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On a Solvable Class of Product-type Systems of Difference Equations

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Abstract. It is shown that the following class of systems of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_n^c z_{n-2}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$, is solvable, continuing our investigation of classification of solvable product-type systems with two dependent variables. We present closed form formulas for solutions to the systems in all the cases. In the main case, when $bd \neq 0$, a detailed investigation of the form of the solutions is presented in terms of the zeros of an associated polynomial whose coefficients depend on some of the parameters of the system.

1. Introduction

The area of difference equations and systems is of a great interest. For some classical results see [6]-[9]. Many concrete types of the equations and systems have been considerably studied recently (see, for example, [1], [2], [4], [5], [10]-[16], [18]-[49]). After some initial studies of symmetric systems by Papaschinopoulos and Schinas in [12]-[14], these and some other experts have continued the investigation in several directions (see, for example, [2], [4], [10], [11], [16], [18], [19], [21], [23]-[29], [31]-[33], [36], [38]-[49]). Among others, the solvability problem has been studied considerably recently (see, for example, [2], [15], [21]-[38], [40]-[49] and the references therein), since some new interesting solvable classes of equations and systems have appeared recently. Many equations and systems are solved by reducing them to known solvable ones by employing some transformations (see, for example, [2], [15], [21], [22], [24]-[27], [35], [40]-[45]). For many of them the transformations are not so obvious, for example, for the systems in [2] and [21], or for some cases of the equation in [35], where can be found several methods for solving difference equations. For some classes of partial difference equations, which can be solved by some related methods, see [30, 34, 37].

Some classes of the equations and systems contain product-type ones as their special cases. We will mention only paper [20] on a class of such equations and [39] on a class of such systems (an interested reader can find many related equations and systems in the lists of the references in these two papers).

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Having studied these types of equations and systems and noticing their connection to product-type ones we came up with an idea to study the later ones, but on the set of real or complex numbers, since the case of positive initial values and parameters is well-known (the system studied in [23] was solved by transforming it to a product-type one with such initial values). However, working on the domains causes some typical problems, usually connected to the multi-valuedness of complex functions.

Bearing in mind that the related product-type system to the one in [39] has the following form

$$z_n = z_{n-k}^p w_{n-l}^q, \quad w_n = w_{n-m}^r z_{n-s}^t, \quad n \in \mathbb{N}_0,$$

for some $k, l, m, s \in \mathbb{N}$, $p, q, r, t \in \mathbb{R}$, it is naturally imposed to study the problem of solvability for special cases of the system or for some of its extensions. It turned out that the problem was not so easy. The case $k = m = 2$, $l = s = 1$, was the first one that was studied (see [38]). The corresponding three-dimensional system was studied in [31]. Somewhat later in [46] was studied the case $k = m = 1$, $l = s = 2$. Motivated by [35], in the course of the study we have noticed that introducing some multipliers will not violate the solvability. The first product-type system with multipliers was considered in [29], while in [47] was considered the corresponding extension of the system in [38]. We have also realized that in some technically complicated cases the structure of the solutions to some product-type systems can be described in detail, which was first successfully done for the systems in [33] and [49]. For the system studied in [48] such analysis was not necessary, since for its solutions are found closed-form formulas in a reasonably concise way. Another method for solving product-type systems has been recently presented in [32]. We would also like to mention that paper [35] deals with some product-type equations, which are special cases of the equation studied there.

Analysing the methods used in papers [29], [32], [33], [38], [46]-[49] it can be seen that the number of solvable product-type systems of the above form is finite, the fact that is connected to the impossibility of solving the polynomial equations of degree five or more.

Here we continue studying the solvability of product-type systems by investigating the following one:

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_n^c z_{n-2}^d, \quad n \in \mathbb{N}_0, \tag{1}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C}$.

We show the solvability of system (1) by developing our previous methods in [29], [32], [33], [38], [46]-[49], especially for the most difficult case $bd \neq 0$, where we conduct a detailed analysis of the structure of the solutions to the system by analysing the nature of the roots of a polynomial of the fourth degree whose coefficients depend on parameters a, b, c, d.

Note that if any of initial values z_{-2}, z_{-1}, z_0, w_0 is equal to zero, then if some of parameters a, b, c, d is negative it will produce a not well-defined solution. If all the parameters are positive then such initial values will produce eventually trivial solutions. The same situation appears if $\alpha = 0$ or $\beta = 0$. This is why we consider the case $z_{-2}, z_{-1}, z_0, w_0, \alpha, \beta \in \mathbb{C} \setminus \{0\}$. As usual, we regard $\sum_{i=k}^{l} a_i = 0$, if $l < k$.

2. Auxiliary Results

This section quotes three lemmas. The first one is well-known (see, e.g., [7, 9]).

Lemma 1. *Let* $i \in \mathbb{N}_0$ *and*

$$s_n^{(i)}(z) = 1 + 2^i z + 3^i z^2 + \cdots + n^i z^{n-1}, \quad n \in \mathbb{N}, \tag{2}$$

where $z \in \mathbb{C}$.

Then

$$s_n^{(0)}(z) = \frac{1-z^n}{1-z},$$
$$s_n^{(1)}(z) = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2},$$
$$s_n^{(2)}(z) = \frac{1+z-(n+1)^2z^n + (2n^2+2n-1)z^{n+1} - n^2z^{n+2}}{(1-z)^3},$$
$$s_n^{(3)}(z) = \frac{n^3z^n(z-1)^3 - 3n^2z^n(z-1)^2 + 3nz^n(z^2-1) - (z^n-1)(z^2+4z+1)}{(1-z)^4},$$

for every $z \in \mathbb{C} \setminus \{1\}$ *and* $n \in \mathbb{N}$.

The following lemma is also classical one. For an elementary proof see, for example, [3, 7]. For a proof which uses a complex analysis method, see, for example, [46].

Lemma 2. *If* λ_j, $j = \overline{1,k}$, *are mutually different zeros of the polynomial*

$$P(t) = a_kt^k + a_{k-1}t^{k-1} + \cdots + a_1t + a_0,$$

with $a_ka_0 \neq 0$, *then*

$$\sum_{j=1}^k \frac{\lambda_j^l}{P'(\lambda_j)} = 0$$

for $l = \overline{0, k-2}$, *and*

$$\sum_{j=1}^k \frac{\lambda_j^{k-1}}{P'(\lambda_j)} = \frac{1}{a_k}.$$

The results proved in [17] can be summarized into the following lemma.

