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On the Strong Convergence for Weighted Sums of Negatively Superadditive Dependent Random Variables

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Abstract. In this paper, we present some results on the complete convergence for arrays of rowwise negatively superadditive dependent (NSD, in short) random variables by using the Rosenthal-type maximal inequality, Kolmogorov exponential inequality and the truncation method. The results obtained in the paper extend the corresponding ones for weighted sums of negatively associated random variables with identical distribution to the case of arrays of rowwise NSD random variables without identical distribution.

1. Introduction

Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ and $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers. It is well known that the limiting behavior for the maximum of weighted sums $\max_{1\le m\le n} \sum_{i=1}^{m} b_{ni}X_{ni}$ is very useful in many probabilistic derivations and stochastic models. There exist several versions of the limiting behavior available in the literature for independent random variables with assumption of control on their moments. If the independent case is classical in the literature, the treatment of dependent variables is more recent.

One of the dependence structure that has attracted the interest of those who are specialized for probability and statisticians is negative association. The concept of negatively associated random variables, which was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [10] is as follows.

A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA, in short) if for every pair of disjoint subsets $A, B \subset \{1, 2, \cdots, n\}$,

$$\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \le 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The next dependence notion is negatively superadditive dependence, which is weaker than negative association. To introduce the concept of negatively superadditive dependence, we first recall the class of superadditive functions as follows.

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Definition 1.1. *(cf. Kemperman [11]). A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called superadditive if $\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where $\vee$ stands for componentwise maximum and $\wedge$ stands for componentwise minimum.*

Based on the class of superadditive functions, Hu [9] introduced the concept of negatively superadditive dependence as follows.

Definition 1.2. *(cf. Hu [9]). A random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ is said to be negatively superadditive dependent (NSD, in short) if*

$$E\phi(X_1, X_2, \cdots, X_n) \leq E\phi(X_1^*, X_2^*, \cdots, X_n^*), \tag{1.1}$$

where $X_1^*, X_2^*, \cdots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.1) exist.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be NSD if for all $n \geq 1$, $(X_1, X_2, \cdots, X_n)$ is NSD.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be rowwise NSD if for all $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is NSD.

Since the concept of NSD random variables was introduced by Hu [9], many authors devoted to studying the probability limit theory for NSD random variables. See for example, Hu [9] pointed out that NSD does not imply NA, and posed an open problem whether NA implies NSD. Christofides and Vaggelatou [6] gave the answer to the question posed by Hu [9] and indicated that NA implies NSD. Hence, NSD is a class of random variables that includes independent sequence and NA sequence as special cases. Studying the limiting behavior of NSD random variables and its applications are of great interest. Eghbal et al. [8] established the Kolmogorov inequality for quadratic forms $T_n = \sum_{1 \leq i < j \leq n} X_i X_j$ and weighted quadratic forms $Q_n = \sum_{1 \leq i < j \leq n} a_{ij} X_i X_j$, where $\{X_i, i \geq 1\}$ is a sequence of nonnegative NSD uniformly bounded random variables. Shen et al. [16] studied the almost sure convergence theorem and strong stability for weighted sums of NSD random variables. Wang et al. [17] established some results on complete convergence for arrays of rowwise NSD random variables and gave applications to nonparametric regression model. Shen [12] studied the asymptotic approximation of inverse moments for nonnegative NSD random variables. Shen et al. [15] gave some applications of the Rosenthal-type inequality for NSD random variables. Wang et al. [18] established the complete consistency for the estimators in the EV regression model. Xue et al. [20] obtained the complete moment convergence for weighted sums of NSD random variables. Amini et al. [2] studied the complete convergence of moving average processes based on NSD sequences. The main purpose of this work is to further study the complete convergence for weighted sums of arrays of rowwise NSD random variables without identical distribution, while the condition of stochastic domination is needed.

The definition of stochastic domination below will play an important role throughout the paper.

Definition 1.3. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$. An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leq CP(|X| > x)$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

Throughout the paper, C denotes a positive constant not depending on n, which may be different in various places. $a_n = O(b_n)$ represents $a_n \leq Cb_n$ for all $n \geq 1$ and $I(A)$ is the indicator function of the set A. Set $\log x = \ln \max(x, e)$.

2. Preliminary Lemmas

In this section, we will provide some preliminary facts needed for the proofs of our main results. The first one comes from Hu [9], or Wang et al. [18].

Lemma 2.1. *If* $(X_1, X_2, \cdots, X_n)$ *is NSD and* $g_1, g_2, \cdots, g_n$ *are all nondecreasing (or all nonincreasing), then* $(g_1(X_1), g_2(X_2), \cdots, g_n(X_n))$ *is NSD.*

The following two lemmas come from Wang et al. [17]. One is the moment inequality for NSD random variables, including Marcinkiewicz-Zygmund type inequality and Rosenthal type inequality, the other is the Kolmogorov type exponential inequality for NSD random variables.

