{\mathrm{T}} \gg 0$ such that $G(\theta)\overrightarrow{p} \gg \overrightarrow{0}$, namely,

$$\nu_i(\theta)p_i - \sum_{j=1}^{S} \gamma_{ij}p_j > 0,\ \ 1 \leq i \leq S. \tag{46}$$

From (46) we derive that there exists a constant $\kappa > 0$ such that

$$\nu_i(\theta)p_i - \sum_{j=1}^{S} \gamma_{ij}p_j - \kappa p_i > 0,\ \ 1 \leq i \leq S. \tag{47}$$

Applying the generalized Itô's formula again, we compute

$$\mathcal{L}\left[\mathrm{e}^{\kappa t} p_i\,(1 + U(t))^{\theta}\right] = \mathrm{e}^{\kappa t}\left\{\kappa p_i\,(1 + U(t))^{\theta} + \mathcal{L}\left[p_i\,(1 + U(t))^{\theta}\right]\right\}, \tag{48}$$

where

$$\begin{aligned} &\kappa p_i\,(1 + U(t))^{\theta} + \mathcal{L}\left[p_i\,(1 + U(t))^{\theta}\right] \\ =&\kappa p_i\,(1 + U(t))^{\theta} + p_i\theta\,(1 + U(t))^{\theta-1}\,\mathcal{L}\,[U(t)] + \frac{1}{2}p_i\theta(\theta - 1)\sigma^2(\rho(t))U^2(t)\,(1 + U(t))^{\theta-2} + \sum_{j=1}^{S} \gamma_{ij}p_j\,(1 + U(t))^{\theta} \\ &+ \int_{\mathbb{Z}}\left[p_i\left(1 + \tfrac{U(t)}{1+\gamma(\mu,\rho(t))}\right)^{\theta} - p_i\,(1 + U(t))^{\theta} + p_i\theta U(t)\,(1 + U(t))^{\theta-1}\,\tfrac{\gamma(\mu,\rho(t))}{1+\gamma(\mu,\rho(t))}\right]\lambda(\mathrm{d}\mu). \end{aligned}$$

(49)

Based on (10) and (45), we compute

$$\begin{aligned}
&\kappa p_i\left(1+U(t)\right)^{\theta}+\mathcal{L}\left[p_i\left(1+U(t)\right)^{\theta}\right]\\
=&\kappa p_i\left(1+U(t)\right)^{\theta}+p_i\theta U(t)\left(1+U(t)\right)^{\theta-1}\left[-\alpha(\rho(t))+\frac{1}{2}\sigma^2(\rho(t))+\int_{\mathbb{Z}}\ln\left(1+\gamma(\mu,\rho(t))\right)\lambda(\mathrm{d}\mu)+\frac{a(\rho(t))}{U(t)}\right]\\
&+\frac{1}{2}p_i\theta(\theta-1)\sigma^2(\rho(t))U^2(t)\left(1+U(t)\right)^{\theta-2}+\sum_{j=1}^{S}\gamma_{ij}p_j\left(1+U(t)\right)^{\theta}+\int_{\mathbb{Z}}\left[p_i\left(1+\tfrac{U(t)}{1+\gamma(\mu,\rho(t))}\right)^{\theta}-p_i\left(1+U(t)\right)^{\theta}\right]\lambda(\mathrm{d}\mu)\\
=&O(U^{\theta}(t))U^{\theta}(t)+F(U(t)),
\end{aligned}\tag{50}$$

where $\lim_{U\to+\infty}\frac{F(U)}{U^{\theta}}=0$ and

$$\begin{aligned}
O(U^{\theta}(t))=&\kappa p_i+p_i\theta\left[-\alpha(\rho(t))+\frac{1}{2}\sigma^2(\rho(t))+\int_{\mathbb{Z}}\ln\left(1+\gamma(\mu,\rho(t))\right)\lambda(\mathrm{d}\mu)\right]\\
&+\frac{1}{2}p_i\theta(\theta-1)\sigma^2(\rho(t))+\sum_{j=1}^{S}\gamma_{ij}p_j+\int_{\mathbb{Z}}\left[p_i\left(\tfrac{1}{1+\gamma(\mu,\rho(t))}\right)^{\theta}-p_i\right]\lambda(\mathrm{d}\mu)\\
=&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\nu_i(\theta).
\end{aligned}\tag{51}$$