Lemma 3. *Let*

$$P_4(t) = t^4 + bt^3 + ct^2 + dt + e,$$
$$\Delta_0 = c^2 - 3bd + 12e, \quad \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27d^2 - 72ce, \quad \Delta = \frac{1}{27}(4\Delta_0^3 - \Delta_1^2),$$
$$P = 8c - 3b^2, \quad Q = b^3 + 8d - 4bc, \quad D = 64e - 16c^2 + 16b^2c - 16bd - 3b^4.$$

(a) *If* $\Delta < 0$, *then two zeros of* P_4 *are real and different, and two are non-real complex conjugate;*

(b) *If* $\Delta > 0$, *then all the zeros of* P_4 *are real or none is. More precisely,*

1° *if* $P < 0$ *and* $D < 0$, *then all four zeros of* P_4 *are real and different;*

2° *if* $P > 0$ *or* $D > 0$, *then there are two pairs of complex conjugate zeros of* P_4.

(c) *If* $\Delta = 0$, *then and only then* P_4 *has a multiple zero. The following cases can occur:*

1° *if* $P < 0$, $D < 0$ *and* $\Delta_0 \neq 0$, *then two zeros of* P_4 *are real and equal and two are real and simple;*

2° *if* $D > 0$ *or* ($P > 0$ *and* ($D \neq 0$ *or* $Q \neq 0$)), *then two zeros of* P_4 *are real and equal and two are complex conjugate;*

3° *if* $\Delta_0 = 0$ *and* $D \neq 0$, *there is a triple zero of* P_4 *and one simple, all real;*

4° *if* $D = 0$, *then*

4.1° *if* $P < 0$ *there are two double real zeros of* P_4*;*

4.2° *if* $P > 0$ *and* $Q = 0$ *there are two double complex conjugate zeros of* P_4*;*

4.3° *if* $\Delta_0 = 0$, *then all four zeros of* P_4 *are real and equal to* $-b/4$.

3. Main Results

The main results in this paper are proved in this section.

Theorem 1. *Assume that* $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof. Since $b = 0$ system (1) is

$$z_{n+1} = \alpha z_n^a, \quad w_{n+1} = \beta w_n^c z_{n-2}^d, \quad n \in \mathbb{N}_0. \tag{3}$$

From (3), we obtain

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N}. \tag{4}$$

Hence

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{5}$$

when $a \neq 1$, and

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}, \tag{6}$$

when $a = 1$.

From the second equation in (3) and (4), it follows that

$$w_n = \beta \alpha^{d \sum_{j=0}^{n-4} a^j} z_0^{da^{n-3}} w_{n-1}^c, \quad n \geq 4. \tag{7}$$

Assume that

$$w_n = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{j=0}^{k-1} c^j \sum_{i=0}^{n-j-4} a^i} z_0^{d \sum_{j=0}^{k-1} c^j a^{n-j-3}} w_{n-k}^{c^k}, \tag{8}$$

for some k and all $n \geq k + 3$.

If we replace n by $n - k$ in (7) and use it in (8), we obtain

$$\begin{aligned} w_n &= \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{j=0}^{k-1} c^j \sum_{i=0}^{n-j-4} a^i} z_0^{d \sum_{j=0}^{k-1} c^j a^{n-j-3}} \left(\beta \alpha^{d \sum_{j=0}^{n-k-4} a^j} z_0^{da^{n-k-3}} w_{n-k-1}^c\right)^{c^k} \\ &= \beta^{\sum_{j=0}^{k} c^j} \alpha^{d \sum_{j=0}^{k} c^j \sum_{i=0}^{n-j-4} a^i} z_0^{d \sum_{j=0}^{k} c^j a^{n-j-3}} w_{n-k-1}^{c^{k+1}}, \end{aligned}$$

for $n \geq k + 4$. This inductive argument shows that (8) holds for every $k, n \in \mathbb{N}$ such that $n \geq k + 3$.

By taking $k = n - 3$ in (8), and using the following fact

$$w_3 = \beta^{1+c+c^2} w_0^{c^3} z_{-2}^{dc^2} z_{-1}^{dc} z_0^d,$$

we have

$$\begin{aligned} w_n &= \beta^{\sum_{j=0}^{n-4} c^j} \alpha^{d \sum_{j=0}^{n-4} c^j \sum_{i=0}^{n-j-4} a^i} z_0^{d \sum_{j=0}^{n-4} c^j a^{n-j-3}} w_3^{c^{n-3}} \\ &= \beta^{\sum_{j=0}^{n-4} c^j} \alpha^{d \sum_{j=0}^{n-4} c^j \sum_{i=0}^{n-j-4} a^i} z_0^{d \sum_{j=0}^{n-4} c^j a^{n-j-3}} \left(\beta^{1+c+c^2} w_0^{c^3} z_{-2}^{dc^2} z_{-1}^{dc} z_0^d\right)^{c^{n-3}} \\ &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{j=0}^{n-4} c^j \sum_{i=0}^{n-j-4} a^i} z_0^{d \sum_{j=0}^{n-3} c^j a^{n-j-3}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n}, \end{aligned} \tag{9}$$

for $n \geq 4$.

Case $a \neq 1 \neq c \neq a$. From (9) and some calculation, we get

$$\begin{aligned} w_n &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{j=0}^{n-4} c^j \frac{1-a^{n-j-3}}{1-a}} z_0^{d\frac{a^{n-2}-c^{n-2}}{a-c}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-a}(\frac{1-c^{n-3}}{1-c}-a\frac{a^{n-3}-c^{n-3}}{a-c})} z_0^{d\frac{a^{n-2}-c^{n-2}}{a-c}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(a-c+(1-a)c^{n-2}+(c-1)a^{n-2})}{(1-a)(1-c)(a-c)}} z_0^{d\frac{a^{n-2}-c^{n-2}}{a-c}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n}. \end{aligned} \tag{10}$$

Case $a = c \neq 1$. From (9) and some calculation, we get

$$\begin{aligned} w_n &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-4} c^j \sum_{i=0}^{n-j-4} c^i} z_0^{d\sum_{j=0}^{n-3} c^j c^{n-j-3}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{j=0}^{n-4} c^j \frac{1-c^{n-j-3}}{1-c}} z_0^{d(n-2)c^{n-3}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-c}(\frac{1-c^{n-3}}{1-c}-(n-3)c^{n-3})} z_0^{d(n-2)c^{n-3}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(1-(n-2)c^{n-3}+(n-3)c^{n-2})}{(1-c)^2}} z_0^{d(n-2)c^{n-3}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n}. \end{aligned} \tag{11}$$

