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Some Existence of Coincidence Point and Approximate Solution Method for Generalized Weak Contraction in b-generalized Pseudodistance Fiunctions†

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Abstract. The purpose of this article is to prove some coincidence point and approximate solution method for generalized weak contraction mapping in $b-$metric spaces by using the concept of b-generalized pseudodistance. Also, we give some examples to illustrate our main results.

1. Introduction

In 1993, Bakhtin[5], (see also Czerwik [8]) was introduced the concept b-metric space and show that the class of b-metric spaces contains the class of metric spaces. Later, in 1996 Kada, Suzuki and Takahashi [11] defined the concept of w-distance in a metric space which is a generalized distance difference way from b-metric spaces. In 2010, Wlodarczyk and Plebaniak [20] introduced the concept of generalized pseudodistance which is generalized w-distance. Recently, in 2014, Plebaniak [14] introduce the concept of b-generalized pseudodistance which is an extension of the b-metric and generalized pseudodistances. They are also gave the sufficient conditions that as certain the existence of an optimal solution and fixed point problem. On the other hand, the concept of weak contraction was introduced by Alber and Guerre - Delabriere [3] in 1997 in Hilbert spaces. Later, in 2001 Rhoades [16] has show that the result which Alber et al. is also valid in complete metric spaces. In 2008 Dutta and Choudhury [10] extended the notion of weak contraction by using the concept of two altering distance functions. Afterward, Dorić [9] (see also [1]) replaced the continuity of φ by "lower semi-continuous" and proved fixed point theorems for such mapping. In 2014 Aghajani et al. [2] proved some common fixed point results for four mappings satisfying generalized weak contractive condition in partially ordered complete b-metric spaces. In the same year, Roshan et al. [17] presented some coincidence point results for four mappings satisfying generalized weak contraction mapping in ordered b-metric spaces. Fixed point results involving weak contraction and generalized weak contraction mappings have extensively been studied in the literature (see, e.g., [4, 7, 13, 15, 19, 21] and references therein).

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From above mentioned, the main purpose of this article is to prove the existence of coincidence points theorem for generalized weak contraction in b-metric spaces via b-generalized pseudodistance. Furthermore, we also give some examples to illustrate our main results.

2. Preliminaries

In this section, we give some notations and basic knowledge for our consideration. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

2.1. b-metric Spaces

Definition 2.1 ([5, 8]). *Let X be a nonempty set and a $b-$metric is a function $d : X \times X \to [0, \infty)$ satisfying*

(bM1) $d(x, y) = 0$ *if and only if* $x = y$

(bM2) $d(x, y) = d(y, x)$

(bM3) there exists a real number $s \geq 1$ *such that* $d(x, y) \leq s[d(x, z) + d(z, y)]$,

for all $x, y, z \in X$. Then (X, d) is called $b-$metric space with coefficient s.

It is obvious that the class of b-metric spaces is effectively larger than that of metric spaces since any metric space is a b-metric space with $s = 1$. The following examples show that, in general, a b-metric space need not necessarily be a metric space.

Example 2.2. *Let $X = \mathbb{R}$ and the mapping $d : X \times X \to \mathbb{R}_+$ be defined by*

$$d(x, y) = (x - y)^2 \quad \text{for all } x, y \in X.$$

Then (X, d) is a b-metric space with coefficient $s = 2$.

Example 2.3. *([18]) Let (X, d) be a metric space and the mapping $\sigma_d : X \times X \to \mathbb{R}_+$ be defined by*

$$\sigma_d(x, y) = [d(x, y)]^p \quad \text{for all } x, y \in X,$$

where $p > 1$ is a fixed real number. Then (X, σ_d) is a b-metric space with coefficient $s = 2^{p-1}$.

Next, we recall the concepts of b-convergence, b-Cauchy sequence, b-continuity and b-completeness in a b-metric spaces.

Definition 2.4 ([6]). *Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*

(a) *b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.*

(b) *b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Each b-convergent sequence in b-metric spaces has a unique limit and it is also a b-Cauchy sequence. Moreover, in general, a b-metric is not continuous.

Lemma 2.5 ([2]). *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Definition 2.6 ([6]). *Let (X, d_X) and (Y, d_Y) be two b-metric spaces.*

1. *The space (X, d_X) is b-complete if every b-Cauchy sequence in X b-converges.*
2. *A function $f : X \to Y$ is b-continuous at a point $x \in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.*

Definition 2.7 ([6]). *Let Y be a nonempty subset of a b-metric space (X, d). The closure of Y is denoted by $\overline{Y}$ and it is the set of limits of all convergent sequences of points in Y, i.e.,*

$$\overline{Y} := \{x \in X : \text{ there exists a sequence } \{x_n\} \text{ in } Y \text{ such that } \lim_{n\to\infty} x_n = x\}.$$

Definition 2.8 ([6]). *A subset Y of a b-metric space (X, d) is called* closed *if and only if for each sequence $\{x_n\}$ in Y which b-converges to an element $x \in X$, we have $x \in Y$ (i.e. $Y = \overline{Y}$).*

2.2. *b-generalized Pseudodistance Function*

In the rest of the paper we assume that the b-metric $d : X \times X \to [0, \infty)$ is continuous on X^2. Now, we recal the concept of a generalized pseudodistance and b-generalized pseudodistance as follow.

Definition 2.9 ([14]). *Let X be a b-metric space (with constant $s \geq 1$. The map $J : X \times X \to [0, \infty)$ is said to be a* b-generalized pseudodistance *on X if the following two conditions hold:*

(J1) $J(x, y) \leq s[J(x, z) + J(z, y)]$ *for all $x, y, z \in X$,*

(J2) for any sequences $\{x_m\}$ and $\{y_m\}$ in X such that

$$\lim_{n\to\infty} \sup_{m>n} J(x_n, x_m) = 0 \tag{1}$$

and

$$\lim_{m\to\infty} J(x_m, y_m) = 0 \tag{2}$$

we have

$$\lim_{m\to\infty} d(x_m, y_m) = 0. \tag{3}$$

In definition 2.9, if $s = 1$, then J is called *generalized pseudodistance* which defined by Włodarczyk and Plebaniak [20]. Now, we give some examples of b-generalized pseudodistance.

Example 2.10. *([15]) Let X be a b-metric space (with a constant $s \geq 1$) equipped in b-metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ Let the closed set $E \subset X$, containing at least two different points, be arbitrary and fixed. Let $c > 0$ be such that $c > \delta(E)$, where $\delta(E) = \sup\{d(x, y) : x, y \in X\}$ is arbitrary and fixed. Define $J : X \times X \to [0, \infty)$ by*

$$J(x, y) = \begin{cases} d(x, y) & E \cap \{x, y\} = \{x, y\} \\ c & E \cap \{x, y\} \neq \{x, y\}. \end{cases}$$

Then J is b-generalized pseudodistance in X.