Lemma 2.2. *Let* $p > 1$, *and* $\{X_n, n \geq 1\}$ *be a sequence of NSD random variables with* $E|X_i|^p < \infty$ *for each* $i \geq 1$. *Then for all* $n \geq 1$,

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq 2^{3-p}\sum_{i=1}^{n} E|X_i|^p, \text{ for } 1 < p \leq 2, \tag{2.2}$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq 2\left(\frac{15p}{\ln p}\right)^p\left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right], \text{ for } p > 2. \tag{2.3}$$

Lemma 2.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of NSD random variables with zero means and finite second moments. Denote* $S_n = \sum_{i=1}^n X_i$ *and* $B_n = \sum_{i=1}^n EX_i^2$ *for each* $n \geq 1$. *Then for all* $x > 0$, $y > 0$, *and* $n \geq 1$,

$$P\left(\max_{1\leq k\leq n}|S_k| \geq x\right) \leq 2P\left(\max_{1\leq k\leq n}|X_k| \geq y\right) + 8\left(\frac{2B_n}{3xy}\right)^{x/12y}.$$

The next one is a basic property for NSD random variables, which plays an important role to prove the main results of the paper.

Lemma 2.4. *Let* $\{X_n, n \geq 1\}$ *be a sequence of NSD random variables. Then there exists a positive constant* C *independent of* n *such that for any* $\epsilon \geq 0$ *and all* $n \geq 1$,

$$\left[1 - P\left(\max_{1\leq i\leq n}|X_i| > \epsilon\right)\right]^2 \sum_{i=1}^{n} P(|X_i| > \epsilon) \leq CP\left(\max_{1\leq i\leq n}|X_i| > \epsilon\right). \tag{2.4}$$

Proof. Let $A_i = (|X_i| > \epsilon)$ and

$$a_n = 1 - P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P\left(\max_{1\leq i\leq n}|X_i| > \epsilon\right).$$

Without loss of generality, assume that $a_n > 0$. Note that $\{I(X_i > \epsilon) - EI(X_i > \epsilon), i \geq 1\}$ and $\{I(X_i < -\epsilon) - EI(X_i < -\epsilon), i \geq 1\}$ are both NSD by Lemma 2.1. We have by (2.1) that

$$\begin{aligned} E\left(\sum_{i=1}^{n}(I(A_i) - EI(A_i))\right)^2 &\leq 2E\left(\sum_{i=1}^{n}(I(X_i > \epsilon) - EI(X_i > \epsilon))\right)^2 \\ &\quad +2E\left(\sum_{i=1}^{n} I(X_i < -\epsilon) - EI(X_i < -\epsilon)\right)^2 \\ &\leq C\sum_{i=1}^{n} P(A_i), \end{aligned}$$

which together with Canchy-Schwarz inequality yields that

$$\begin{aligned}\sum_{i=1}^{n} P(A_i) &= E\left(\sum_{i=1}^{n}(I(A_i)-EI(A_i))\right)I(\cup_{j=1}^{n}A_j)+\sum_{i=1}^{n}P(A_i)P\left(\cup_{j=1}^{n}A_j\right)\\ &\le \left(E\left(\sum_{i=1}^{n}(I(A_i)-EI(A_i))\right)^2 EI(\cup_{j=1}^{n}A_j)\right)^{1/2}+(1-a_n)\sum_{i=1}^{n}P(A_i)\\ &\le \left(\frac{C(1-a_n)}{a_n}a_n\sum_{i=1}^{n}P(A_i)\right)^{1/2}+(1-a_n)\sum_{i=1}^{n}P(A_i)\\ &\le \frac{1}{2}\left(\frac{C(1-a_n)}{a_n}+a_n\sum_{i=1}^{n}P(A_i)\right)+(1-a_n)\sum_{i=1}^{n}P(A_i).\end{aligned}$$

Then

$$a_n^2\sum_{i=1}^{n}P(A_i)\le C(1-a_n),$$

which implies (2.4). □

The following one is a basic property for stochastic domination. For the proof, one can refer to Wu [19], or Shen and Wu [14].

Lemma 2.5. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$\begin{aligned} E|X_{ni}|^\alpha I\left(|X_{ni}|\le b\right) &\le C_1\left[E|X|^\alpha I\left(|X|\le b\right)+b^\alpha P\left(|X|>b\right)\right],\\ E|X_{ni}|^\alpha I\left(|X_{ni}|> b\right) &\le C_2 E|X|^\alpha I\left(|X|> b\right),\end{aligned}$$

where C_1 and C_2 are positive constants.

Similarly to the proof of Lemma 3.1 of Shen [13] and applying Lemma 2.3, we can get the following result on complete convergence for arrays of rowwise NSD random variables.