In view of (47), (48), (49), (50) and (51), we deduce that there exists a constant $\mathcal{H}(\theta)>0$ such that

$$\mathcal{L}\left[\mathrm{e}^{\kappa t}p_i\left(1+U(t)\right)^{\theta}\right]\le\mathcal{H}(\theta)\mathrm{e}^{\kappa t}.\tag{52}$$

On the basis of (52), integrating $\mathrm{d}\left[\mathrm{e}^{\kappa t}p_i\left(1+U(t)\right)^{\theta}\right]$ from 0 to t and then taking the expectations of both sides yield

$$\mathbb{E}\left[p_i\mathrm{e}^{\kappa t}\left[1+U(t)\right]^{\theta}\right]-p_i\left[1+U(0)\right]^{\theta}\le\tfrac{\mathcal{H}(\theta)}{\kappa}\left(\mathrm{e}^{\kappa t}-1\right).\tag{53}$$

Based on (53), we deduce

$$\mathbb{E}\left[1+U(t)\right]^{\theta}\le\tfrac{\mathcal{H}(\theta)}{\kappa\min_{1\le i\le S}p_i}+\left(1+\tfrac{1}{x_0}\right)^{\theta}\mathrm{e}^{-\kappa t}.\tag{54}$$

Define

$$H(\theta)=\tfrac{\mathcal{H}(\theta)}{\kappa\min_{1\le i\le S}p_i}.\tag{55}$$

From (44), (54) and (55) we have

$$\limsup_{t\to+\infty}\mathbb{E}\left[\tfrac{1}{x^{\theta}(t)}\right]\le H(\theta).\tag{56}$$

The proof is therefore complete. □

Theorem 4.7. *Under assumptions 1, 2, 3 and 4, system (4) is stochastically permanent.*

Proof. By Chebyshev's inequality, for any $\epsilon > 0$, there exists $\delta_* = \left(\frac{\epsilon}{H(\theta)}\right)^{\frac{1}{\theta}} > 0$ such that

$$\limsup_{t\to+\infty} P\left\{x(t) < \delta_*\right\} = \limsup_{t\to+\infty} P\left\{\frac{1}{x(t)} > \frac{1}{\delta_*}\right\} \le (\delta_*)^\theta \limsup_{t\to+\infty} \mathbb{E}\left[\frac{1}{x^\theta(t)}\right] \le \epsilon. \tag{57}$$

Therefore,

$$\liminf_{t\to+\infty} P\left\{x(t) \ge \delta_*\right\} \ge 1-\epsilon. \tag{58}$$

The second part of (23) follows from combining Lemma 4.2 with Chebyshev's inequality. Hence, system (4) is stochastically permanent. □

Corollary 4.8. *Under assumptions 1 and 3. If $\sum_{i=1}^S \pi_i \left[r(i) - \frac{1}{2}\sigma^2(i)\right] > 0$, then system (20) is stochastically permanent.*

Remark 4.9. *Corollary 4.8 implies that Theorem 4.7 contains Theorem 3.2 in [8] as a special case.*

Corollary 4.10. *Assume that for some $i \in \mathbb{S}$, $\alpha(i) > 0$. Then system (5) is stochastically permanent.*

Remark 4.11. *Corollary 4.10 implies that Theorem 4.7 contains Theorem 1 in [10] as a special case.*

5. Asymptotic Properties

Lemma 5.1. *Under assumptions 1 and 2, the solution $x(t)$ of system (4) with initial value $x_0 \in \mathbb{R}_+$ has the property that*

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{\ln t} \le 1 \ a.s. \tag{59}$$