Example 2.11. *Let $X = [0, 2]$ and $d(x, y) = (x - y)^2$. Defined $J : X \times X \to [0, \infty)$ by*

$$J(x, y) = \begin{cases} 0, & x - y = -2 \\ d(x, y), & -2 < x - y \leq 0 \\ x + 2^2, & 0 < x - y \leq 2. \end{cases}$$

We will show that J is b-generalized pseudodistance wit $s = 2$. Suppose that $x, y, z \in X$, We consider the following cases.

case I *If $x - y = -2$, then it is clear that $J(x,y) = 0 \leq 2J(x,z) + 2J(z,y)$ for all $z \in X$.*
case II *Suppose that $-2 < x - y \leq 0$.*
If $x \leq z$ and $z \leq y$, then

$$J(x,y) = d(x,y) \leq 2d(x,z) + 2d(z,y) = 2J(x,z) + 2J(z,y).$$

If $x \leq z$ and $z > y$, then

$$\begin{aligned} J(x,y) = d(x,y) &= x^2 - 2xy + y^2 \\ &< x^2 - 2xy + z^2 + 2xz - 2xz \\ &= (x-z)^2 + 2xz - 2xy \\ &< 2(x-z)^2 + 2(z+2^2) \\ &= 2J(x,z) + 2J(z,y). \end{aligned}$$

If $x > z$ and $z \leq y$ then

$$\begin{aligned} J(x,y) = d(x,y) &< 4 - 2yz + y^2 \\ &< x + 2^2 + (z^2 - 2yz + y^2) \\ &< 2(x + 2^2) + 2(z-y)^2 \\ &= 2J(x,z) + 2J(z,y). \end{aligned}$$

case III *Suppose that $0 < x - y \leq 2$.*
If $x > z$ and $z \leq y$ then

$$\begin{aligned} J(x,y) = x + 2^2 &\leq (x + 2^2) + (z-y)^2 \\ &< 2J(x,z) + 2J(z,y). \end{aligned}$$

If $x > z$ and $z > y$ then

$$\begin{aligned} J(x,y) = x + 2^2 &< (x + 2^2) + (z + 2^2) \\ &< 2J(x,z) + 2J(z,y). \end{aligned}$$

If $x \leq z$ and $z > y$ then

$$\begin{aligned} J(x,y) = x + 2^2 &\leq (x-z)^2 + (z + 2^2) \\ &< 2J(x,z) + 2J(z,y). \end{aligned}$$

There fore the condition $(J1)$ hold. Next we will show that J is satisfies $(J2)$. Let $\{x_n\}$ and $\{y_n\}$ be sequence in X such that

$$\lim_{n\to\infty} \sup_{m>n} J(x_n, x_m) = 0 \ \text{ and } \ \lim_{m\to\infty} J(x_m, y_m) = 0.$$

Since $J(x,y) > 2^2$ for all $x > y$ and $\lim_{m\to\infty} J(x_m, y_m) = 0$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$ and thus $(J2)$ hold. Therefore J is b-generalized pseudodistance wit $s = 2$. Further, since $J(0,0) = 0 = J(0,2)$, but $0 \neq 2$ and hence J is not a b-metric.

Remark 2.12. *Let (X,d) be a complete b-metric space (with $s \geq 1$) and J is a b-generalized pseudodistance on X.*

(A) d is a b-generalized pseudodistance on X. However, there exists a b-generalized pseudodistance on X which is not b-metric.
(B) Let $X_J^0 = \{x \in X : J(x,x) = 0\}$ and $X_J^+ = \{x \in X : J(x,x) > 0\}$, then $X = X_J^0 \cup X_J^+$.
(C) For each $x, y \in X$ with $x \neq y$, then $J(x,y) > 0 \vee J(y,x) > 0$(for more detail see ([14]).

Definition 2.13 ([15]). *Let X be a b-metric space (with constant $s \geq 1$). The map $J : X \times X \to [0,\infty)$ is a* b-generalized pseudodistance *on X. We call that X is J-complete if for all sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} \sup_{m>n} J(x_n, x_m) = 0$$

there exists $x \in X$ such that

$$\lim_{m\to\infty} J(x_m, x) = \lim_{m\to\infty} J(x, x_m) = 0.$$

Remark 2.14. *If we take $J = d$, then the definitions of J-completeness and completeness are identical.*

Lemma 2.15 ([14]). *Let $J : X \times X \to [0,\infty)$ be a b-generalized pseudodistance on b-metric space X (with $s \geq 1$) and let $\{x_n\}$ be the sequence in X. If*

$$\lim_{n\to\infty} \sup_{m>n} J(x_n, x_m) = 0,$$

then $\{x_n\}$ is a Cauchy sequence on X.

2.3. *Some History of Various Types of Weak Contraction*

In 1984, Khan et al. [12] introduced the concept of an altering distance function as follows.

Definition 2.16 ([12]). *A function $\psi : [0,\infty) \to [0,\infty)$ is called an* altering distance function *if the following properties are satisfied:*

1. *ψ is continuous and monotone nondecreasing;*
2. *$\psi(t) = 0$ if and only if $t = 0$.*

Here, we recall some examples of an altering distance function given in [18] as follow.

Example 2.17. *Let $\varphi_i : [0,\infty) \to [0,\infty), i \in \{1,2,...,5\}$ be defined by*

(φ_1) $\varphi_1(t) = kt$, *where* $k > 0$,

(φ_2) $\varphi_2(t) = t^k$, *where* $k > 0$,

(φ_3) $\varphi_3(t) = \begin{cases} t^2, & t \in [0,1] \\ 1 + \sqrt{t-1}, & t \in (1,\infty). \end{cases}$

(φ_4) $\varphi_4(t) = a^t - 1$, *where* $a > 0$ *and* $a \neq 1$,

(φ_5) $\varphi_5(t) = \log(kt+1)$, *where* $k > 0$.

Then φ_i is an altering distance function for each $i \in \{1,2,...,5\}$.

In 1997 Alber and Guerre-Delabriere [3](see also [16]) using the concept of altering distance function established the notion weak contraction as follow.

Definition 2.18 ([3, 16]). *A mapping $T : X \to X$ is said to be* weak contraction, *if for each $x, y \in X$,*

$$d(Tx,Ty) \leq d(x,y) - \psi(d(x,y)), \tag{4}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a altering distance functions and $\lim_{t\to\infty} \psi(t) = \infty$. If X is bounded, then the infinity condition can be omitted (see [3, 16]).

If we take $\psi(t) = (1-k)t$, then the inequality (4), reduce to Banach contraction mapping.

In 2008, Dutta and Choudhury [10] extended the concept of weak contraction by using the concept of two altering distance functions and proved the fixed point results for such contractions as follow.

Theorem 2.19. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)) \tag{5}$$

for all $x, y \in X$, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are two altering distance functions. Then T has a unique fixed point.

Clearly, if we take ψ is identity mapping, then the inequality (5), reduce to (4).

Afterward, Dorić [9] (see also [1]) replaced the continuity of φ by ”lower semi-continuous” which is include the classes in (4) and (5). Also, they are proved the following theorem.