Case $a = 1 \neq c$. From (9) and some calculation, we get

$$\begin{aligned} w_n &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-4} c^j (n-j-3)} z_0^{d\sum_{j=0}^{n-3} c^j} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\left((n-3)\frac{1-c^{n-3}}{1-c}-c\sum_{j=1}^{n-4} jc^{j-1}\right)} z_0^{d\frac{1-c^{n-2}}{1-c}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\left((n-3)\frac{1-c^{n-3}}{1-c}-c\frac{1-(n-3)c^{n-4}+(n-4)c^{n-3}}{(1-c)^2}\right)} z_0^{d\frac{1-c^{n-2}}{1-c}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(n-3-(n-2)c+c^{n-2})}{(1-c)^2}} z_0^{d\frac{1-c^{n-2}}{1-c}} z_{-2}^{dc^{n-1}} z_{-1}^{dc^{n-2}} w_0^{c^n}. \end{aligned} \tag{12}$$

Case $a \neq 1 = c$. From (9) and some calculation, we get

$$\begin{aligned} w_n &= \beta^n \alpha^{d\sum_{j=0}^{n-4} \frac{1-a^{n-j-3}}{1-a}} z_0^{d\sum_{j=0}^{n-3} a^{n-j-3}} z_{-2}^d z_{-1}^d w_0 \\ &= \beta^n \alpha^{\frac{d}{1-a}(n-3-a\frac{1-a^{n-3}}{1-a})} z_0^{d\frac{a^{n-2}-1}{a-1}} z_{-2}^d z_{-1}^d w_0 \\ &= \beta^n \alpha^{\frac{d(a^{n-2}-(n-2)a+n-3)}{(1-a)^2}} z_0^{d\frac{a^{n-2}-1}{a-1}} z_{-2}^d z_{-1}^d w_0. \end{aligned} \tag{13}$$

Case $a = c = 1$. From (9) in this case, we have

$$\begin{aligned} w_n &= \beta^n \alpha^{d\sum_{j=0}^{n-4}(n-j-3)} z_0^{d(n-2)} z_{-2}^d z_{-1}^d w_0 \\ &= \beta^n \alpha^{d\frac{(n-3)(n-2)}{2}} z_0^{d(n-2)} z_{-2}^d z_{-1}^d w_0. \end{aligned} \tag{14}$$

From all the above the theorem follows. □

Theorem 1 yields the following corollary.

Corollary 1. *Assume that* $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $a \neq 1 \neq c \neq a$, *then the general solution to system* (1) *is given by* (5) *and* (10).
(b) *If* $a = c \neq 1$, *then the general solution to system* (1) *is given by* (5) *and* (11).
(c) *If* $a = 1 \neq c$, *then the general solution to system* (1) *is given by* (6) *and* (12).
(d) *If* $a \neq 1 = c$, *then the general solution to system* (1) *is given by* (5) *and* (13).
(e) *If* $a = c = 1$, *then the general solution to system* (1) *is given by* (6) *and* (14).

The following result was proved in [49]. Hence, we will only sketch the proof for the completeness and the benefit of the reader.

Theorem 2. *Assume that* $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof. Since $d = 0$ system (1) is

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_n^c, \quad n \in \mathbb{N}_0. \tag{15}$$

The second equation in (15) yields

$$w_n = \beta^{\sum_{i=0}^{n-1} c^i} w_0^{c^n}, \quad n \in \mathbb{N}. \tag{16}$$

Hence, for $c \neq 1$

$$w_n = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n}, \quad n \in \mathbb{N}, \tag{17}$$

while for $c = 1$

$$w_n = \beta^n w_0, \quad n \in \mathbb{N}. \tag{18}$$

From the first equation in (15) and (16), is obtained

$$z_n = \alpha \beta^{b \sum_{i=0}^{n-2} c^i} w_0^{bc^{n-1}} z_{n-1}^a, \tag{19}$$

for $n \geq 2$.

By using the induction is obtained

$$z_n = \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b \sum_{j=0}^{k-1} \left(a^j \sum_{i=0}^{n-j-2} c^i\right)} w_0^{b \sum_{i=0}^{k-1} a^i c^{n-i-1}} z_{n-k}^{a^k}, \tag{20}$$

for $n \geq k + 1$.

By taking $k = n - 1$ into (20) and using the fact $z_1 = \alpha z_0^a w_0^b$, we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} c^i\right)} w_0^{b \sum_{i=0}^{n-1} a^i c^{n-i-1}} z_0^{a^n}, \tag{21}$$

for $n \geq 2$.

Subcase $a \neq c$. In this case from (21), we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} c^i\right)} z_0^{a^n} w_0^{b \frac{a^n - c^n}{a-c}}, \quad n \in \mathbb{N}. \tag{22}$$

If $a \neq 1$ and $c \neq 1$, then by Lemma 1 is obtained

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b \frac{a-c+c^n-a^n+ca^n-ac^n}{(1-a)(1-c)(a-c)}} z_0^{a^n} w_0^{b \frac{a^n-c^n}{a-c}}, \quad n \in \mathbb{N}. \tag{23}$$

If $a \neq c$ and $a = 1$, then by Lemma 1, we get

$$z_n = \alpha^n \beta^{b \frac{n-1-nc+c^n}{(1-c)^2}} z_0 w_0^{b \frac{c^n-1}{c-1}}, \quad n \in \mathbb{N}, \tag{24}$$

while if $a \neq c$ and $c = 1$, then by Lemma 1, we get

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b \frac{n-1-na+a^n}{(1-a)^2}} z_0^{a^n} w_0^{b \frac{a^n-1}{a-1}}, \quad n \in \mathbb{N}. \tag{25}$$

Subcase $a = c$. In this case from (21), we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} a^i\right)} z_0^{a^n} w_0^{bna^{n-1}}, \quad n \geq 2. \tag{26}$$

If $a = c \neq 1$, then by using Lemma 1, (26) becomes

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{1-na^{n-1}+(n-1)a^n}{(1-a)^2}} z_0^{a^n} w_0^{bna^{n-1}}, \quad n \geq 2, \tag{27}$$

while if $a = c = 1$, then by using Lemma 1, we get

$$z_n = \alpha^n \beta^{b\frac{(n-1)n}{2}} z_0 w_0^{bn}, \quad n \in \mathbb{N}, \tag{28}$$

completing the proof. □

Corollary 2. *Assume that $a, b, c \in \mathbb{Z}$, $d = 0$ and $\alpha, \beta, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If $a \neq c$, $a \neq 1$ and $c \neq 1$, then the general solution to system* (1) *is given by* (17) *and* (23).
(b) *If $a \neq c$ and $a = 1$, then the general solution to system* (1) *is given by* (17) *and* (24).
(c) *If $a \neq c$ and $c = 1$, then the general solution to system* (1) *is given by* (18) *and* (25).
(d) *If $a = c \neq 1$, then the general solution to system* (1) *is given by* (17) *and* (27).
(e) *If $a = c = 1$, then the general solution to system* (1) *is given by* (18) *and* (28).