Proof. It is easy to see that

$$\mathrm{d}x(t) \le r(\rho(t))x(t)\mathrm{d}t + \sigma(\rho(t))x(t)\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma(\mu,\rho(t))x(t)\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu). \tag{60}$$

Integrating both sides of (60) from t to $u\,(u > t)$ yields

$$x(u) - x(t) \le \int_t^u r(\rho(s))x(s)\mathrm{d}s + \int_t^u \sigma(\rho(s))x(s)\mathrm{d}B(s) + \int_t^u \int_{\mathbb{Z}} \gamma(\mu,\rho(s))x(s)\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu). \tag{61}$$

Denote $H_0 = \max_{1\le i\le S} r(i)$. According to (61), we compute

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le u\le t+1} x(u)\right] \le&\mathbb{E}\left[x(t)\right] + H_0 \int_t^{t+1} \mathbb{E}\left[x(s)\right]\mathrm{d}s + \mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u \sigma(\rho(s))x(s)\mathrm{d}B(s)\right]\\ &+ \mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u\int_{\mathbb{Z}} \gamma(\mu,\rho(s))x(s)\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right].\end{aligned} \tag{62}$$

Using the Burkholder-Davis-Gundy inequality (see e.g. pp.264-265 in [20]) and Young inequality, we deduce

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u \sigma(\rho(s))x(s)\mathrm{d}B(s)\right] \le J\mathbb{E}\left(\int_t^{t+1}\left[\sigma(\rho(s))x(s)\right]^2\mathrm{d}s\right)^{0.5} \le J\mathbb{E}\left(\int_t^{t+1}\sigma^2x^2(s)\mathrm{d}s\right)^{0.5}\\ \le& J\mathbb{E}\left(\sigma^2\sup_{t\le u\le t+1}x(u)\int_t^{t+1}x(s)\mathrm{d}s\right)^{0.5} \le J\mathbb{E}\left(\frac{1}{2J}\sup_{t\le u\le t+1}x(u) + \frac{\sigma^2 J}{2}\int_t^{t+1}x(s)\mathrm{d}s\right)\\ =&\frac{1}{2}\mathbb{E}\left(\sup_{t\le u\le t+1}x(u)\right) + \frac{\sigma^2J^2}{2}\int_t^{t+1}\mathbb{E}\left[x(s)\right]\mathrm{d}s.\end{aligned} \tag{63}$$

Combining the Burkholder-Davis-Gundy inequality with Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u\int_{\mathbb{Z}}\gamma(\mu,\rho(s))x(s)\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right]\le J\mathbb{E}\left(\int_t^{t+1}\int_{\mathbb{Z}}[\gamma(\mu,\rho(s))x(s)]^2N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\
\le& J\mathbb{E}\left(\int_t^{t+1}\int_{\mathbb{Z}}[\gamma^\star]^2x^2(s)N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\le J\left(\mathbb{E}\int_t^{t+1}\int_{\mathbb{Z}}[\gamma^\star]^2x^2(s)N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\
=&J\left(\int_{\mathbb{Z}}[\gamma^\star]^2\lambda(\mathrm{d}\mu)\right)^{0.5}\left(\mathbb{E}\int_t^{t+1}x^2(s)\mathrm{d}s\right)^{0.5}.
\end{aligned}\tag{64}$$

Substituting (63) and (64) into (62), we deduce

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\le u\le t+1}x(u)\right)\le&2\mathbb{E}[x(t)]+2H_0\int_t^{t+1}\mathbb{E}[x(s)]\,\mathrm{d}s+\sigma^2J^2\int_t^{t+1}\mathbb{E}[x(s)]\,\mathrm{d}s\\
&+2J\left(\int_{\mathbb{Z}}[\gamma^\star]^2\lambda(\mathrm{d}\mu)\right)^{0.5}\left(\int_t^{t+1}\mathbb{E}\left[x^2(s)\right]\mathrm{d}s\right)^{0.5}.
\end{aligned}\tag{65}$$