Theorem 2.20 ([9]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(N(x, y)) - \varphi(N(x, y)) \tag{6}$$

for all $x, y \in X$, where N is given by

$$N(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

$\psi : [0, \infty) \to [0, \infty)$ is an altering distance function and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

3. Main Results

In this section, we establish the existence theorems of coincidence points for generalized weak contraction mapping in $b-$metric spaces via b-generalized pseudodistance function. First, we give some useful lemmas for consideration of our main result as follow.

Lemma 3.1. *Let X be b-metric space (with $s \geq 1$) and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Let $f, g, T, S : X \to X$ is a four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ with $f(X) \subseteq X_j^0$ and $g(X) \subseteq X_j^0$. Assume that there exist altering distance function ψ and φ such that for all $x, y \in X$ satisfies*

$$\psi(s^4 M(fx, gy)) \leq \psi(M_s^J(x, y)) - \varphi(M_s^J(x, y)), \tag{7}$$

where

$$M(fx, gy) = \max\{J(fx, gy), J(gy, fx)\}$$

and

$$M_s^J(x, y) = \max\{\min\{J(Sx, Ty), J(Ty, Sx)\}, J(Sx, fx), J(Ty, gy), \frac{J(Sx, gy) + J(Ty, fx)}{2s}\}.$$

Then

$$\lim_{n\to\infty} J(y_n, y_{n+1}) = 0 \tag{8}$$

and

$$\lim_{n\to\infty} J(y_{n+1}, y_n) = 0 \tag{9}$$

when, the sequence $\{y_n\}$ generated by $x_0 \in X$ such that

$$y_{2n+1} := fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} := gx_{2n+1} = Sx_{2n+2} \quad \text{for all } n \geq 0.$$

Proof. Let $x_0 \in X$ be an arbitrary and fixed. Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. We can define inductively the sequences $\{x_n\}$ and $\{y_n\}$ by

$$y_{2n+1} := fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} := gx_{2n+1} = Sx_{2n+2} \quad \text{for all } n \geq 0.$$

From (7) and property of ψ implies that

$$J(fx, gy) \leq \psi(s^4 J(fx, gy)) \leq \psi(M_s^J(x, y)) - \varphi(M_s^J(x, y)), \tag{10}$$

and

$$J(gy, fx) \leq \psi(s^4 J(gy, fx)) \leq \psi(M_s^J(x, y)) - \varphi(M_s^J(x, y)). \tag{11}$$

First, let us to show (8) hold, by prove that

$$\lim_{n\to\infty} J(y_{2n+1}, y_{2n+2}) = 0 \quad \text{and} \quad \lim_{n\to\infty} J(y_{2n}, y_{2n+1}) = 0. \tag{12}$$

Putting $x := x_{2n}$ and $y := x_{2n+1}$ in (10), we have

$$\begin{aligned} \psi(J(y_{2n+1}, y_{2n+2})) &\leq \psi(s^4 J(y_{2n+1}, y_{2n+2})) \\ &= \psi(s^4 J(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M_s^J(x_{2n}, x_{2n+1})) - \varphi(M_s^J(x_{2n}, x_{2n+1})), \end{aligned} \tag{13}$$

where

$$\begin{aligned} M_s^J(x_{2n}, x_{2n+1}) &= \max\Big\{ \min\{J(Sx_{2n}, Tx_{2n+1}), J(Tx_{2n+1}, Sx_{2n})\}, \\ &\quad J(Sx_{2n}, fx_{2n}), J(Tx_{2n+1}, gx_{2n+1}), \frac{J(Sx_{2n}, gx_{2n+1}) + J(Tx_{2n+1}, fx_{2n})}{2s}\Big\} \\ &= \max\{\min\{J(y_{2n}, y_{2n+1}), J(y_{2n+1}, y_{2n})\}, \\ &\quad J(y_{2n}, y_{2n+1}), J(y_{2n+1}, y_{2n+2}), \frac{J(y_{2n}, y_{2n+2}) + J(y_{2n+1}, y_{2n+1})}{2s}\Big\}. \end{aligned} \tag{14}$$

Let us consider the following three cases.

case I . If

$$\begin{aligned} M_s^J(x_{2n}, x_{2n+1}) &= \max\{\min\{J(y_{2n}, y_{2n+1}), J(y_{2n+1}, y_{2n})\}, J(y_{2n}, y_{2n+1})\} \\ &= J(y_{2n}, y_{2n+1}). \end{aligned}$$

By (13), we get

$$\psi(J(y_{2n+1}, y_{2n+2})) \quad \leq \psi(J(y_{2n}, y_{2n+1})) - \psi(J(y_{2n}, y_{2n+1})) \quad \leq \psi(J(y_{2n}, y_{2n+1})).$$

From fact that ψ is non-decreasing, we have $J(y_{2n+1}, y_{2n+2}) \leq J(y_{2n}, y_{2n+1})$. This mean that the sequence $\{J(y_{2n+1}, y_{2n+2})\}$ is non-increasing and converges to some $j \geq 0$. Again, by (13) which give

$$\psi(j) \leq \psi(j) - \psi(j) \leq \psi(j)$$

and hence $j = 0$.

case II. If $M_s^J(x_{2n}, x_{2n+1}) = J(y_{2n+1}, y_{2n+2})$, then by (13), we get

$$\psi(J(y_{2n+1}, y_{2n+2})) \quad \leq \psi(J(y_{2n+1}, y_{2n+2})) - \psi(J(y_{2n+1}, y_{2n+2})) \quad \leq \psi(J(y_{2n+1}, y_{2n+2})).$$

for all $n \geq 0$. Then, we must have $J(y_{2n+1}, y_{2n+2}) = 0$ for all $n \geq 0$.

case III. If

$$M_s^J(x_{2n}, x_{2n+1}) = \frac{J(y_{2n}, y_{2n+2}) + J(y_{2n+1}, y_{2n+1})}{2s}.$$

In fact that $f(X) \subseteq X_j^0$, we have

$$M_s^J(x_{2n}, x_{2n+1}) = \frac{J(y_{2n}, y_{2n+2})}{2s}. \tag{15}$$

Hence, by (13) we get

$$\begin{aligned}\psi(J(y_{2n+1}, y_{2n+2})) &\leq \psi(\frac{J(y_{2n}, y_{2n+2})}{2s}) - \varphi(\frac{J(y_{2n}, y_{2n+2})}{2s}) \\ &\leq \psi(\frac{J(y_{2n}, y_{2n+2})}{2s}).\end{aligned} \tag{16}$$

It follow from the property of ψ and (J1), we have

$$\begin{aligned}J(y_{2n+1}, y_{2n+2}) &\leq \frac{J(y_{2n}, y_{2n+2})}{2s} \\ &\leq \frac{sJ(y_{2n}, y_{2n+1}) + sJ(y_{2n+1}, y_{2n+2})}{2s} \\ &= \frac{J(y_{2n}, y_{2n+1}) + J(y_{2n+1}, y_{2n+2})}{2}.\end{aligned} \tag{17}$$