Theorem 3. *Assume that $a, b, c, d \in \mathbb{Z}$, $bd \neq 0$, $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (1) *is solvable in closed form.*

Proof. Since $z_{-2}, z_{-1}, z_0, w_0, \alpha, \beta \in \mathbb{C} \setminus \{0\}$, from (1) it easily follows that $z_n w_n \neq 0$ for $n \in \mathbb{N}_0$. Hence, from (1) we have

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{29}$$

and

$$w_{n+1}^b = \beta^b w_n^{bc} z_{n-2}^{bd}, \quad n \in \mathbb{N}_0. \tag{30}$$

Using (29) in (30) is obtained

$$z_{n+2} = \alpha^{1-c} \beta^b z_{n+1}^{a+c} z_n^{-ac} z_{n-2}^{bd}, \quad n \in \mathbb{N}_0. \tag{31}$$

Let $\delta = \alpha^{1-c}\beta^b$,

$$a_1 = a + c, \quad b_1 = -ac, \quad c_1 = 0, \quad d_1 = bd, \quad y_1 = 1. \tag{32}$$

Then (31) is presented in the following form

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \quad n \in \mathbb{N}_0. \tag{33}$$

We have

$$\begin{aligned} z_{n+2} &= \delta^{y_1} (\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1})^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \\ &= \delta^{y_1+a_1} z_n^{a_1a_1+b_1} z_{n-1}^{b_1a_1+c_1} z_{n-2}^{c_1a_1+d_1} z_{n-3}^{d_1a_1} \\ &= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2} z_{n-3}^{d_2}, \end{aligned} \tag{34}$$

for $n \in \mathbb{N}$, where

$$a_2 := a_1a_1 + b_1, \; b_2 := b_1a_1 + c_1, \; c_2 := c_1a_1 + d_1, \; d_2 := d_1a_1, \; y_2 := y_1 + a_1. \tag{35}$$

Assume that

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k}, \tag{36}$$

for a $k \in \mathbb{N} \setminus \{1\}$ and all $n \geq k-1$, and that

$$a_k = a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1} + c_{k-1},$$
$$c_k = c_1 a_{k-1} + d_{k-1}, \quad d_k = d_1 a_{k-1}, \tag{37}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{38}$$

Then by using (33) where n is replaced by $n-k$ in (36), we obtain

$$\begin{aligned} z_{n+2} &= \delta^{y_k}(\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1} z_{n-k-2}^{d_1})^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k} \\ &= \delta^{y_k+a_k} z_{n+1-k}^{a_1a_k+b_k} z_{n-k}^{b_1a_k+c_k} z_{n-k-1}^{c_1a_k+d_k} z_{n-k-2}^{d_1a_k} \\ &= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}} z_{n-k-2}^{d_{k+1}}, \end{aligned}$$

for $n \geq k$, where

$$a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k + c_k,$$
$$c_{k+1} := c_1 a_k + d_k, \quad d_{k+1} := d_1 a_k,$$
$$y_{k+1} := y_k + a_k.$$

This along with (34), (35) and the induction shows that (36)-(38), hold for every $k, n \in \mathbb{N}$ such that $2 \leq k \leq n+1$ (note that (36) holds for $1 \leq k \leq n+1$).

By taking $k = n+1$ in (36), using the fact $z_1 = \alpha z_0^a w_0^b$, (37) and (38), we have

$$\begin{aligned} z_{n+2} &= \delta^{y_{n+1}} z_1^{a_{n+1}} z_0^{b_{n+1}} z_{-1}^{c_{n+1}} z_{-2}^{d_{n+1}} \\ &= (\alpha^{1-c}\beta^b)^{y_{n+1}} (\alpha z_0^a w_0^b)^{a_{n+1}} z_0^{b_{n+1}} z_{-1}^{c_{n+1}} z_{-2}^{d_{n+1}} \\ &= \alpha^{(1-c)y_{n+1}+a_{n+1}} \beta^{by_{n+1}} z_0^{aa_{n+1}+b_{n+1}} z_{-1}^{c_{n+1}} z_{-2}^{d_{n+1}} w_0^{ba_{n+1}} \\ &= \alpha^{y_{n+2}-cy_{n+1}} \beta^{by_{n+1}} z_0^{a_{n+2}-ca_{n+1}} z_{-1}^{bda_{n-1}} z_{-2}^{bda_n} w_0^{ba_{n+1}}, \quad n \geq 2. \end{aligned} \tag{39}$$

From (37) we see that $(a_k)_{k\geq 5}$ is a solution to

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}, \tag{40}$$

which along with the relations $b_k = a_{k+1} - a_1 a_k$, $c_k = b_{k+1} - b_1 a_k$, $d_k = d_1 a_{k-1}$ shows that $(b_k)_{k\geq 5}$, $(c_k)_{k\geq 5}$ and $(d_k)_{k\geq 5}$ are also solutions to (40).

From (37) with $k = 1, 0, -1, -2$, some calculation and by using the assumption $bd \neq 0$, it is not difficult to see that

$$\begin{aligned} &a_{-3} = 0, \quad a_{-2} = 0, \quad a_{-1} = 0, \quad a_0 = 1; \\ &b_{-3} = 0, \quad b_{-2} = 0, \quad b_{-1} = 1, \quad b_0 = 0; \\ &c_{-3} = 0, \quad c_{-2} = 1, \quad c_{-1} = 0, \quad c_0 = 0; \\ &d_{-3} = 1, \quad d_{-2} = 0, \quad d_{-1} = 0, \quad d_0 = 0. \end{aligned} \tag{41}$$

Hence, $(a_k)_{k\geq -3}$, is the solution to (40) with the initial conditions $a_{-3} = a_{-2} = a_{-1} = 0$, $a_0 = 1$, whereas $(y_k)_{k\geq -3}$ satisfies (38) and

$$y_{-3} = y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1, \tag{42}$$

from which along with $a_0 = 1$ is obtained

$$y_k = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{43}$$

The solvability of (40) implies that a closed form formula for $(a_k)_{k\geq-3}$ can be found, from which along with (43) and by the formulas in Lemma 1 a closed-form formula for $(y_k)_{k\geq-3}$ is found. Employing such obtained formulas in (39) shows the solvability of (31).