Lemma 2.6. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise NSD random variables, where $\{k_n, n \ge 1\}$ is a sequence of positive integers such that $k_n \uparrow \infty$ as $n \to \infty$. Assume that $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ is an array of nonnegative constants and $\{a_n, n \ge 1\}$ is a sequence of positive constants. Assume that the following conditions are satisfied:*

(i) for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \varepsilon\right) < \infty$;
(ii) for some $\delta > 0$, there exists $q \ge 1$ such that

$$\sum_{n=1}^{\infty} a_n\left(\sum_{i=1}^{k_n} Var\left(a_{ni}X_{ni}I(|a_{ni}X_{ni}|\le\delta)\right)\right)^q<\infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1\le m\le k_n}\left|\sum_{i=1}^{m} a_{ni}\left(X_{ni}-EX_{ni}I(|a_{ni}X_{ni}|\le\delta)\right)\right|>\epsilon\right)<\infty,\quad \forall\, \epsilon>0. \tag{2.5}$$

Proof. Note that for any fixed $\epsilon > 0$ and $n \geq 1$,

$$\begin{aligned} & P\left(\max_{1\leq m\leq k_n}\left|\sum_{i=1}^{m} a_{ni}\left(X_{ni} - EX_{ni}I(|a_{ni}X_{ni}| \leq \delta)\right)\right| > \epsilon\right) \\ \leq & \sum_{i=1}^{k_n} P(|a_{ni}X_{ni}| > \delta) + P\left(\max_{1\leq m\leq k_n}\left|\sum_{i=1}^{m} a_{ni}\left(X_{ni}I(|a_{ni}X_{ni}| \leq \delta) - EX_{ni}I(|a_{ni}X_{ni}| \leq \delta)\right)\right| > \epsilon\right). \end{aligned}$$

By condition (i), it suffices to show that

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{i=1}^{m} a_{ni}(X_{ni}I(|a_{ni}X_{ni}| \leq \delta) - EX_{ni}I(|a_{ni}X_{ni}| \leq \delta))\right| > \epsilon\right) < \infty.$$

Denote for $1 \leq i \leq k_n$ and $n \geq 1$ that,

$$\begin{aligned} Y_{ni} &= \delta I(a_{ni}X_{ni} > \delta) + a_{ni}X_{ni}I(|a_{ni}X_{ni}| \leq \delta) - \delta I(a_{ni}X_{ni} < -\delta), \\ Y'_{ni} &= \delta I(a_{ni}X_{ni} > \delta) - \delta I(a_{ni}X_{ni} < -\delta). \end{aligned}$$

We have

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{i=1}^{m} a_{ni}(X_{ni}I(|a_{ni}X_{ni}| \leq \delta) - EX_{ni}I(|a_{ni}X_{ni}| \leq \delta))\right| > \epsilon\right) \\ \leq & \sum_{n=1}^{\infty} a_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{i=1}^{m}(Y'_{ni} - EY'_{ni})\right| > \epsilon/2\right) + \sum_{n=1}^{\infty} a_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{i=1}^{m}(Y_{ni} - EY_{ni})\right| > \epsilon/2\right) \\ \doteq & \; I + J\,. \end{aligned}$$

For I, by Markov's inequality and condition (i), it is easy to obtain that

$$I \leq C\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \delta\right) < \infty.$$

For J, let $M_{2,n} = \sum_{i=1}^{k_n} \mathrm{Var}(Y_{ni})$. It is easy to check that

$$M_{2,n} \leq \sum_{i=1}^{k_n} \mathrm{Var}\left(a_{ni}X_{ni}I(|a_{ni}X_{ni}| \leq \delta)\right) + 8\delta^2 \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \delta\right).$$

For any $y > 0$, let $d = \min\{1, y/6\delta\}$,

$$\begin{aligned} \mathbf{N_1} &= \left\{n : \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \min\{\delta, y/6\}\right) > d\right\}, \\ \mathbf{N_2} &= \mathbf{N}\backslash\mathbf{N_1} = \left\{n : \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \min\{\delta, y/6\}\right) \leq d\right\}. \end{aligned}$$

Note that

$$\sum_{n\in\mathbf{N_1}} a_n P\left(\left|\sum_{i=1}^{k_n}(Y_{ni} - EY_{ni})\right| > \epsilon/2\right) \leq \frac{1}{d}\sum_{n\in\mathbf{N_1}} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \min\{\delta, y/6\}\right) < \infty.$$

In the following, we prove that

$$\sum_{n\in\mathbf{N_2}} a_n P\left(\left|\sum_{i=1}^{k_n}(Y_{ni} - EY_{ni})\right| > \epsilon/2\right) < \infty.$$

By Lemma 2.1, we know that $\{Y_{ni} - EY_{ni}, 1 \le i \le k_n, n \ge 1\}$ is an array of rowwise NSD random variables. By Lemma 2.3 we have that

$$\begin{aligned}
&\sum_{n\in\mathbf{N_2}} a_n P\left(\max_{1\le m\le k_n}\left|\sum_{i=1}^{m}(Y_{ni}-EY_{ni})\right| > \epsilon/2\right)\\
\le\;& C\sum_{n\in\mathbf{N_2}} a_n \sum_{i=1}^{k_n} P\left(|Y_{ni}-EY_{ni}|\ge y\right) + C\sum_{n\in\mathbf{N_2}} a_n (M_{2,n})^{\epsilon/24y}\\
\doteq\;& CJ_1 + CJ_2.
\end{aligned}$$