Consequently, we obtain that $J(y_{2n+1}, y_{2n+2}) \leq J(y_{2n}, y_{2n+1})$. That is the sequence $\{J(y_{2n}, y_{2n+1})\}$ is non-increasing and converges to some nonnegative real number j'. By (10), (15), (17) and φ, ψ are non-decreasing function, we have

$$\begin{aligned}\psi(J(y_{2n+1}, y_{2n+2})) &\leq \psi(\frac{J(y_{2n}, y_{2n+2})}{2s}) - \varphi(\frac{J(y_{2n}, y_{2n+2})}{2s}) \\ &\leq \psi(\frac{J(y_{2n}, y_{2n+1}) + J(y_{2n+1}, y_{2n+2})}{2}) - \varphi(J(y_{2n+1}, y_{2n+2})).\end{aligned} \tag{18}$$

Letting $n \to \infty$ in (18) we obtain that

$$\psi(j') \leq \psi(\frac{j' + j'}{2}) - \varphi(j') \leq \psi(j')$$

and hence $j' = 0$. Now, letting $x := x_{2n}$ and $y := x_{2n-1}$ in (11), then we have

$$\begin{aligned}\psi(J(y_{2n}, y_{2n+1})) &\leq \psi(s^4 J(y_{2n}, y_{2n+1})) \\ &= \psi(s^4 J(gx_{2n-1}, fx_{2n})) \\ &\leq \psi(M_s^J(x_{2n}, x_{2n-1})) - \varphi(M_s^J(x_{2n}, x_{2n-1})),\end{aligned} \tag{19}$$

where

$$\begin{aligned}M_s^J(x_{2n}, x_{2n-1}) &= \max\Big\{ \min\{J(Sx_{2n}, Tx_{2n-1}), J(Tx_{2n-1}, Sx_{2n})\}, \\ &\quad J(Sx_{2n}, fx_{2n}), J(Tx_{2n-1}, gx_{2n-1}), \frac{J(Sx_{2n}, gx_{2n-1}) + J(Tx_{2n-1}, fx_{2n})}{2s}\} \\ &= \max\{\min\{J(y_{2n}, y_{2n-1}), J(y_{2n-1}, y_{2n})\}, \\ &\quad J(y_{2n}, y_{2n+1}), J(y_{2n-1}, y_{2n}), \frac{J(y_{2n}, y_{2n}) + J(y_{2n-1}, y_{2n+1})}{2s}\Big\}.\end{aligned} \tag{20}$$

We distinguish three cases.

case I . If

$$\begin{aligned}M_s^J(x_{2n}, x_{2n-1}) &= \max\{\min\{J(y_{2n}, y_{2n-1}), J(y_{2n-1}, y_{2n})\}, J(y_{2n-1}, y_{2n})\} \\ &= J(y_{2n-1}, y_{2n}).\end{aligned}$$

By (19), we get

$$\psi(J(y_{2n}, y_{2n+1})) \leq \psi(J(y_{2n-1}, y_{2n})) - \varphi(J(y_{2n-1}, y_{2n})) \leq \psi(J(y_{2n-1}, y_{2n})).$$

It follow from the property of ψ, we have $J(y_{2n}, y_{2n+1}) \leq J(y_{2n-1}, y_{2n})$. This mean that the sequence $\{J(y_{2n}, y_{2n+1})\}$ is non-increasing and converges to some $j_1 \geq 0$. Letting $n \to \infty$ in (19), which give

$$\psi(j_1) \leq \psi(j_1) - \psi(j_1) \leq \psi(j_1).$$

So, we have must $j_1 = 0$.

case II. If $M_s^J(x_{2n}, x_{2n-1}) = J(y_{2n}, y_{2n+1})$, then by (19), we get

$$\psi(J(y_{2n}, y_{2n+1})) \quad \leq \psi(J(y_{2n}, y_{2n+1})) - \varphi(J(y_{2n}, y_{2n+1})) \quad \leq \psi(J(y_{2n}, y_{2n+1}))$$

which implies that $J(y_{2n}, y_{2n+1}) = 0$ for all $n \geq 0$.

case III. If

$$M_s^J(x_{2n}, x_{2n-1}) = \frac{J(y_{2n}, y_{2n}) + J(y_{2n-1}, y_{2n+1})}{2s}.$$

In fact that $g(X) \subseteq X_j^0$, we have

$$M_s^J(x_{2n}, x_{2n+1}) = \frac{J(y_{2n-1}, y_{2n+1})}{2s}. \tag{21}$$

Hence, by (19) we get

$$\begin{aligned}\psi(J(y_{2n}, y_{2n+1})) &\leq \psi(\frac{J(y_{2n-1}, y_{2n+1})}{2s}) - \varphi(\frac{J(y_{2n-1}, y_{2n+1})}{2s}) \\ &\leq \psi(\frac{J(y_{2n-1}, y_{2n+2})}{2s})\end{aligned} \tag{22}$$

it follow that

$$\begin{aligned}J(y_{2n}, y_{2n+1}) &\leq \frac{J(y_{2n-1}, y_{2n+1})}{2s} \\ &\leq \frac{sJ(y_{2n-1}, y_{2n}) + sJ(y_{2n}, y_{2n+1})}{2s} \\ &= \frac{J(y_{2n-1}, y_{2n}) + J(y_{2n}, y_{2n+1})}{2}.\end{aligned} \tag{23}$$

Consequently, we obtain that $J(y_{2n}, y_{2n+1}) \leq J(y_{2n-1}, y_{2n})$. That is the sequence $\{J(y_{2n}, y_{2n+1})\}$ is non-increasing and converges to some nonnegative real number j_1'. By (11), (21) and (23) with φ and ψ are non-decreasing function, we have

$$\begin{aligned}\psi(J(y_{2n}, y_{2n+1})) &\leq \psi(\frac{J(y_{2n-1}, y_{2n+1})}{2s}) - \varphi(\frac{J(y_{2n-1}, y_{2n+1})}{2s}) \\ &\leq \psi(\frac{J(y_{2n-1}, y_{2n}) + J(y_{2n}, y_{2n+1})}{2}) - \varphi(J(y_{2n}, y_{2n+1})).\end{aligned} \tag{24}$$

Letting $n \to \infty$ in (24) we obtain that

$$\psi(j_1') \leq \psi(\frac{j_1' + j_1'}{2}) - \varphi(j_1') < \psi(j_1')$$

and hence $j_1' = 0$. Therefore (12) hold and consequently (8) hold

Next, we will show that (9) hold, by prove that

$$\lim_{n\to\infty} J(y_{2n+1}, y_{2n}) = 0 \quad \text{and} \quad \lim_{n\to\infty} J(y_{2n+2}, y_{2n+1}) = 0. \tag{25}$$

Given $x := x_{2n}$ and $y := x_{2n-1}$ in (10), we have

$$\begin{aligned} \psi(J(y_{2n+1}, y_{2n})) &\le \psi(s^4 J(y_{2n+1}, y_{2n})) \\ &= \psi(s^4 J(fx_{2n}, gx_{2n-1})) \\ &\le \psi(M_s^J(x_{2n}, x_{2n-1})) - \varphi(M_s^J(x_{2n}, x_{2n-1})), \end{aligned} \tag{26}$$

where $M_s^J(x_{2n}, x_{2n-1})$ satisfies (20). We distinguish three cases.