Further, we have

$$z_{n-2}^d = \frac{w_{n+1}}{\beta w_n^c}, \quad n \in \mathbb{N}_0, \tag{44}$$

and

$$z_{n+1}^d = \alpha^d z_n^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0. \tag{45}$$

Combining (44) and (45) we obtain

$$w_{n+4} = \alpha^d \beta^{1-a} w_{n+3}^{a+c} w_{n+2}^{-ac} w_n^{bd}, \quad n \in \mathbb{N}_0. \tag{46}$$

Note also that

$$w_1 = \beta w_0^c z_{-2}^d, \quad w_2 = \beta^{1+c} w_0^{c^2} z_{-2}^{cd} z_{-1}^d \quad \text{and} \quad w_3 = \beta^{1+c+c^2} w_0^{c^3} z_{-2}^{c^2 d} z_{-1}^{cd} z_0^d. \tag{47}$$

Following the lines of the method for getting the closed-form formula for z_n is obtained that for all $k, n \in \mathbb{N}$, $1 \leq k \leq n+1$

$$w_{n+4} = \eta^{y_k} w_{n+4-k}^{a_k} w_{n+3-k}^{b_k} w_{n+2-k}^{c_k} w_{n+1-k}^{d_k}, \tag{48}$$

where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are solutions to the problem (37), (41), whereas $(y_k)_{k\in\mathbb{N}}$ satisfies (38), the condition $y_1 = 1$, and (43).

By taking $k = n+1$ in (48) and employing (47), it follows that

$$\begin{aligned} w_{n+4} =& \eta^{y_{n+1}} w_3^{a_{n+1}} w_2^{b_{n+1}} w_1^{c_{n+1}} w_0^{d_{n+1}} \\ =& (\alpha^d \beta^{1-a})^{y_{n+1}} (\beta^{1+c+c^2} w_0^{c^3} z_{-2}^{c^2 d} z_{-1}^{cd} z_0^d)^{a_{n+1}} (\beta^{1+c} w_0^{c^2} z_{-2}^{cd} z_{-1}^d)^{b_{n+1}} (\beta w_0^c z_{-2}^d)^{c_{n+1}} w_0^{d_{n+1}} \\ =& \alpha^{d y_{n+1}} \beta^{(1-a)y_{n+1} + (1+c+c^2)a_{n+1} + (1+c)b_{n+1} + c_{n+1}} w_0^{c^3 a_{n+1} + c^2 b_{n+1} + c c_{n+1} + d_{n+1}} z_{-2}^{c^2 d a_{n+1} + cd b_{n+1} + d c_{n+1}} z_{-1}^{cd a_{n+1} + d b_{n+1}} z_0^{d a_{n+1}} \\ =& \alpha^{d y_{n+1}} \beta^{y_{n+4} - a y_{n+3}} w_0^{a_{n+4} - a a_{n+3}} z_{-2}^{d(a_{n+3} - a a_{n+2})} z_{-1}^{d(a_{n+2} - a a_{n+1})} z_0^{d a_{n+1}}, \end{aligned} \tag{49}$$

for $n \in \mathbb{N}_0$.

Since (40) is solvable we have that closed-form formulas for $(a_k)_{k\geq-3}$ and $(y_k)_{k\geq-3}$ can be found. Using such obtained formulas in (49) shows the solvability of (46). By some calculation it is shown that (39) and (49) are solutions to (1), completing the proof. □

Corollary 3. *Assume that $a, b, c, d \in \mathbb{Z}$, $bd \neq 0$, $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the general solution to system* (1) *is given by* (39) *and* (49), *where sequence $(a_k)_{k\geq-3}$ is given by* (40) *and $a_{-3} = a_{-2} = a_{-1} = 0$, $a_0 = 1$, while $(y_k)_{k\geq-3}$ is given by* (42) *and* (43).

3.1. *Structure of sequence a_n in the case $bd \neq 0$*

Equation (40) is solvable, since its characteristic polynomial

$$p_4(\lambda) = \lambda^4 - (a+c)\lambda^3 + ac\lambda^2 - bd, \tag{50}$$

for the case $bd \neq 0$, is of the forth degree, so, solvable by radicals. By using the Ferrari-type argument given in [3] we can write the equation $p_4(\lambda) = 0$ in the following form

$$\left(\lambda^2 - \frac{a+c}{2}\lambda + \frac{s}{2}\right)^2 - \left(\left(\frac{(a-c)^2}{4} + s\right)\lambda^2 - \frac{(a+c)s}{2}\lambda + \frac{s^2}{4} + bd\right) = 0, \tag{51}$$

and choose parameter s so that the expression in the second bracket in (51) is a perfect square, i.e., $((a+c)s)^2 = ((a-c)^2+4s)(s^2+4bd)$, which is equivalent to

$$s^3 - acs^2 + 4bds + bd(a-c)^2 = 0, \tag{52}$$

while equation (51) becomes

$$\Big(\lambda^2 - \frac{a+c}{2}\lambda + \frac{s}{2}\Big)^2 - \Big(\frac{\sqrt{(a-c)^2+4s}}{2}\lambda - \frac{(a+c)s}{2\sqrt{(a-c)^2+4s}}\Big)^2 = 0, \tag{53}$$

which is equivalent to the following two quadratic equations

$$\lambda^2 - \Big(\frac{a+c}{2} + \frac{\sqrt{(a-c)^2+4s}}{2}\Big)\lambda + \frac{s}{2} + \frac{(a+c)s}{2\sqrt{(a-c)^2+4s}} = 0, \tag{54}$$

$$\lambda^2 - \Big(\frac{a+c}{2} - \frac{\sqrt{(a-c)^2+4s}}{2}\Big)\lambda + \frac{s}{2} - \frac{(a+c)s}{2\sqrt{(a-c)^2+4s}} = 0. \tag{55}$$

By using the change of variables $s = t + \frac{ac}{3}$ equation (52) becomes

$$t^3 + pt + q = 0, \tag{56}$$

where

$$p = \frac{12bd - a^2c^2}{3} \quad \text{and} \quad q = \frac{(27a^2+27c^2-18ac)bd - 2a^3c^3}{27}.$$