Note that

$$\begin{aligned}
P\left(|Y_{ni}-EY_{ni}|\ge y\right) \;\le\;& P\left(|a_{ni}X_{ni}I(|a_{ni}X_{ni}|\le\delta) - Ea_{ni}X_{ni}I(|a_{ni}X_{ni}|\le\delta)| > \frac{y}{2}\right)\\
&+P\left(|Y'_{ni}-EY'_{ni}| > \frac{y}{2}\right).
\end{aligned}$$

It is easy to check that for $n \in \mathbf{N_2}$,

$$\begin{aligned}
|Ea_{ni}X_{ni}I(|a_{ni}X_{ni}|\le\delta)| \;\le\;& E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|\le y/6) + E|a_{ni}X_{ni}|I(y/6\le|a_{ni}X_{ni}|\le\delta)\\
\le\;& y/6 + \delta\sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \min\{y/6,\delta\}\right)\\
\le\;& y/6 + \delta d \le y/3,
\end{aligned}$$

which implies that for $n \in \mathbf{N_2}$,

$$P\left(|Y_{ni}-EY_{ni}|\ge y\right) \le P\left(|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|\le\delta) > \frac{y}{6}\right) + P\left(\left|Y'_{ni}-EY'_{ni}\right| > \frac{y}{2}\right).$$

Therefore, by Markov's inequality and condition (i) we obtain

$$\begin{aligned}
J_1 \;\le\;& \sum_{n\in\mathbf{N_2}} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|\le\delta) > \frac{y}{6}\right) + \sum_{n\in\mathbf{N_2}} a_n \sum_{i=1}^{k_n} P\left(|Y'_{ni}-EY'_{ni}| > \frac{y}{2}\right)\\
\le\;& \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \frac{y}{6}\right) + \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \delta\right)\\
<\;& \infty.
\end{aligned}$$

Now we prove $J_2 < \infty$. When $n \in \mathbf{N_2}$, we have that $\sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}| > \delta\right) \le 1$. Let $y = \epsilon/24q$, we have by conditions (i) and (ii) that

$$\begin{aligned}
J_2 \;\le\;& C\sum_{n\in\mathbf{N_2}} a_n \left(\sum_{i=1}^{k_n} \mathrm{Var}(a_{ni}X_{ni}I(|a_{ni}X_{ni}|\le\delta))\right)^q + C\sum_{n\in\mathbf{N_2}} a_n \left(\sum_{i=1}^{k_n} P(|a_{ni}X_{ni}|>\delta)\right)^q\\
\le\;& C\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} \mathrm{Var}(a_{ni}X_{ni}I(|a_{ni}X_{ni}|\le\delta))\right)^q + C\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\left(|a_{ni}X_{ni}|>\delta\right)\\
<\;& \infty.
\end{aligned}$$

This completes the proof of the lemma. □

3. Main Results

Our main results are as follows. The first one is the complete convergence for weighted sums of arrays of rowwise NSD random variables.

Theorem 3.1. *Let $\beta \in \mathbb{R}$, and $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise NSD random variables which is stochastically dominated by a random variable X with $EX_{ni} = 0$ and $E|X|^p/(\log|X|)^r < \infty$ for some $1 < p \le 2$ and $r > 0$. Assume that $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of constants satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} n^\beta \sum_{i=1}^n P(|b_{ni}X_{ni}| > \epsilon) < \infty,$ *for all* $\epsilon > 0$;

(ii) $\sum_{n=1}^{\infty} n^\beta (\sum_{i=1}^n |b_{ni}|^p (\log n)^r)^j < \infty,$ *for some* $j \ge 1$;

(iii) $(\log n)^r \sum_{i=1}^n |b_{ni}|^p \to 0$ *as* $n \to \infty$.

Then

$$\sum_{n=1}^{\infty} n^\beta P\left(\max_{1\le m\le n}\left|\sum_{i=1}^m b_{ni}X_{ni}\right| > \epsilon\right) < \infty \quad for\ all\ \epsilon > 0. \tag{3.0}$$

Proof. The result is trivial if $\beta < -1$. So we assume that $\beta \ge -1$. Noting that $b_{ni} = b_{ni}^+ - b_{ni}^-$, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^\beta P\left(\max_{1\le m\le n}\left|\sum_{i=1}^m b_{ni}X_{ni}\right| > \epsilon\right) \\ \le & \sum_{n=1}^{\infty} n^\beta P\left(\max_{1\le m\le n}\left|\sum_{i=1}^m b_{ni}^+X_{ni}\right| > \frac{\epsilon}{2}\right) + \sum_{n=1}^{\infty} n^\beta P\left(\max_{1\le m\le n}\left|\sum_{i=1}^m b_{ni}^-X_{ni}\right| > \frac{\epsilon}{2}\right). \end{aligned}$$