case I . If

$$\begin{aligned} M_s^J(x_{2n}, x_{2n-1}) &= \max\{\min\{J(y_{2n}, y_{2n-1}), J(y_{2n-1}, y_{2n})\}, J(y_{2n-1}, y_{2n})\} \\ &= J(y_{2n-1}, y_{2n}). \end{aligned}$$

By (11) and property of ψ, we have

$$J(y_{2n+1}, y_{2n}) \le J(y_{2n-1}, y_{2n}). \tag{27}$$

case II. If $M_s^J(x_{2n}, x_{2n-1}) = J(y_{2n}, y_{2n+1})$, then by (13), we get

$$\psi(J(y_{2n+1}, y_{2n})) \le \psi(J(y_{2n}, y_{2n+1})) - \psi(J(y_{2n}, y_{2n+1})) \le \psi(J(y_{2n}, y_{2n+1})),$$

then

$$J(y_{2n+1}, y_{2n}) \le J(y_{2n}, y_{2n+1}) \tag{28}$$

case III. If

$$M_s^J(x_{2n}, x_{2n-1}) = \frac{J(y_{2n}, y_{2n}) + J(y_{2n-1}, y_{2n+1})}{2s} = \frac{J(y_{2n-1}, y_{2n+1})}{2s}.$$

By the same argument as case III of step I, we get

$$J(y_{2n+1}, y_{2n}) \le J(y_{2n-1}, y_{2n}). \tag{29}$$

Letting $n \to \infty$ in (27), (28) and (29), by using (8), we have

$$\lim_{n\to\infty} J(y_{2n+1}, y_{2n}) = 0.$$

Similarly, by given $x := x_{2n}$ and $y := x_{2n+1}$ in (10), we can see that

$$\lim_{n\to\infty} J(y_{2n+2}, y_{2n+1}) = 0.$$

This complete the proof. □

Lemma 3.2. *Let X be b-metric space (with $s > 1$) and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Assume that $f, g, T, S : X \to X$ is a four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ with $f(X) \subseteq X_j^0$ and $g(X) \subseteq X_j^0$. Assume that, the mappings f, g, T and S satisfies (7) and for $x_0 \in X$ defined the sequence $\{y_n\}$ by*

$$y_{2n+1} := fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} := gx_{2n+1} = Sx_{2n+2} \quad \text{for all } n \ge 0.$$

Then

$$\lim_{m\to\infty} \sup_{n>m} J(y_m, y_n) = 0. \tag{30}$$

That is, the sequence $\{y_n\}$ is Cauchy.

Proof. Suppose the contrary, then there exist subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_n\}$ such that $2n(k) > 2m(k) \geq k$ and

$$J(y_{2m(k)}, y_{2n(k)}) > 1 \tag{31}$$

and $n(k)$ is the smallest in such that (31) holds. From (31), we have

$$J(y_{2m(k)}, y_{2n(k)-2}) \leq 1. \tag{32}$$

By property (J1), (31) and (32), we get

$$\begin{aligned} 1 &< J(y_{2m(k)}, y_{2n(k)}) \\ &\leq s[J(y_{2m(k)}, y_{2n(k)-2}) + J(y_{2n(k)-2}, y_{2n(k)})] \\ &\leq s + s^2 sJ(y_{2n(k)-2}, y_{2n(k)-1}) + s^2 J(y_{2n(k)-1}, y_{2n(k)}). \end{aligned} \tag{33}$$

Letting limit supremum as $k \to \infty$ in (33), by Lemma 3.1 we get

$$1 \leq \limsup_{k\to\infty} J(y_{2m(k)}, y_{2n(k)}) \leq s. \tag{34}$$

Again, by (J1), we obtain that

$$J(y_{2m(k)}, y_{2n(k)}) \leq s[J(y_{2m(k)}, y_{2n(k)+1}) + J(y_{2n(k)+1}, y_{2n(k)})] \tag{35}$$

and

$$J(y_{2m(k)}, y_{2n(k)+1}) \leq sJ(y_{2m(k)}, y_{2n(k)}) + sJ(y_{2n(k)}, y_{2n(k)+1}). \tag{36}$$

Taking limit supremum as $k \to \infty$ in (35) and (36), by using Lemma 3.1 and (34), we get

$$1 \leq s\Big(\limsup_{k\to\infty} J(y_{2m(k)}, y_{2n(k)+1})\Big)$$

and

$$\limsup_{k\to\infty} sJ(y_{2m(k)}, y_{2n(k)+1}) \leq s^2.$$

Thus

$$\frac{1}{s} \leq \limsup_{k\to\infty} J(y_{2m(k)}, y_{2n(k)+1}) \leq s^2. \tag{37}$$

By (31) and (J1), we have

$$\begin{aligned} 1 < J(y_{2m(k)}, y_{2n(k)}) &\leq s[J(y_{2m(k)}, y_{2n(k)+2}) + J(y_{2n(k)+2}, y_{2n(k)})] \\ &\leq sJ(y_{2m(k)}, y_{2n(k)+2}) + \\ &\quad s^2 J(y_{2n(k)+2}, y_{2n(k)+1}) + s^2 J(y_{2n(k)+1}, y_{2n(k)}) \end{aligned} \tag{38}$$

and

$$J(y_{2m(k)}, y_{2n(k)+2}) \leq sJ(y_{2m(k)}, y_{2n(k)+1}) + s[J(y_{2n(k)+1}, y_{2n(k)+2})]. \tag{39}$$

Taking limit supremum as $k \to \infty$ in (38) and (39), by using Lemma 3.1 and right hand side of (37), we have

$$1 \leq s\Big(\limsup_{k\to\infty} J(y_{2m(k)}, y_{2n(k)+2})\Big)$$

and

$$\left(\limsup_{k\to\infty} J(y_{2m(k)}, y_{2n(k)+2})\right) \leq s^3.$$

Equivalently,

$$\frac{1}{s} \leq \limsup_{k\to\infty} J(y_{2m(k)}, y_{2n(k)+2}) \leq s^3. \tag{40}$$

Again, by (31) and (J1), we have

$$\begin{aligned} 1 < J(y_{2m(k)}, y_{2n(k)}) &\leq s[J(y_{2m(k)}, y_{2n(k)+1}) + J(y_{2n(k)+1}, y_{2n(k)})] \\ &\leq sJ(y_{2m(k)}, y_{2n(k)+1}) \\ &\quad +s^2 J(y_{2n(k)+1}, y_{2m(k)}) + s^2 J(y_{2m(k)}, y_{2n(k)}) \end{aligned}$$

it follow that

$$\frac{1-2s^3}{s^2} \leq \limsup_{k\to\infty} J(y_{2n(k)+1}, y_{2m(k)}). \tag{41}$$