In the light of Lemma 4.2, we derive that there exists a constant $K^*(\theta)>0$ such that $\sup_{t\ge0}\mathbb{E}\left[x^\theta(t)\right]\le K^*(\theta)$. Hence,

$$\mathbb{E}\left(\sup_{t\le u\le t+1}x(u)\right)\le 2K^*(1)+2H_0K^*(1)+\sigma^2J^2K^*(1)+2J\left(K^*(2)\int_{\mathbb{Z}}[\gamma^\star]^2\lambda(\mathrm{d}\mu)\right)^{0.5}=:\widetilde{K}.\tag{66}$$

Therefore, from (66) we get

$$\mathbb{E}\left(\sup_{k\le u\le k+1}x(u)\right)\le\widetilde{K},\ k=1,2,\ldots\tag{67}$$

Then by Chebyshev's inequality, we observe that for arbitrary $\epsilon>0$,

$$P\left(\omega:\sup_{k\le t\le k+1}x(t)>k^{1+\epsilon}\right)\le\frac{\widetilde{K}}{k^{1+\epsilon}},\ k=1,2,\ldots\tag{68}$$

Using Borel-Cantelli's lemma, we obtain that there exists a set $\Omega_o\in\mathcal{F}$ with $P(\Omega_o)=1$ and an integer-valued random variable k_o such that for every $\omega\in\Omega_o$,

$$\sup_{k\le t\le k+1}x(t)\le k^{1+\epsilon}\tag{69}$$

holds whenever $k\ge k_o(\omega)$. Thus, for almost all $\omega\in\Omega$, if $k\ge k_o$ and $k\le t\le k+1$,

$$\frac{\ln x(t)}{\ln t}\le\frac{\ln\left(\sup_{k\le t\le k+1}x(t)\right)}{\ln t}\le\frac{\ln k^{1+\epsilon}}{\ln t}\le 1+\epsilon.\tag{70}$$

Therefore, we have

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t}\le 1+\epsilon\ a.s.\tag{71}$$

So the desired assertion (59) follows from letting $\epsilon\to0^+$. □

Lemma 5.2. *If there exists a constant $\theta>0$ such that $G(\theta)$ is a nonsingular M-matrix, then the solution $x(t)$ of system (4) with initial value $x_0\in\mathbb{R}_+$ has the property that*

$$\liminf_{t\to+\infty}\frac{\ln x(t)}{\ln t}\ge-\frac{1}{\theta}\ a.s.\tag{72}$$

Proof. For convenience, denote

$$\Phi\left(U(t)\right) = -\,\theta\left(1+U(t)\right)^{\theta-1}\sigma(\rho(t))U(t),\ \ \Psi\left(U(t)\right) = \left(1+\tfrac{U(t)}{1+\gamma(\mu,\rho(t))}\right)^{\theta} - \left(1+U(t)\right)^{\theta}. \tag{73}$$

By the generalized Itô's formula, we compute

$$\mathrm{d}\left[\left(1+U(t)\right)^{\theta}\right] = \mathcal{L}\left[\left(1+U(t)\right)^{\theta}\right]\mathrm{d}t + \Phi\left(U(t)\right)\mathrm{d}B(t) + \int_{\mathbb{Z}}\Psi\left(U(t)\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu), \tag{74}$$

where

$$\begin{aligned}\mathcal{L}\left[\left(1+U(t)\right)^{\theta}\right] =& \theta\left(1+U(t)\right)^{\theta-1}\mathcal{L}[U(t)] + \frac{1}{2}\theta(\theta-1)\left(1+U(t)\right)^{\theta-2}\left[\sigma(\rho(t))U(t)\right]^2 \\ &+ \int_{\mathbb{Z}}\left[\Psi\left(U(t)\right) + \theta\left(1+U(t)\right)^{\theta-1}\tfrac{\gamma(\mu,\rho(t))}{1+\gamma(\mu,\rho(t))}U(t)\right]\lambda(\mathrm{d}\mu).\end{aligned} \tag{75}$$