So without loss of generality, we assume that $b_{ni} \ge 0$. Otherwise, we will use b_{ni}^+ and b_{ni}^- instead of b_{ni} respectively. For $n \ge 1$, define for $1 \le i \le n$ that

$$\begin{aligned} Y_{ni} &= b_{ni}[X_{ni}I(|X_{ni}| \le f(n)) + f(n)I(X_{ni} > f(n)) - f(n)I(X_{ni} < -f(n))], \\ Z_{ni} &= b_{ni}[(X_{ni} - f(n))I(X_{ni} > f(n)) + (X_{ni} + f(n))I(X_{ni} < -f(n))], \end{aligned}$$

where $f(x)$ is an increasing function defined on $[0, \infty)$ satisfying $f(0) = 0$ and $f(n) = n^{(\beta+2)/p}(\log n)^{r/p}$ for all large n. For fixed $n \ge 1$, Lemma 2.1 yields that $\{Y_{ni}, 1 \le i \le n\}$ are NSD random variables. Noting that $Y_{ni} + Z_{ni} = b_{ni}X_{ni}$, to prove (3.1), we only need to show that

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} n^\beta P\left(\max_{1\le m\le n}\left|\sum_{i=1}^m Z_{ni}\right| > \epsilon\right) < \infty, \\ I_2 &= \sum_{n=1}^{\infty} n^\beta P\left(\max_{1\le m\le n}\left|\sum_{i=1}^m Y_{ni}\right| > \epsilon\right) < \infty. \end{aligned}$$

Firstly, we will prove $I_1 < \infty$. Noting that $f(n) = n^{(\beta+2)/p}(\log n)^{r/p}$ for all large n, we can obtain that

$$\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{n} P(|X_{ni}| > f(n)) \\
&\le C\sum_{n=1}^{\infty} n^{\beta+1} P(|X| > f(n)) \\
&= C\sum_{n=1}^{\infty} n^{\beta+1} \sum_{i=n}^{\infty} P(f(i) < |X| \le f(i+1)) \\
&= C\sum_{i=1}^{\infty} P(f(i) < |X| \le f(i+1)) \sum_{n=1}^{i} n^{\beta+1} \\
&\le C\sum_{i=1}^{\infty} P(f(i) < |X| \le f(i+1)) i^{\beta+2} \\
&\le C\sum_{i=1}^{\infty} P(f(i) < |X| \le f(i+1))(f(i))^p/(\log f(i))^r \\
&\le CE|X|^p/(\log|X|)^r < \infty. \qquad (3.1)
\end{aligned}$$

To prove $I_2 < \infty$, we will apply Lemma 2.6 to the array $\{Y_{ni}\}$ with $a_n = n^{\beta}$, $k_n = n$ and $a_{ni} \equiv 1$. By (3.1) and condition (i), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{n} P(|Y_{ni}| > \epsilon) &\le \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{n} [P(|X_{ni}| > f(n)) + P(|X_{ni}| \le f(n), |b_{ni}X_{ni}| > \epsilon)] \\
&\le CE|X|^p/(\log|X|)^r + \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{n} P(|b_{ni}X_{ni}| > \epsilon) < \infty,
\end{aligned}$$

which implies that the condition (i) in Lemma 2.6 is satisfied.

Noting that $1 < p \le 2$, we have by Lemma 2.5 and C_r-inequality that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} \mathrm{Var}(Y_{ni} I(|Y_{ni}| \le 1)) \right)^j &\le \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}| \le 1) \right)^j \\
&\le \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} E|Y_{ni}|^p \right)^j \\
&\le C\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} |b_{ni}|^p E|X_{ni}|^p I(|X_{ni}| \le f(n)) \right)^j \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} |b_{ni}|^p (f(n))^p P(|X_{ni}| > f(n)) \right)^j \\
&\le C\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} |b_{ni}|^p E|X|^p I(|X| \le f(n)) \right)^j \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} |b_{ni}|^p (f(n))^p P(|X| > f(n)) \right)^j \\
&\doteq I_3 + I_4.
\end{aligned}$$

It follows from $E|X|^p/(\log|X|)^r < \infty$ and condition (*ii*) that

$$\begin{aligned} I_3 &\leq C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{n}|b_{ni}|^p(\log f(n))^r E|X|^p/(\log|X|)^r I(|X|\leq f(n))\right)^j \\ &\leq C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{n}|b_{ni}|^p(\log f(n))^r\right)^j < \infty. \end{aligned} \tag{3.4}$$

Noting that $(\log x)^r/x^p$ is a decreasing function on $[M,\infty)$ for some large $M > 0$, we have by $E|X|^p/(\log|X|)^r < \infty$ and condition (*ii*) again that

$$\begin{aligned} I_4 &\leq C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{n}|b_{ni}|^p(\log f(n))^r E|X|^p/(\log|X|)^r I(|X|> f(n))\right)^j \\ &\leq C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{n}|b_{ni}|^p(\log f(n))^r\right)^j < \infty. \end{aligned} \tag{3.5}$$