Since

$$J(y_{2n(k)+1}, y_{2m(k)}) \leq sJ(y_{2n(k)+1}, y_{2n(k)}) + sJ(y_{2n(k)}, y_{2m(k)}). \tag{42}$$

and

$$\begin{aligned} J(y_{2n(k)}, y_{2m(k)}) &\leq sJ(y_{2n(k)}, y_{2n(k)+1}) + sJ(y_{2n(k)+1}, y_{2m(k)}) \\ &\leq sJ(y_{2n(k)}, y_{2n(k)+1}) + s^2 J(y_{2n(k)+1}, y_{2n(k)}) + s^2 J(y_{2n(k)}, y_{2m(k)}), \end{aligned}$$

then

$$(1-s^2)J(y_{2n(k)}, y_{2m(k)}) \leq sJ(y_{2n(k)}, y_{2n(k)+1}) + s^2 J(y_{2n(k)+1}, y_{2n(k)}).$$

which implies, by Lemma 3.1 and $s > 1$ that

$$\lim_{k\to\infty} J(y_{2n(k)}, y_{2m(k)}) = 0. \tag{43}$$

Hence, by (41), (42), (43) and Lemma 3.1 , we have

$$\frac{1-2s^3}{s^2} \leq \limsup_{k\to\infty} J(y_{2n(k)+1}, y_{2m(k)}) \leq 0.$$

Thus, by definition of J we can conclude that

$$\lim_{k\to\infty} J(y_{2n(k)+1}, y_{2m(k)}) = 0. \tag{44}$$

Further,

$$J(y_{2n(k)+1}, y_{2m(k)+1}) \leq sJ(y_{2n(k)+1}, y_{2m(k)}) + sJ(y_{2m(k)}, y_{2m(k)+1}). \tag{45}$$

Also, we have

$$\lim_{k\to\infty} J(y_{2n(k)+1}, y_{2m(k)+1}) = 0. \tag{46}$$

Since

$$\begin{aligned}\psi(s^4 J(y_{2m(k)+1}, y_{2n(k)+2})) &\leq \psi(s^4 J(fx_{2m(k)}, gx_{2n(k)+1})) \\ &\leq \psi\Big(s^4 M(x_{2m(k)}, x_{2n(k)+1})\Big) \\ &\leq \psi(M_s^J(x_{2m(k)}, x_{2n(k)+1})) - \varphi(M_s^J(x_{2m(k)}, x_{2n(k)+1})),\end{aligned} \tag{47}$$

where

$$\begin{aligned}M_s^J(x_{2m(k)}, x_{2n(k)+1}) &\leq \max\Big\{\min\{J(Sx_{2m(k)}, Tx_{2n(k)+1}), J(Tx_{2n(k)+1}, Sx_{2m(k)})\}, \\ &\quad J(Sx_{2m(k)}, fx_{2m(k)}), J(Tx_{2n(k)+1}, gx_{2n(k)+1}), \\ &\quad \frac{J(Sx_{2m(k)}, gx_{2n(k)+1}) + J(Tx_{2n(k)+1}, fx_{2m(k)})}{2s}\Big\} \\ &= \max\Big\{\min\{J(y_{2m(k)}, y_{2n(k)+1}), J(y_{2n(k)+1}, y_{2m(k)})\}, \\ &\quad J(y_{2m(k)}, y_{2m(k)+1}), J(y_{2n(k)+1}, y_{2n(k)+2}), \\ &\quad \frac{J(y_{2m(k)}, y_{2n(k)+2}) + J(y_{2n(k)+1}, y_{2m(k)+1})}{2s}\Big\}.\end{aligned} \tag{48}$$

By taking limit supremum as $k \to \infty$ in (48), by (37), (40), (44), (46) and Lemma 3.1 , we have

$$\begin{aligned}\frac{1}{2s^2} = \max\Big\{\min\{\frac{1}{s}, 0)\}, 0, 0, \frac{\frac{1}{s}+0}{2s}\Big\} &\leq \limsup_{k\to\infty} M_s^J(x_{2m(k)}, x_{2n(k)+1}) \\ &\leq \max\Big\{\min\{s^2, 0)\}, 0, 0, \frac{s^3+0}{2s}\Big\} \\ &= \frac{s^2}{2}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2s^2} = \max\Big\{\min\{\frac{1}{s}, 0)\}, 0, 0, \frac{\frac{1}{s}+0}{2s}\Big\} &\leq \liminf_{k\to\infty} M_s^J(x_{2m(k)}, x_{2n(k)+1}) \\ &\leq \max\Big\{\min\{s^2, 0)\}, 0, 0, \frac{s^3+0}{2s}\Big\} \\ &= \frac{s^2}{2}.\end{aligned}$$

Therefore, by taking limit supremum as $k \to \infty$ in (47), we have

$$\begin{aligned}
\psi(\frac{s^2}{2}) &= \psi\Big(s^4\Big(\frac{1}{2s^2}\Big)\Big) \\
&\leq \psi\Big(s^4 \limsup_{k\to\infty} J(x_{2m(k)+1}, x_{2n(k)+2})\Big) \qquad (49)\\
&\leq \psi\Big(\limsup_{k\to\infty} M_s^J(x_{2m(k)}, x_{2n(k)+1})\Big) - \varphi\Big(\liminf_{k\to\infty} M_s^J(x_{2m(k)}, x_{2n(k)+1})\Big) \\
&\leq \psi(\frac{s^2}{2}) - \varphi(\frac{1}{2s^2}) \\
&\leq \psi(\frac{s^2}{2}). \qquad (50)
\end{aligned}$$

This implies that $\varphi(\frac{1}{2s^2}) = 0$ and hence $\frac{1}{2s^2} = 0$ which is a contradiction. Therefore

$$\lim_{m\to\infty} \sup_{n>m} J(y_m, y_n) = 0.$$

By Lemma 2.15, the sequence $\{y_n\}$ is Cauchy. This complete the proof. □

Theorem 3.3. *Let X be a b-metric space (with $s > 1$) and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Assume that $f, g, T, S : X \to X$ is a four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ with $f(X) \subseteq X_J^0$ and $g(X) \subseteq X_J^0$. Assume that the mappings f, g, T and S satisfies (7) and the images under mapping S or T is J-complete. Then*

(a) f and S have a coincidence point;
(b) g and T have a coincidence point.

Moreover, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by

$$y_{2n+1} := fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} := gx_{2n+1} = Sx_{2n+2} \quad \text{for all } n \geq 0,$$

converges to unique point $y_\star \in X$ with $y_\star = fx_\star = Sx_\star = Tz_\star = gz_\star$ fore some $x_\star, z_\star \in X$.