Clearly, one can see that there exists a constant $\beta^* > 0$ such that

$$\mathcal{L}\left[\left(1+U(t)\right)^{\theta}\right] \le \beta^*\left(1+U(t)\right)^{\theta}. \tag{76}$$

Substituting (76) into (74) yields

$$\mathrm{d}\left[\left(1+U(t)\right)^{\theta}\right] \le \beta^*\left(1+U(t)\right)^{\theta}\mathrm{d}t + \Phi\left(U(t)\right)\mathrm{d}B(t) + \int_{\mathbb{Z}}\Psi\left(U(t)\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu). \tag{77}$$

For sufficiently small constant $\delta > 0$, integrating both sides of (77) from $(k-1)\delta$ to t, then taking supremums of both sides on $[(k-1)\delta, k\delta]$, and then taking the expectations of both sides yield

$$\begin{aligned}\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}\left(1+U(t)\right)^{\theta}\right) \le& \mathbb{E}\left(\left[1+U((k-1)\delta)\right]^{\theta}\right) + \beta^*\int_{(k-1)\delta}^{k\delta}\mathbb{E}\left(\left(1+U(s)\right)^{\theta}\right)\mathrm{d}s \\ &+ \mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}\int_{(k-1)\delta}^{t}\Phi\left(U(s)\right)\mathrm{d}B(s)\right) + \mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}\int_{(k-1)\delta}^{t}\int_{\mathbb{Z}}\Psi\left(U(s)\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right).\end{aligned} \tag{78}$$

By the Burkholder-Davis-Gundy inequality and Young inequality, we derive

$$\begin{aligned}&\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}\int_{(k-1)\delta}^{t}\Phi\left(U(s)\right)\mathrm{d}B(s)\right) \le J\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\Phi^2\left(U(s)\right)\mathrm{d}s\right)^{0.5} \\ \le& J\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\sigma^2\theta^2\left(1+U(s)\right)^{2\theta}\mathrm{d}s\right)^{0.5} \le J\mathbb{E}\left(\sigma^2\theta^2\sup_{(k-1)\delta\le t\le k\delta}\left(1+U(t)\right)^{\theta}\int_{(k-1)\delta}^{k\delta}\left(1+U(s)\right)^{\theta}\mathrm{d}s\right)^{0.5} \\ \le& \frac{1}{4}\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}\left(1+U(t)\right)^{\theta}\right) + \sigma^2\theta^2J^2\int_{(k-1)\delta}^{k\delta}\mathbb{E}\left[\left(1+U(t)\right)^{\theta}\right]\mathrm{d}t.\end{aligned} \tag{79}$$

Noting that there exists a constant $\Lambda > 0$ such that $\Psi^2(U) \le (1+\Lambda)^2(1+U)^{2\theta}$, in view of the Burkholder-

Davis-Gundy inequality and Young inequality again, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}\int_{(k-1)\delta}^{t}\int_{\mathbb{Z}}\Psi\left(U(s)\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right)\le J\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}}\Psi^{2}\left(U(s)\right)N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\
\le& J\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}}(1+\Lambda)^{2}\left(1+U(s)\right)^{2\theta}N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\
\le& J\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}}(1+\Lambda)^{2}\left(1+U(s)\right)^{\theta}N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\
\le&\frac{1}{4}\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right)+J^{2}\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}}(1+\Lambda)^{2}\left(1+U(s)\right)^{\theta}N(\mathrm{d}s,\mathrm{d}\mu)\right)\\
=&\frac{1}{4}\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right)+J^{2}\int_{\mathbb{Z}}(1+\Lambda)^{2}\lambda(\mathrm{d}\mu)\int_{(k-1)\delta}^{k\delta}\mathbb{E}\left(1+U(s)\right)^{\theta}\mathrm{d}s.
\end{aligned}
\tag{80}
$$