By (3.4) and (3.5) we can see that the condition (*ii*) in Lemma 2.6 is also satisfied. Hence, we have by Lemma 2.6 that

$$\sum_{n=1}^{\infty} n^{\beta}P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}(Y_{ni}-EY_{ni}I(|Y_{ni}|\leq 1))\right| > \epsilon\right) < \infty.$$

To prove $I_2 < \infty$, it suffices to show that

$$I_5 \doteq \max_{1\leq m\leq n}\left|\sum_{i=1}^{m} EY_{ni}I(|Y_{ni}|\leq 1)\right| \to 0, \text{ as } n\to\infty. \tag{3.6}$$

Noting that $Y_{ni}I(|Y_{ni}|\leq 1) + Y_{ni}I(|Y_{ni}|>1) + Z_{ni} = b_{ni}X_{ni}$ and $EX_{ni} = 0$, we can get that

$$I_5 \leq \sum_{i=1}^{n} E|Y_{ni}|I(|Y_{ni}|>1) + \sum_{i=1}^{n} E|Z_{ni}| \doteq I_6 + I_7.$$

It is easily seen that if $|b_{ni}|f(n) > 1$, then

$$\{|Y_{ni}|>1\} = \{|X_{ni}|>f(n)\}\cup\{|X_{ni}|\leq f(n), |b_{ni}X_{ni}|>1\}.$$

Noting that $(\log x)^r/x^p$ is a decreasing function on $[M,\infty)$ for some large $M > 0$, $(\log x)^r$ is an increasing function and $|Y_{ni}| \le |b_{ni}|f(n)$, we have by $E|X|^p/(\log|X|)^r < \infty$, condition (*iii*) and Lemma 2.5 that

$$\begin{aligned}
I_6 &= \sum_{i:|b_{ni}|f(n)>1} E|Y_{ni}|I(|Y_{ni}|>1) + \sum_{i:|b_{ni}|f(n)\le 1} E|Y_{ni}|I(|Y_{ni}|>1) \\
&= \sum_{i:|b_{ni}|f(n)>1} \{E|Y_{ni}|I(|X_{ni}|>f(n)) + E|Y_{ni}|I(|X_{ni}|\le f(n)), |b_{ni}X_{ni}|>1\} \\
&\le \sum_{i=1}^{n} |b_{ni}f(n)|^p P(|X_{ni}|>f(n)) + \sum_{i=1}^{n} |b_{ni}|E|X_{ni}|I(|X_{ni}|\le f(n), |b_{ni}X_{ni}|>1) \\
&\le \sum_{i=1}^{n} |b_{ni}f(n)|^p P(|X_{ni}|>f(n)) + \sum_{i=1}^{n} |b_{ni}|^p E|X_{ni}|^p I(|X_{ni}|\le f(n)) \\
&\le C\sum_{i=1}^{n} |b_{ni}|^p(\log f(n))^r E|X|^p/(\log|X|)^r I(|X|>f(n)) \\
&\quad +C\sum_{i=1}^{n} |b_{ni}|^p(\log f(n))^r E|X|^p/(\log|X|)^r I(|X|\le f(n)) \\
&\le C\sum_{i=1}^{n} |b_{ni}|^p(\log f(n))^r \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.7}$$

It follows from Hölder's inequality, condition (*iii*) and Lemma 2.5 again that

$$\begin{aligned}
I_7 &\le \sum_{i=1}^{n} |b_{ni}|E|X_{ni}|I(|X_{ni}|>f(n)) \\
&\le C\sum_{i=1}^{n} |b_{ni}|(\log f(n))^r (f(n))^{-(p-1)} E|X|^p/(\log|X|)^r I(|X|>f(n)) \\
&\le C\left(\sum_{i=1}^{n} |b_{ni}|^p\right)^{1/p} n^{1-1/p}(\log f(n))^r (f(n))^{-(p-1)} E|X|^p/(\log|X|)^r I(|X|>f(n)) \\
&\le C\left(\sum_{i=1}^{n} |b_{ni}|^p\right)^{1/p} n^{-(\beta+1)(p-1)/p}(\log n)^{r/p} E|X|^p/(\log|X|)^r I(|X|>f(n)) \\
&\to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.8}$$

Hence, (3.6) follows from (3.7) and (3.8). This completes the proof of the theorem. □

Theorem 3.2. *Let $1<\alpha\le 2$, $\alpha>\gamma>0$, and $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of rowwise NSD random variables which is stochastically dominated by a random variable X with $EX_{ni}=0$ and $E|X|^\alpha/(\log|X|)^{\alpha/\gamma-\delta}<\infty$ for some $0<\delta<\alpha/\gamma$. Assume that $\{a_{ni}, 1\le i\le n, n\ge 1\}$ is an array of constants satisfying $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n} P\left(|a_{ni}X_{ni}|>b_n\epsilon\right)<\infty \quad \textit{for all } \epsilon>0 \tag{3.9}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_{ni}\right|>b_n\epsilon\right)<\infty \quad \textit{for all } \epsilon>0. \tag{3.10}$$

are equivalent.