Proof. As the proof of Lemma 3.1 and Lemma 3.2, we have two sequences $\{x_n\}$ and $\{y_n\}$. Furthermore, $\{y_n\}$ is a b-Cauchy sequence. Suppose that $S(X)$ is a J-complete, then the sequence $\{y_n\}$ converges to some element $y_\star \in S(X)$ and

$$\lim_{k\to\infty} fx_{2n} = \lim_{k\to\infty} Tx_{2n+1} = \lim_{k\to\infty} gx_{2n+1} = \lim_{k\to\infty} Sx_{2n+2} = y_\star. \qquad (51)$$

with

$$\lim_{n\to\infty} J(y_n, y_\star) = \lim_{n\to\infty} J(y_\star, y_n) = 0. \qquad (52)$$

Moreover, $y_\star \in X_J^0$, i.e $J(y_\star, y_\star) = 0$. Indeed,

$$0 \leq J(y_\star, y_\star) \leq s\Big(\lim_{n\to\infty} J(y_\star, y_n)\Big) + s\Big(\lim_{n\to\infty} J(y_n, y_\star)\Big) = 0.$$

First, we will show that $y_\star$ is unique. Suppose that there exist $y \in X$ with $y \neq y_\star$ such that

$$\lim_{n\to\infty} J(y_n, y) = \lim_{n\to\infty} J(y, y_n) = 0. \qquad (53)$$

Let $v_n = y$ and $u_n = y_\star$ for all $n \in \mathbb{N}$. By (30), (52), (53) and property ($J2$), we have

$$\lim_{n\to\infty} d(y_n, y_\star) = \lim_{n\to\infty} d(y_n, u_n) = 0 \tag{54}$$

and

$$\lim_{n\to\infty} d(y_n, y) = \lim_{n\to\infty} d(y_n, v_n) = 0. \tag{55}$$

By (54), (55) and Remark 2.12 with $y \neq y_\star$, we get

$$0 < d(y, y_\star) \leq \lim_{n\to\infty}\Big(sd(y, y_n) + sd(y_n, y_\star)\Big) = 0$$

which is a contradiction and hence $y_\star$ is unique. Now, we will prove S and f have coincidence point. Let $x_\star = S^{-1}y_\star$, then $Sx_\star = y_\star$. Putting $x := x_\star$ and $y := x_{2n+1}$ in (11), we have

$$\begin{aligned} \psi(J(y_{2n+2}, fx_\star)) &\leq \psi(s^4 J(gx_{2n+1}, fx_\star)) \\ &\leq \psi(M_s^J(x_\star, x_{2n+1})) - \varphi(M_s^J(x_\star, x_{2n+1})), \end{aligned} \tag{56}$$

where

$$\begin{aligned} M_s^J(x_\star, x_{2n+1}) &\leq \max\Big\{ \min\{J(Sx_\star, Tx_{2n+1}), J(Tx_{2n+1}, Sx_\star)\}, J(Sx_\star, fx_\star), J(Tx_{2n+1}, gx_{2n+1}), \\ &\quad \frac{J(Sx_\star, gx_{2n+1}) + J(Tx_{2n+1}, fx_\star)}{2s}\Big\} \\ &= \max\Big\{ \min\{J(y_\star, Tx_{2n+1}), J(Tx_{2n+1}, y_\star)\}, J(y_\star, fx_\star), J(Tx_{2n+1}, gx_{2n+1}), \\ &\quad \frac{J(y_\star, gx_{2n+1}) + J(Tx_{2n+1}, fx_\star)}{2s}\Big\}. \end{aligned} \tag{57}$$

Taking limit supremum as $\to \infty$ in (57) and (56) by using (51) and $y_\star \in X_J^0$, we get

$$\lim_{n\to\infty} M_s^J(x_\star, x_{2n+1}) = J(y_\star, fx_\star). \tag{58}$$

Then by (11), we have

$$\begin{aligned} \psi(J(y_\star, fx_\star)) &\leq \psi(s^4 J(y_\star, fx_\star)) \\ &\leq \psi(J(y_\star, fx_\star)) - \varphi(J(y_\star, fx_\star))) \\ &\leq \psi(J(y_\star, fx_\star)) \end{aligned} \tag{59}$$

which implies that $J(y, fx_\star) = 0$. Further, by (7), (10) and (58) we have

$$\psi(J(fx_\star, y_\star)) \leq \psi(J(y_\star, fx_\star)) - \varphi(J(y_\star, fx_\star))) \leq \psi(0) - \varphi(0) = 0$$

and hence $J(fx_\star, y_\star) = 0$. Therefore, from Remark 2.12, we obtain that

$$fx_\star = Sx_\star. \tag{60}$$

Next, we will prove g and T have coincidence. Since, $y_\star = Sx_\star = fx_\star$ and $f(X) \subseteq T(X)$, then there exist $z_\star \in X$ such that $y_\star = Tz_\star$. Putting $x := x_{2n}$ and $y := z_\star$ in (10), we have

$$\begin{aligned} \psi(J(y_{2n+1}, gz_\star)) &\leq \psi(s^4 J(fx_{2n}, gz_\star)) \\ &\leq \psi(M_s^J(x_{2n}, z_\star)) - \varphi(M_s^J(x_{2n}, z_\star)), \end{aligned} \tag{61}$$

where

$$\begin{aligned} M_s^J(x_{2n}, z_\star) &\leq \max\left\{ \min\{J(Sx_{2n}, Tz_\star), J(Tz_\star, Sx_{2n})\}, J(Sx_{2n}, fx_{2n}), J(Tz_\star, gz_\star), \frac{J(Sx_{2n}, gz_\star) + J(Tz_\star, fx_{2n})}{2s} \right\} \\ &= \max\left\{ \min\{J(y_{2n}, y_\star), J(y_\star, y_{2n})\}, J(y_{2n}, y_{2n+1}), J(y_\star, gz_\star), \frac{J(y_{2n}, gz_\star) + J(y_\star, y_{2n+1})}{2s} \right\}. \end{aligned} \tag{62}$$

Taking limit supremum as $\to \infty$ in (62) and (61), by using Lemma 3.1 and $y_\star \in X_J^0$, we get

$$\lim_{n\to\infty} M_s^J(z_\star, x_{2n+1}) = J(y_\star, gz_\star) \tag{63}$$

and hence

$$\psi(J(y_\star, gz_\star)) \leq \psi(J(y_\star, gz_\star)) - \varphi(J(y_\star, gz_\star)), \tag{64}$$

which implies that $J(y, gz_\star) = 0$. Further, by (61) and (63) we have

$$\psi(J(gz_\star, y_\star)) \leq \psi(J(y_\star, gz_\star)) - \varphi(J(y_\star, gz_\star)) \leq \psi(0) - \varphi(0) = 0$$

Therefore $J(gz_\star, y_\star) = 0$. Again, from Remark 2.12(C), we obtain that

$$y_\star = Tz_\star = gz_\star. \tag{65}$$

Furthermore, since $Sx_\star = y_\star = Tz_\star$, then by (60) and (65), we obtain that

$$y_\star = fx_\star = Sx_\star = Tz_\star = gz_\star.$$

This complete the proof. □

Next, we give some consequence of our main result.