A solution to equation (56) is found in the following form $t = u + v$. If we put it into (56) and request that $uv = -p/3$, it is obtained that $u^3 + v^3 = -q$ and $u^3v^3 = -p^3/27$, which implies that u^3 and v^3 are solutions to the quadratic equation $z^2 + qz - p^3/27$, so consequently they are equal to $(-q \pm \sqrt{q^2+4p^3/27})/2$. Hence

$$t = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{57}$$

If $p = -\Delta_0/3$ and $q = -\Delta_1/27$, from (57), we have

$$t = \frac{1}{3\sqrt[3]{2}}\Big(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\Big). \tag{58}$$

For such chosen t, that is, s, equations (54) and (55) are easily solved and by some calculation it is obtained that the zeros of polynomial (50) are the following:

$$\lambda_1 = \frac{a+c}{4} + \frac{1}{2}\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t} + \frac{1}{2}\sqrt{\frac{(a+c)^2}{2} - \frac{4ac}{3} - t - \frac{Q}{4\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t}}}, \tag{59}$$

$$\lambda_2 = \frac{a+c}{4} + \frac{1}{2}\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t} - \frac{1}{2}\sqrt{\frac{(a+c)^2}{2} - \frac{4ac}{3} - t - \frac{Q}{4\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t}}}, \tag{60}$$

$$\lambda_3 = \frac{a+c}{4} - \frac{1}{2}\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t} + \frac{1}{2}\sqrt{\frac{(a+c)^2}{2} - \frac{4ac}{3} - t + \frac{Q}{4\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t}}}, \tag{61}$$

$$\lambda_4 = \frac{a+c}{4} - \frac{1}{2}\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t} - \frac{1}{2}\sqrt{\frac{(a+c)^2}{2} - \frac{4ac}{3} - t - \frac{Q}{4\sqrt{\frac{(a+c)^2}{4} - \frac{2ac}{3} + t}}}, \tag{62}$$

where

$$\Delta_0 := a^2c^2 - 12bd, \tag{63}$$

$$\Delta_1 := 2a^3c^3 - 27(a+c)^2bd + 72acbd, \tag{64}$$

$$Q := -(a+c)(a-c)^2. \tag{65}$$

From Lemma 3 we see that the nature of the zeros is determined by the discriminant

$$\Delta := \frac{1}{27}(4\Delta_0^3 - \Delta_1^2), \tag{66}$$

and the signs of

$$P : -3a^2 + 2ac - 3c^2 \tag{67}$$

and

$$D := -64bd - 16a^2c^2 + 16ac(a+c)^2 - 3(a+c)^4. \tag{68}$$

First note that $P < 0$ for every $(a, c) \neq (0, 0)$, since the polynomial $-3t^2 + 2t - 3$ is negative for all $t \in \mathbb{R}$, and that $P = 0$ if and only if $a = c = 0$.

All the zeros of p_4 are different and none of them is equal to 1. If, for example, $a = 1$, $c = 2$ and $bd = 3$, polynomial (50) becomes

$$p_4(\lambda) = \lambda^4 - 3\lambda^3 + 2\lambda^2 - 3.$$

Since $\Delta < 0$, by Lemma 3 we see that p_4 has four different zeros (two real and two complex-conjugate). Note that the same situation always appears when $\Delta_0 < 0$, that is, if $a^2c^2 < 12bd$, since this implies $\Delta < 0$.

If $a = c$ and $a^4 > 16bd > 0$, then by some calculation it is shown that $\Delta > 0$ and $D < 0$, which along with the fact $P < 0$ and by using Lemma 3 shows that in this case polynomial p_4 have four different real zeros. Since the case $P > 0$ is not possible, p_4 cannot have two pairs of complex-conjugate zeros.

In these cases the zeros λ_j, $j = \overline{1,4}$, of (50) are different, so the general solution to (40) has the form

$$a_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n + \alpha_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{69}$$

where α_j, $j = \overline{1,4}$, are arbitrary constants.

Employing Lemma 2 to p_4, we have

$$\sum_{j=1}^{4} \frac{\lambda_j^l}{p_4'(\lambda_j)} = 0,$$

for $l = \overline{0,2}$, and

$$\sum_{j=1}^{4} \frac{\lambda_j^3}{p_4'(\lambda_j)} = 1,$$

where λ_j, $j = \overline{1,4}$, are given by (59)-(62).

From this, since $a_{-3} = a_{-2} = a_{-1} = 0$ and $a_0 = 1$, and the form of the solution to (40), it follows that

$$a_n = \sum_{j=1}^{4} \frac{\lambda_j^{n+3}}{p_4'(\lambda_j)} = \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}$$
$$+ \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}, \tag{70}$$

for $n \geq -3$.

Employing (70) into (43), is obtained

$$y_n = \sum_{j=0}^{n-1} \sum_{i=1}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \sum_{i=1}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \quad n \in \mathbb{N}, \tag{71}$$

when $\lambda_j \neq 1$, $j = \overline{1,4}$, which is equivalent to $p_4(1) \neq 1$. It is easily verified by Lemma 2 that (71) also holds for $n = -j$, $j = \overline{0,3}$.

All the zeros of p_4 are different and one of them is equal to 1. Polynomial p_4 will have a zero equal to 1 if $p_4(1) = 1 - a - c + ac - bd = 0$, that is, if

$$(a-1)(c-1) = bd. \tag{72}$$

Hence

$$\begin{aligned} p_4(\lambda) =& \lambda^4 - (a+c)\lambda^3 + ac\lambda^2 - (a-1)(c-1) \\ =& (\lambda - 1)(\lambda^3 - (a+c-1)\lambda^2 + (a-1)(c-1)\lambda + (a-1)(c-1)). \end{aligned}$$

By using the change of variables $\lambda = t + (a+c-1)/3$, the equation

$$\lambda^3 - (a+c-1)\lambda^2 + (a-1)(c-1)\lambda + (a-1)(c-1) = 0,$$

is transformed to $t^3 + \hat{p}t + \hat{q} = 0$ with

$$\hat{p} = -\frac{(a+c-1)^2}{3} + (a-1)(c-1) \quad \text{and} \quad \hat{q} = -\frac{2(a+c-1)^3}{27} + \frac{(a-1)(c-1)(a+c+2)}{3}.$$