Substituting (79) and (80) into (78) leads to

$$
\begin{aligned}
\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right)\le&2\mathbb{E}\left([1+U((k-1)\delta)]^{\theta}\right)+2\beta^{*}\int_{(k-1)\delta}^{k\delta}\mathbb{E}\left((1+U(s))^{\theta}\right)\mathrm{d}s\\
&+2\sigma^{2}\theta^{2}J^{2}\int_{(k-1)\delta}^{k\delta}\mathbb{E}\left[(1+U(t))^{\theta}\right]\mathrm{d}t+2J^{2}\int_{\mathbb{Z}}(1+\Lambda)^{2}\lambda(\mathrm{d}\mu)\int_{(k-1)\delta}^{k\delta}\mathbb{E}\left(1+U(s)\right)^{\theta}\mathrm{d}s.
\end{aligned}
\tag{81}
$$

According to Lemma 4.6, we obtain that there exists a constant $H^{*}(\theta)>0$ such that $\sup_{t\ge0}\mathbb{E}\left[(1+U(t))^{\theta}\right]\le H^{*}(\theta)$. Hence, we compute

$$
\mathbb{E}\left(\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right)\le 2H^{*}(\theta)+2\beta^{*}\delta H^{*}(\theta)+2\sigma^{2}\theta^{2}J^{2}\delta H^{*}(\theta)+2J^{2}\int_{\mathbb{Z}}(1+\Lambda)^{2}\lambda(\mathrm{d}\mu)\delta H^{*}(\theta)=:\widetilde{H}. \tag{82}
$$

Then by Chebyshev's inequality, we observe that for arbitrary $\epsilon>0$,

$$
P\left(\omega:\sup_{(k-1)\delta\le t\le k\delta}[1+U(t)]^{\theta}>(k\delta)^{1+\epsilon}\right)\le\frac{\widetilde{H}}{(k\delta)^{1+\epsilon}},\ k=1,2,\dots \tag{83}
$$

Using Borel-Cantelli's lemma, we obtain that there exists a set $\Omega^{*}\in\mathcal{F}$ with $P(\Omega^{*})=1$ and an integer-valued random variable k^{*} such that for every $\omega\in\Omega^{*}$,

$$
\sup_{(k-1)\delta\le t\le k\delta}[1+U(t)]^{\theta}\le(k\delta)^{1+\epsilon} \tag{84}
$$

holds whenever $k\ge k^{*}(\omega)$. Hence, for almost all $\omega\in\Omega$, if $k\ge k^{*}$ and $(k-1)\delta\le t\le k\delta$,

$$
\frac{\ln[1+U(t)]^{\theta}}{\ln t}\le\frac{\ln(k\delta)^{1+\epsilon}}{\ln t}\le1+\epsilon. \tag{85}
$$

Based on (85) and the arbitrariness of ϵ, we deduce

$$
\limsup_{t\to+\infty}\frac{\ln\left[\frac{1}{x(t)}\right]}{\ln t}\le\frac{1}{\theta}\ a.s. \tag{86}
$$

In other words, we have

$$\liminf_{t\to+\infty}\frac{\ln x(t)}{\ln t}\geq-\frac{1}{\theta}\ a.s. \tag{87}$$

Hence, the proof is complete. □

Theorem 5.3. *Under assumptions 1, 2, 3 and 4, the solution $x(t)$ of system (4) with initial value $x_0\in\mathbb{R}_+$ has the property that*

$$\begin{cases}\limsup_{t\to+\infty}\dfrac{1}{t}\displaystyle\int_0^t x(s)\mathrm{d}s\leq\dfrac{1}{\min_{1\leq i\leq S}a(i)}\sum_{i=1}^{S}\pi_i\alpha(i),\\ \liminf_{t\to+\infty}\dfrac{1}{t}\displaystyle\int_0^t x(s)\mathrm{d}s\geq\dfrac{1}{\max_{1\leq i\leq S}a(i)}\sum_{i=1}^{S}\pi_i\alpha(i).\end{cases} \tag{88}$$