Corollary 3.4. *Let X be a b-metric space (with $s > 1$). Assume that $f, g, T, S : X \to X$ is a four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$. Assume that the mappings f, g, T and S satisfies.*

$$\psi(s^4 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \tag{66}$$

where

$$M_s(x, y) = \max\{J(Sx, Ty), J(Sx, fx), J(Ty, gy), \frac{J(Sx, gy) + J(Ty, fx)}{2s}\}.$$

Then

- *(a) f and S have a coincidence point;*
- *(b) g and T have a coincidence point.*

Moreover, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by

$$x_{2n+1} := fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} := gx_{2n+1} = Sx_{2n+2} \quad \text{for all } n \geq 0,$$

converges to unique point $y \in X$ with $y_\star = fx_\star = Sx_\star = Tz_\star = gz_\star$ fore some $x_\star, z_\star \in X$.

The following, we give some illustrative example for support our main result.

Example 3.5. *Let $X = [0,5]$, $E = [0,3]$ and $d(x,y) = (x-y)^2$, then (X,d) is b-metric spaces wit $s = 2$. Define $J : X \times X \to [0,\infty)$ by*

$$J(x,y) = \begin{cases} d(x,y) & E \cap \{x,y\} = \{x,y\} \\ 26 & E \cap \{x,y\} \neq \{x,y\}. \end{cases}$$

By Example 2.10, J is b-generalized pseudodistance. Let $f, g, T, S : X \to X$ is a four mappings define by

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{2}}\sinh^{-1}(\frac{x}{4}), & \text{if } x \in E, \\ 0, & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} \dfrac{1}{2\sqrt{2}}\sinh^{-1}(\frac{x}{8}), & \text{if } x \in E, \\ 0, & \text{otherwise} \end{cases}$$

$$T(x) = \begin{cases} \dfrac{1}{2\sqrt{2}}\sinh x, & \text{if } x \in E, \\ 0, & \text{otherwise} \end{cases}$$

and

$$S(x) = \begin{cases} \dfrac{1}{2\sqrt{2}}\sinh 2x, & \text{if } x \in E, \\ 0, & \text{otherwise}. \end{cases}$$

with altering distance functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \lambda t$ and $\varphi(t) = (\lambda - 1)t$ for all $t \in [0,\infty)$, where $\lambda \in (1,2)$. Then we can see that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$. Now we will show that $f, g, T, S : X \to X$ are satisfies (7). Let $x, y \in X$. In case, $x, y \notin E$, easy to see that $f, g, T, S : X \to X$ are satisfies (7).
If $x, y \in E$, then $fx, gy \in E$ and $J(gy, fx) = J(fx, gy)$. Furthermore

$$\begin{aligned}
\psi(2^4 M(fx,gy)) &= 16\lambda(fx-gy)^2 \\
&= 16\lambda\Big(\frac{1}{2\sqrt{2}}\sinh^{-1}\frac{x}{4} - \frac{1}{2\sqrt{2}}\sinh^{-1}\frac{y}{8}\Big)^2 \\
&\leq 16\lambda\Big(\frac{2x}{16\sqrt{2}} - \frac{y}{16\sqrt{2}}\Big)^2 \\
&= \frac{\lambda}{32}(2x-y)^2 \\
&\leq \frac{\lambda}{4}(\frac{1}{2\sqrt{2}}\sinh 2x - \frac{1}{2\sqrt{2}}\sinh y)^2 \\
&\leq (\frac{1}{2\sqrt{2}}\sinh 2x - \frac{1}{2\sqrt{2}}\sinh y)^2 \\
&= J(Sx,Ty) \\
&\leq M_s^J(x,y) \\
&\leq \psi(M_s^J(x,y)) - \varphi(M_s^J(x,y)).
\end{aligned}$$

If $x \in E$ and $y \notin E$, then $g(y) = 0 \in E$ and

$$J(fx,gy) = J(\frac{1}{2\sqrt{2}}\sinh^{-1}\frac{x}{4}, 0) = J(0, \frac{1}{2\sqrt{2}}\sinh^{-1}\frac{x}{4}) = J(gy,fx).$$

Furthermore

$$\begin{aligned}
\psi(2^4 M(fx, gy)) &= 16\lambda(fx - gy)^2 \\
&= 16\lambda\Big(\frac{1}{2\sqrt{2}}\sinh^{-1}\frac{x}{4} - 0\Big)^2 \\
&\leq 16\lambda\Big(\frac{x}{16\sqrt{2}}\Big)^2 \\
&= \frac{\lambda}{4}(\frac{1}{2\sqrt{2}}\sinh 2x - 0)^2 \\
&\leq (\frac{1}{2\sqrt{2}}\sinh 2x - 0)^2 \\
&= J(Sx, Ty) \\
&\leq M_s^J(x, y) \\
&\leq \psi(M_s^J(x, y)) - \varphi(M_s^J(x, y)).
\end{aligned}$$

If $x \notin E$ and $y \in E$, similarly we see that (7). Then the mapping f, g, T and S satisfies all conditions of Theorem 3.3. Furthermore, 0 is coincidence point of f, g, T and S.

Now, we give the solution path in Example 3.5 as follow.

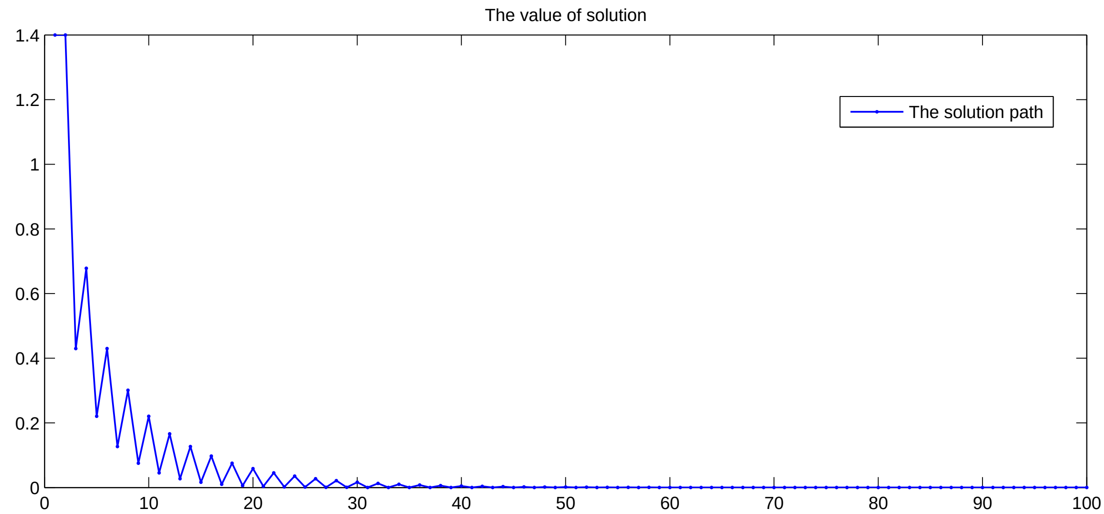

Fig. 1. The value of the sequences $\{y_n\}$ with $x_0 = 1$, functions f, g, S and T at Example 3.5.

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