Hence, as in (57), it is obtained that the zeros of the last equation are:

$$t_j = \varepsilon^j \sqrt[3]{-\frac{\hat{q}}{2} - \sqrt{\frac{\hat{q}^2}{4} + \frac{\hat{p}^3}{27}}} + \overline{\varepsilon}^j \sqrt[3]{-\frac{\hat{q}}{2} + \sqrt{\frac{\hat{q}^2}{4} + \frac{\hat{p}^3}{27}}}, \quad j = \overline{0,2}, \tag{73}$$

where $\varepsilon^3 = 1$ and $\varepsilon \neq 1$, and consequently

$$\lambda_j = \frac{a+c-1}{3} + \varepsilon^{j-2} \sqrt[3]{-\frac{\hat{q}}{2} - \sqrt{\frac{\hat{q}^2}{4} + \frac{\hat{p}^3}{27}}} + \overline{\varepsilon}^{j-2} \sqrt[3]{-\frac{\hat{q}}{2} + \sqrt{\frac{\hat{q}^2}{4} + \frac{\hat{p}^3}{27}}}, \quad j = \overline{2,4}. \tag{74}$$

If, for example, $a = 3$ and $c = 2$, then $bd = 2$, $\Delta \neq 0$. Thus, p_4 has four different zeros one of which is equal to 1, and

$$p_4(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 - 2 = (\lambda - 1)(\lambda^3 - 4\lambda^2 + 2\lambda + 2). \tag{75}$$

Formula (70) also holds but with $\lambda_1 = 1$. Further, we have

$$\begin{aligned} y_n =& \sum_{j=0}^{n-1} \frac{1}{p_4'(1)} + \sum_{j=0}^{n-1} \sum_{i=2}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} \\ =& \frac{n}{4 - 3a - 3c + 2ac} + \sum_{i=2}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \end{aligned} \tag{76}$$

since $p_4'(1) = 4 - 3a - 3c + 2ac$. Some calculation shows that (76) holds for $n \geq -3$.

From this and by Corollary 3 we get the following result.

Corollary 4. *Consider system* (1) *with $a, b, c, d \in \mathbb{Z}$ and $bd \neq 0$. Assume that $z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta \neq 0$. Then the following statements are true.*

(a) *If none of the zeros of polynomial* (50) *is equal to 1, i.e., if* $(a-1)(c-1) \neq bd$, *then the general solution to system* (1) *is given by formulas* (39) *and* (49), *where the sequence* $(a_n)_{n\geq -3}$ *is given by* (70), $(y_n)_{n\geq -3}$ *is given by* (71), *while* λ_j*-s,* $j = \overline{1,4}$, *are given by* (59)-(62).

(b) *If (exactly) one of the zeros of polynomial* (50) *is equal to 1, say* λ_1, *i.e., if* $(a-1)(c-1) = bd$ *and* $4-3a-3c+2ac \neq 0$, *then the general solution to system* (1) *is given by formulas* (39) *and* (49), *where the sequence* $(a_n)_{n\geq -3}$ *is given by* (70) *with* $\lambda_1 = 1$, $(y_n)_{n\geq -3}$ *is given by* (76), *while* λ_j*-s,* $j = \overline{2,4}$, *are given by* (74).

Case when p_4 *has exactly one double zero which is different from 1.* If $a = 4$, $c = 0$ and $bd = -27$, then (50) is

$$p_4(\lambda) = \lambda^4 - 4\lambda^3 + 27 = (\lambda - 3)^2(\lambda^2 + 2\lambda + 3).$$

So there is a polynomial with exactly one double zero (here it is $\lambda_{1,2} = 3$) different from one. The other two zeros are complex conjugate $\lambda_{3,4} = -1 \pm i\sqrt{2}$.

In the case when only two zeros are equal, say λ_1 and λ_2, then the general solution has the following form

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{77}$$

where γ_j, $j = \overline{1,4}$, are arbitrary constants.

The solution satisfying conditions $a_{-3} = a_{-2} = a_{-1} = 0$ and $a_0 = 1$ is obtained, for example, by letting $\lambda_1 \to \lambda_2$ in (70). It is shown that

$$\begin{aligned} a_n = \lim_{\lambda_1\to\lambda_2}\Bigg(&\frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &+ \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\Bigg) \\ =&\frac{\lambda_2^{n+2}((n+3)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4) - \lambda_2(2\lambda_2-\lambda_3-\lambda_4))}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2} \\ &+ \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)}. \end{aligned} \tag{78}$$

Employing (78) in (43) and using Lemma 1, we get

$$\begin{aligned} y_n =& \sum_{j=0}^{n-1}\Bigg(\frac{\lambda_2^{j+2}((j+3)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4) - \lambda_2(2\lambda_2-\lambda_3-\lambda_4))}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2} + \frac{\lambda_3^{j+3}}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)}\Bigg) \\ =&\frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(1-\lambda_2)^2} + \frac{(\lambda_2^4 - 2\lambda_2^3\lambda_3 - 2\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_3\lambda_4)(\lambda_2^n - 1)}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2(\lambda_2-1)} \\ &+ \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)(\lambda_3-1)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)(\lambda_4-1)}. \end{aligned} \tag{79}$$

Case when p_4 *has three different zeros and* 1 *is a double zero.* Polynomial p_4 has a double zero equal to 1 if (72) holds and if

$$p_4'(1) = 4 - 3(a+c) + 2ac = 0, \tag{80}$$

that is, if and only if

$$(2a-3)(2c-3) = 1. \tag{81}$$

From (81) we have that it must be $a = c = 2$ or $a = c = 1$. If $a = c = 1$, then $bd = 0$, which is impossible. If $a = c = 2$, then $bd = 1 \neq 0$, from which it follows that

$$p_4(\lambda) = \lambda^4 - 4\lambda^3 + 4\lambda^2 - 1 = (\lambda - 1)^2(\lambda^2 - 2\lambda - 1). \tag{82}$$

From (82) we have that

$$\lambda_{1,2} = 1, \quad \lambda_{3,4} = 1 \pm \sqrt{2}. \tag{83}$$

From this, we have proved in passing, that there are no such $a, c \in \mathbb{Z} \setminus \{1\}$, such that 1 is a triple zero of p_4.