Proof. Firstly, we prove (3.9) implies (3.10) by using Theorem 3.1. Let $\beta = -1$, $p = \alpha$, $r = \alpha/\gamma - \delta$, and $b_{ni} = a_{ni}/b_n$. It is easily seen that the condition (*i*) of Theorem 3.1 is satisfied by (3.9). Note that

$$\sum_{i=1}^{n} |b_{ni}|^p (\log n)^r \leq C(\log n)^{-\delta},$$
$$\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{n} |b_{ni}|^p (\log n)^r \right)^j \leq C \sum_{n=1}^{\infty} n^{-1} (\log n)^{-\delta j},$$

which imply that conditions (*ii*) and (*iii*) of Theorem 3.1 are satisfied if we take $j > \max\{1, 1/\delta\}$. Hence (3.10) follows from Theorem 3.1 immediately.

Next we will prove that (3.10) implies (3.9). Without loss of generality, we assume that $b_{ni} \geq 0$. Otherwise, we will use b_{ni}^+ and b_{ni}^- instead of b_{ni} respectively. Noting that

$$\max_{1 \leq i \leq n} |b_{ni} X_{ni}| \leq 2 \max_{1 \leq m \leq n} \left| \sum_{i=1}^{m} b_{ni} X_{ni} \right|,$$

we have by Lemma 2.4 that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq m \leq n} \left| \sum_{i=1}^{m} a_{ni} X_{ni} \right| > b_n \epsilon \right) \\ \geq \; & \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq i \leq n} |b_{ni} X_{ni}| > 2\epsilon \right) \\ \geq \; & C \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - P\left(\max_{1 \leq i \leq n} |b_{ni} X_{ni}| > 2\epsilon \right) \right]^2 \sum_{i=1}^{n} P(|b_{ni} X_{ni}| > 2\epsilon). \end{aligned} \tag{3.12}$$

Now we prove that $P\left(\max_{1 \leq i \leq n} |b_{ni} X_{ni}| > 2\epsilon \right) \to 0$ firstly. Denote Y_{ni} and Z_{ni} as in the proof of Theorem 3.1. It follows from Markov's inequality, the proof of Theorem 3.1 and Lemma 2.5 that

$$\begin{aligned} P\left(\max_{1 \leq i \leq n} |b_{ni} X_{ni}| > 2\epsilon \right) & \leq P\left(\max_{1 \leq i \leq n} |Y_{ni}| > \epsilon \right) + P\left(\max_{1 \leq i \leq n} |Z_{ni}| > \epsilon \right) \\ & \leq \epsilon^{-p} \sum_{i=1}^{n} E|Y_{ni}|^p + \epsilon^{-1} \sum_{i=1}^{n} E|Z_{ni}| \\ & \leq C \sum_{i=1}^{n} |b_{ni}|^p E|X|^p I(|X| \leq f(n)) + C \sum_{i=1}^{n} |b_{ni}|^p (f(n))^p P(|X| > f(n)) \\ & \quad + C \sum_{i=1}^{n} |b_{ni}| E|X| I(|X| > f(n)) \\ & \leq C \sum_{i=1}^{n} |b_{ni}|^p (\log f(n))^r E|X|^p / (\log |X|)^r \\ & \quad + C \left(\sum_{i=1}^{n} |b_{ni}|^p \right)^{1/p} n^{-(\beta+1)(p-1)/p} (\log n)^{r/p} E|X|^p / (\log |X|)^r \\ & \leq C(\log n)^{-\delta} E|X|^{\alpha} / (\log |X|)^{\alpha/\gamma - \delta} + C(\log n)^{-\delta/\alpha} E|X|^{\alpha} / (\log |X|)^{\alpha/\gamma - \delta} \\ & \longrightarrow 0, \text{ as } n \to \infty. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we can obtain that

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le m\le n}\sum_{i=1}^{m}|a_{ni}X_{ni}|>b_n\epsilon\right)\ge C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P\left(|b_{ni}X_{ni}|>2\epsilon\right).$$

Hence, (3.10) implies (3.9). This completes the proof of the theorem. □

By using Theorem 3.2, we can get the following corollary.

Corollary 3.3. *Let $1<\alpha\le 2$, $\alpha>\gamma>0$, and $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of rowwise NSD random variables which is stochastically dominated by a random variable X with $EX_{ni}=0$ and $E|X|^\alpha/(\log|X|)^{\alpha/\gamma-1}<\infty$. Assume that $\{a_{ni}, 1\le i\le n, n\ge 1\}$ is an array of constants satisfying $\sum_{i=1}^{n}|a_{ni}|^\alpha=O(n)$. Let $b_n=n^{1/\alpha}(\log n)^{1/\gamma}$. Then (3.10) holds.*

Proof. By Theorem 3.2, it suffices to show that

$$J\doteq\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>\epsilon n^{1/\alpha}(\log n)^{1/\gamma})<\infty.$$