Proof. An application of Lemma 5.1 and Lemma 5.2 implies that

$$\lim_{t\to+\infty}\frac{\ln x(t)}{t}=0\ a.s. \tag{89}$$

In view of (17), (18) and (89), letting $t\to+\infty$, we have

$$\begin{cases}\limsup_{t\to+\infty}\dfrac{1}{t}\displaystyle\int_0^t x(s)\mathrm{d}s\leq\dfrac{1}{\min_{1\leq i\leq S}a(i)}\limsup_{t\to+\infty}t^{-1}\displaystyle\int_0^t\alpha(\rho(s))\mathrm{d}s=\dfrac{1}{\min_{1\leq i\leq S}a(i)}\sum_{i=1}^{S}\pi_i\alpha(i),\\ \liminf_{t\to+\infty}\dfrac{1}{t}\displaystyle\int_0^t x(s)\mathrm{d}s\geq\dfrac{1}{\max_{1\leq i\leq S}a(i)}\liminf_{t\to+\infty}t^{-1}\displaystyle\int_0^t\alpha(\rho(s))\mathrm{d}s=\dfrac{1}{\max_{1\leq i\leq S}a(i)}\sum_{i=1}^{S}\pi_i\alpha(i).\end{cases} \tag{90}$$

The proof is complete. □

Corollary 5.4. *Under assumptions 1, 3 and 4, the solution $x(t)$ of system (20) with initial value $x_0\in\mathbb{R}_+$ has the property that*

$$\begin{cases}\limsup_{t\to+\infty}\dfrac{1}{t}\displaystyle\int_0^t x(s)\mathrm{d}s\leq\dfrac{1}{\min_{1\leq i\leq S}a(i)}\sum_{i=1}^{S}\pi_i\left[r(i)-\dfrac{1}{2}\sigma^2(i)\right],\\ \liminf_{t\to+\infty}\dfrac{1}{t}\displaystyle\int_0^t x(s)\mathrm{d}s\geq\dfrac{1}{\max_{1\leq i\leq S}a(i)}\sum_{i=1}^{S}\pi_i\left[r(i)-\dfrac{1}{2}\sigma^2(i)\right].\end{cases} \tag{91}$$

Remark 5.5. *Corollary 5.4 implies that Theorem 5.3 contains Theorem 5.1 in [8] as a special case.*

6. Conclusions and an Example

This paper is concerned with stochastic permanence and extinction of a stochastic logistic model with Markovian switching and Lévy noise. Corollary 4.10 tells us that if $\alpha(i)>0$ for some $i\in\mathbb{S}$, then system (5) is stochastically permanent. Theorem 4.7 tells us that if for every $i\in\mathbb{S}$, system (5) is stochastically permanent, then as the result of Markovian switching, system (4) remains stochastically permanent. On the other hand, Corollary 3.2 tells us that if $\alpha(i)<0$ for some $i\in\mathbb{S}$, then system (5) is extinctive. Theorem 3.1 tells us that if for every $i\in\mathbb{S}$, system (5) is extinctive, then as the result of Markovian switching, system (4) remains extinctive. However, Theorem 3.1 and Theorem 4.7 provide a more interesting result that if some subsystems are stochastically permanent while some are extinctive, again as the result of Markovian switching, system (4) may be stochastically permanent or extinctive, depending on the sign of $\sum_{i=1}^{S}\pi_i\alpha(i)$. In order to see this point clearly, we state the following sufficient and necessary conditions for stochastic permanence or extinction of system (4) which follow from Theorem 3.1 and Theorem 4.7.

Theorem 6.1. *Let assumptions 1, 2 and 3 hold and assume that* $\sum_{i=1}^{S}\pi_i\alpha(i)\neq 0$. *Then system (4) is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if* $\sum_{i=1}^{S}\pi_i\alpha(i)>0$, *while it is extinctive if and only if* $\sum_{i=1}^{S}\pi_i\alpha(i)<0$.

Corollary 6.2. *Let assumptions 1 and 3 hold and assume that* $\sum_{i=1}^{S}\pi_i\left[r(i)-\frac{1}{2}\sigma^2(i)\right]\neq 0$. *Then system (20) is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if* $\sum_{i=1}^{S}\pi_i\left[r(i)-\frac{1}{2}\sigma^2(i)\right]>0$, *while it is extinctive if and only if* $\sum_{i=1}^{S}\pi_i\left[r(i)-\frac{1}{2}\sigma^2(i)\right]<0$.