In this case, we have

$$a_n = \frac{n(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} + \frac{\lambda_3^{n+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)}, \tag{84}$$

and

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1}\Big(\frac{j(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} + \frac{\lambda_3^{j+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)}\Big) \\ &= \frac{(n-1)n}{2(1-\lambda_3)(1-\lambda_4)} + \frac{n(3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1)}{(1-\lambda_3)^2(1-\lambda_4)^2} + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3 - 1)^3(\lambda_3 - \lambda_4)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4 - 1)^3(\lambda_4 - \lambda_3)}. \end{aligned} \tag{85}$$

Corollary 5. *Consider system* (1) *with $a, b, c, d \in \mathbb{Z}$ and $bd \neq 0$. Assume that $z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If only one of the zeros of polynomial* (50) *is double and different from* 1, *then the general solution to system* (1) *is given by formulas* (39) *and* (49), *where sequence $(a_n)_{n\geq -3}$ is given by* (78), *while $(y_n)_{n\geq -3}$ is given by* (79).

(b) *If only double zero of polynomial* (50) *is equal to* 1, *say $\lambda_1 = \lambda_2 = 1$, then the general solution to system* (1) *is given by formulas* (39) *and* (49), *where sequence $(a_n)_{n\geq -3}$ is given by* (84), *$(y_n)_{n\geq -3}$ is given by* (85), *while λ_j-s, $j = \overline{1,4}$, are given by* (83).

Case when p_4 has two pairs of different double zeros. According to Lemma 3 such a situation happens if $\Delta = D = 0$. Hence it must be

$$bd = \frac{-16a^2c^2 + 16ac(a+c)^2 - 3(a+c)^4}{64}. \tag{86}$$

Using this in the relation $\Delta = 0$, we get

$$4\Big(2ac - \frac{3}{4}(a+c)^2\Big)^6 = \Big(-2^4a^3c^3 + \frac{99}{4}a^2c^2(a+c)^2 - \frac{3^4}{2^3}ac(a+c)^4 + \frac{3^4}{2^6}(a+c)^6\Big)^2,$$

from which it follows that

$$2048a^3c^3 - 2736a^2c^2(a+c)^2 + 1080ac(a+c)^4 - 135(a+c)^6 = 0 \tag{87}$$

or

$$(a+c)^2(a-c)^4 = 0. \tag{88}$$

If $a = 0$ or $c = 0$, then from (87) and (88), it follows that $a = c = 0$, which implies $bd = 0$, which is impossible. Hence, if $a \neq 0 \neq c$ for the case of equation (87), we can use the change of variables $t = (a + c)^2/ac$, and transform it into the equation

$$135t^3 - 1080t^2 + 2736t - 2048 = 0. \tag{89}$$

Since the discriminant of equation (89) is negative it has only one real zero, which is equal to 4/3, and two-complex conjugate ones. Hence, it must be $3(a + c)^2 = 4ac$, that is, $3a^2 + 2ac + 3c^2 = 0$. Since $a \neq 0 \neq c$, and the polynomial $3t^2 + 2t + 3$ is always positive it follows that such a and c do not exist, in this case.

If $a - c = 0$, then from (86) it follows that $bd = 0$, which is impossible. Finally, if $a + c = 0$, then from (86) it follows that $bd = -a^4/4$, from which it follows that

$$p_4(\lambda) = \lambda^4 - a^2\lambda^2 + \frac{a^4}{4} = \left(\lambda^2 - \frac{a^2}{2}\right)^2.$$

Hence, this is the only case when p_4 has two pairs of double zeros, which are

$$\lambda_{1,2} = a/\sqrt{2} \quad \text{and} \quad \lambda_{3,4} = -a/\sqrt{2}. \tag{90}$$

The characteristic polynomial (50), in this case, has two double zeros, say, $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$, so the general solution to equation (40) has the following form

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + (\gamma_3 + \gamma_4 n)\lambda_4^n, \quad n \in \mathbb{N}, \tag{91}$$

where γ_j, $j = \overline{1,4}$, are arbitrary constants.

In this case, we have

$$a_n = \frac{\lambda_2^{n+2}(n(\lambda_2 - \lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2 - \lambda_4)^4} + \frac{\lambda_4^{n+2}(n(\lambda_4 - \lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4 - \lambda_2)^4} \tag{92}$$

and

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1}\left(\frac{\lambda_2^{j+2}(j(\lambda_2 - \lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2 - \lambda_4)^4} + \frac{\lambda_4^{j+2}(j(\lambda_4 - \lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4 - \lambda_2)^4}\right) \\ &= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2 - \lambda_4)^2(1 - \lambda_2)^2} + \frac{(\lambda_2^4 - 4\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_4^2)(\lambda_2^n - 1)}{(\lambda_2 - \lambda_4)^4(\lambda_2 - 1)} \\ &\quad + \frac{\lambda_4^3 - n\lambda_4^{n+2} + (n-1)\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(1 - \lambda_4)^2} + \frac{(\lambda_4^4 - 4\lambda_2\lambda_4^3 + 3\lambda_2^2\lambda_4^2)(\lambda_4^n - 1)}{(\lambda_4 - \lambda_2)^4(\lambda_4 - 1)}. \end{aligned} \tag{93}$$

Since $a \in \mathbb{Z}$, then numbers $\pm a/\sqrt{2}$ are irrational, so it is not possible that polynomial p_4 has two double zeros, one of which is equal to 1.

Corollary 6. *Consider system* (1) *with* $a, b, c, d \in \mathbb{Z}$ *and* $bd \neq 0$. *Assume* $z_{-2}, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If polynomial* (50) *has two pairs of double zeros both different from* 1, *then the general solution to system* (1) *is given by formulas* (39) *and* (49), *where sequence* $(a_n)_{n\geq-3}$ *is given by* (92), $(y_n)_{n\geq-3}$ *is given by* (93), *while* λ_j*-s,* $j = \overline{1,4}$, *are given by* (90).

(b) *The polynomial* (50) *cannot have two pairs of double zeros such that one of them is equal to* 1.

Triple zero case. If polynomial (50) has a triple zero, then it must be $\Delta = \Delta_0 = 0$ which is equivalent to $\Delta_0 = \Delta_1 = 0$, that is, $bd = a^2c^2/12$ and

$$\Delta_1 = 2a^3c^3 - 27(a+c)^2\frac{a^2c^2}{12} + 72ac\frac{a^2c^2}{12} = -\frac{a^2c^2}{4}(9a^2 - 14ac + 9c^2).$$

Now note that $\Delta_1 < 0$ if $a \neq 0 \neq c$, since the polynomial $9t^2 - 14t + 9$ is positive for every $t \in \mathbb{R}$, and $\Delta_1 = 0$ if and only if $a = 0$ or $c = 0$. If $a = 0$ or $c = 0$, we have that $bd = 0$ which is impossible. Hence, if $bd \neq 0$, polynomial (50) cannot have a triple zero, and consequently it cannot have a zero of the fourth order either.

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