Actually,

$$\begin{aligned}
J &\le C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}P\left(|a_{ni}X|>\epsilon n^{1/\alpha}(\log n)^{1/\gamma}\right)\\
&= C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}P\left(|a_{ni}X|>\epsilon n^{1/\alpha}(\log n)^{1/\gamma}, |X|\le n^{1/\alpha}(\log n)^{1/\gamma-1/\alpha}\right)\\
&\quad +C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}P\left(|a_{ni}X|>\epsilon n^{1/\alpha}(\log n)^{1/\gamma}, |X|> n^{1/\alpha}(\log n)^{1/\gamma-1/\alpha}\right)\\
&\le C\sum_{n=1}^{\infty}n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}|a_{ni}|^\alpha E|X|^\alpha I(|X|\le n^{1/\alpha}(\log n)^{1/r-1/\alpha})\\
&\quad +C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}P\left(|X|>n^{1/\alpha}(\log n)^{1/\gamma-1/\alpha}\right)\\
&\le C\sum_{n=1}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}E|X|^\alpha I(|X|\le n^{1/\alpha}(\log n)^{1/\gamma-1/\alpha})\\
&\quad +C\sum_{n=1}^{\infty}P\left(|X|>n^{1/\alpha}(\log n)^{1/\gamma-1/\alpha}\right)\\
&\le CE|X|^\alpha/(\log|X|)^{\alpha/\gamma-1} < \infty.
\end{aligned}$$

The proof is completed. □

With Corollary 3.3 accounter for, we can get the following Marcinkiewicz-Zygmund type strong law of large number for NSD random variables. The proof is standard, so we omit the details.

Corollary 3.4. *Let $1<\alpha\le 2$, $\alpha>\gamma>0$, and $\{X_n, n\ge 1\}$ be a sequence of NSD random variables which is stochastically dominated by a random variable X with $EX_n=0$ and $E|X|^\alpha/(\log|X|)^{\alpha/\gamma-1}<\infty$. Assume that $\{a_n, n\ge 1\}$ is a sequence of constants satisfying $\sum_{i=1}^{n}|a_i|^\alpha=O(n)$. Let $b_n=n^{1/\alpha}(\log n)^{1/\gamma}$. Then for all $\epsilon>0$,*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}a_iX_i\right|>b_n\epsilon\right)<\infty, \tag{3.14}$$

and thus

$$\frac{1}{n^{1/\alpha}(\log n)^{1/\gamma}}\sum_{i=1}^{n}a_iX_i \to 0 \ \ a.s. \ \ as \ \ n\to\infty. \tag{3.15}$$

Remark 3.5. *The results of Theorems 3.1 and 3.2 generalize the corresponding ones of Chen and Sung [5] for sequences of identically distributed NA random variables to the case of arrays of rowwise NSD random variables without identical distribution.*

Remark 3.6. *Theorem 3.2 and Corollaries 3.3, 3.4 state convergence results for weighted sums of NSD variables normalized by terms of the form $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. Such normalizations have appeared in earlier literature, as Cuzick [7] or Bai and Cheng [3] for independent random variables, and Cai [4] for NA random variables. However, the moment condition $E\exp(h|X|^\gamma) < \infty$ for some $h, \gamma > 0$ in the literatures above is much stronger than $E|X|^\alpha/(\log|X|)^{\alpha/\gamma-\delta} < \infty$ for some $1 < \alpha \le 2$, $\alpha > \gamma > 0$, and $0 < \delta < \alpha/\gamma$. Hence, the results of Theorem 3.2 and Corollaries 3.3, 3.4 generalize and improve the corresponding ones of Cuzick [7] and Bai and Cheng [3] for independent random variables, and Cai [4] for NA random variables to the case of NSD random variables.*

Example 3.7. *Let $a_{ni} = 1$ for $1 \le i \le n$ and $n \ge 1$. Then the array $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfies $\sum_{i=1}^{n}|a_{ni}|^\alpha = O(n)$. Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise NSD random variables which is stochastically dominated by a random variable X with $EX_{ni} = 0$ and $E|X|^\alpha/(\log|X|)^{\alpha/\gamma-\delta} < \infty$ for some $1 < \alpha \le 2$, $\alpha > \gamma > 0$, and $0 < \delta < \alpha/\gamma$. Noting that $E|X|^\alpha/(\log|X|)^{\alpha/\gamma-\delta} < \infty$ is equivalent to $\sum_{n=1}^{\infty}P\left(|X| > \epsilon n^{1/\alpha}(\log n)^{1/\gamma}\right) < \infty$, we have by the definition of stochastic domination that for all $\epsilon > 0$,*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P\left(|a_{ni}X_{ni}| > b_n\epsilon\right) &\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P\left(|X| > \epsilon n^{1/\alpha}(\log n)^{1/\gamma}\right)\\ &= C\sum_{n=1}^{\infty}P\left(|X| > \epsilon n^{1/\alpha}(\log n)^{1/\gamma}\right)\\ &< \infty.\end{aligned}$$

Thus, it follows from Theorem 3.2 and the inequality above that for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}X_{ni}\right| > \epsilon n^{1/\alpha}(\log n)^{1/\gamma}\right) < \infty.$$

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