Remark 6.3. *Corollary 6.2 implies that Theorem 6.1 contains Theorem 6.1 in [8] as a special case.*

Remark 6.4. *Corollary 3.2 and Corollary 4.10 establish sufficient and necessary conditions of stochastic permanence and extinction for system (5), which correspond to Remark 1 in [10].*

Moreover, in the case of stochastic permanence, according to Theorem 5.3, both the superior limit and the inferior limit of the average in time of the sample path of the solution are estimated by two constants related to the stationary probability distribution $(\pi_1,\pi_2,\dots,\pi_S)$ of the Markov chain and the parameters $r(i)$, $a(i)$, $\sigma(i)$, $\gamma(\mu,i)$ of the subsystems, $i\in\mathbb{S}$. Our conclusions are illustrated by considering the following stochastic logistic model with Markovian switching and Lévy noise:

$$\mathrm{d}x(t)=x(t^-)\left\{[r(\rho(t))-a(\rho(t))x(t^-)]\,\mathrm{d}t+\sigma(\rho(t))\mathrm{d}B(t)+\int_{\mathbb{Z}}\gamma(\mu,\rho(t))\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right\},\tag{92}$$

where $\rho(t)$ is a right-continuous Markov chain taking values in $\mathbb{S}=\{1,2\}$. System (92) may be regarded as the result of regime switching, which switches between the following two subsystems:

$$\mathrm{d}x(t)=x(t^-)\left\{[2-x(t^-)]\,\mathrm{d}t+3\mathrm{d}B(t)+\int_{\mathbb{Z}}\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right\},\tag{93}$$

and

$$\mathrm{d}x(t)=x(t^-)\left\{[5-2x(t^-)]\,\mathrm{d}t+\mathrm{d}B(t)+\int_{\mathbb{Z}}\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right\}.\tag{94}$$

Here, $\lambda(\mathbb{Z})=1$ and

$$r(1)=2,\ r(2)=5,\ a(1)=1,\ a(2)=2,\ \sigma(1)=3,\ \sigma(2)=1,\ \gamma(\mu,1)=1,\ \gamma(\mu,2)=1.\tag{95}$$

Based on (95), we compute

$$\alpha(1)=-\frac{7}{2}+\ln 2,\ \alpha(2)=\frac{7}{2}+\ln 2.\tag{96}$$

From Corollary 3.2, system (93) is extinctive. By Corollary 4.10, system (94) is stochastically permanent.

Case 1. Let the generator of the Markov chain $\rho(t)$ be

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix}-5 & 5\\ 1 & -1\end{pmatrix}.\tag{97}$$

Solving equation (8) yields the unique stationary probability distribution

$$\pi=(\pi_1,\pi_2)=\left(\frac{1}{6},\frac{5}{6}\right).\tag{98}$$

Thus, we have

$$\sum_{i=1}^{2}\pi_i\alpha(i)=\frac{7}{3}+\ln 2>0.\tag{99}$$

Therefore, according to Theorem 4.7, system (92) is stochastically permanent. Moreover, in view of Theorem 5.3, we obtain

$$\frac{7}{6}+\frac{\ln 2}{2}\le\liminf_{t\to+\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s\le\limsup_{t\to+\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s\le\frac{7}{3}+\ln 2. \tag{100}$$

Case 2. Let the generator of the Markov chain $\rho(t)$ be

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix}. \tag{101}$$

Solving equation (8) yields the unique stationary probability distribution

$$\pi=(\pi_1,\pi_2)=\left(\frac{3}{4},\frac{1}{4}\right). \tag{102}$$

Then, we get

$$\sum_{i=1}^{2}\pi_i\alpha(i)=-\frac{7}{4}+\ln 2<0. \tag{103}$$

So based on Theorem 3.1, system (92) is extinctive.

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