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New Family of Whitney Numbers

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Abstract. In this paper we give a new family of numbers, called $\bar{\alpha}$-Whitney numbers, which gives generalization of many types of Whitney numbers and Stirling numbers. Some basic properties of these numbers such as recurrence relations, explicit formulas and generating functions are given. Finally many interesting special cases are derived.

1. Introduction

Dowling [10] constructed a class of geometric lattices of rank n based on finite groups G of order $m \geq 1$, called Dowling lattices and denoted by $Q_n(G)$. For $0 \leq k \leq n$, the Whitney numbers of the first and second kind of $Q_n(G)$ denoted by $w_m(n,k)$ and $W_m(n,k)$, respectively, are defined by

$$m^n (x)_n = \sum_{k=0}^{n} w_m(n,k)\,(mx+1)^k, \tag{1}$$

$$(mx+1)^n = \sum_{k=0}^{n} m^k\, W_m(n,k)\,(x)_k, \tag{2}$$

where $(x)_n$ is the falling factorial i.e., $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ and $(x)_0 = 1$. These numbers satisfy the following recurrence relations:

$$w_m(n+1,k) = w_m(n,k-1) - (1+nm)w_m(n,k),$$

$$W_m(n+1,k) = W_m(n,k-1) + (1+km)W_m(n,k),$$

with $w_m(n,n) = W_m(n,n) = 1$ for $n \geq 0$ and $w_m(n,k) = W_m(n,k) = 0$ for $k > n$ or $k < 0$. Benoumhani [4, 5] gave many properties of the whitney numbers of Dowling lattice such as generating functions, explicit formulas, concavity and the relations with the Stirling numbers.

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Mezö [16] introduced the r-Whitney numbers of the first and second kind, $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$, $0 \leq k \leq n$, as a new class of numbers generalizing the Whitney and r-Stirling numbers, they defined by

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)\,(mx+r)^k, \tag{3}$$

$$(mx+r)^n = \sum_{k=0}^{n} m^k\, W_{m,r}(n,k)\,(x)_k. \tag{4}$$

It is clear that at $r = 1$ the r-Whitney numbers are reduced to the Whitney numbers of Dowling lattice. Many properties of r-Whitney numbers can be found in Mezö [16] and Cheon [7].

In this paper, we construct and study a new family of Whitney numbers, called $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the first and second kind, denoted by $w_{m,\bar{\alpha}}(n,k)$ and $W_{m,\bar{\alpha}}(n,k)$, respectively, $0 \leq k \leq n$, where $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ and $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1}$ are real numbers. For the new numbers we give recurrence relations, generating functions, explicit formula, and some special cases are discussed.

Throughout this paper we use the following notations:

$$(x;\bar{\boldsymbol{\alpha}}|m)_n = \prod_{j=0}^{n-1}(x-\alpha_j-jm) = (x-\alpha_0)(x-\alpha_1-m)\cdots(x-\alpha_{n-1}-(n-1)m),$$

for $n = 1, 2, \cdots$ with $(x;\bar{\boldsymbol{\alpha}}|m)_0 = 1$.

If $\alpha_i = 0, i = 0, 1, \cdots, n-1$, hence $(x;\mathbf{0}|m)_n = x(x-m)(x-2m)\cdots(x-(n-1)m) = (x|m)_n$, where $(x|m)_n$ is called the generalized factorial of x of degree n with increment m, with $(x|m)_0 = 1$, in particular we write $(x|1)_n = (x)_n$, and also we use the notation

$$(x;\bar{\boldsymbol{\alpha}})_n = \prod_{j=0}^{n-1}(x-\alpha_j) = (x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1}) \text{ with } (x;\bar{\boldsymbol{\alpha}})_0 = 1.$$

2. $\bar{\boldsymbol{\alpha}}$-Whitney Numbers of the Second Kind

Definition 2.1. *The $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the second kind, $W_{m,\bar{\alpha}}(n,k)$, are defined by*

$$x^n = \sum_{k=0}^{n} W_{m,\bar{\alpha}}(n,k)\,(x;\bar{\boldsymbol{\alpha}}|m)_k\,, \tag{5}$$

where $W_{m,\bar{\alpha}}(0,0) = 1$ and $W_{m,\bar{\alpha}}(n,k) = 0$ for $k > n$ or $k < 0$.

Theorem 2.2. *The $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the second kind satisfy the recurrence relation*

$$W_{m,\bar{\alpha}}(n+1,k) = W_{m,\bar{\alpha}}(n,k-1) + (\alpha_k + km)W_{m,\bar{\alpha}}(n,k), \tag{6}$$

where $n \geq k \geq 1$, $W_{m,\bar{\alpha}}(n,k) = 0$ for $n < k < 0$ and for $k = 0$, we have

$$W_{m,\bar{\alpha}}(n,0) = \alpha_0^n, \text{ for } n \geq 0. \tag{7}$$

Proof. We can write

$$x^{n+1} = x^n(x - \alpha_k - km + \alpha_k + km).$$

Using (5), we get

$$\begin{aligned}
&\sum_{k=0}^{n+1} W_{m,\bar{\alpha}}(n+1,k)\,(x;\bar{\boldsymbol{\alpha}}|m)_k \\
&= \sum_{k=0}^{n} W_{m,\bar{\alpha}}(n,k)\,(x;\bar{\boldsymbol{\alpha}}|m)_k\,[(x-\alpha_k-km)+(\alpha_k+km)] \\
&= \sum_{k=0}^{n} W_{m,\bar{\alpha}}(n,k)\,(x;\bar{\boldsymbol{\alpha}}|m)_{k+1} + \sum_{k=0}^{n}(\alpha_k+km)W_{m,\bar{\alpha}}(n,k)\,(x;\bar{\boldsymbol{\alpha}}|m)_k \\
&= \sum_{k=1}^{n+1} W_{m,\bar{\alpha}}(n,k-1)\,(x;\bar{\boldsymbol{\alpha}}|m)_k + \sum_{k=0}^{n}(\alpha_k+km)W_{m,\bar{\alpha}}(n,k)\,(x;\bar{\boldsymbol{\alpha}}|m)_k\,.
\end{aligned}$$

Equating the coefficients of $(x;\bar{\boldsymbol{\alpha}}|m)_k$ on both sides, we obtain (6).
For $k=0$, we find $W_{m,\bar{\alpha}}(n+1,0)=\alpha_0\,W_{m,\bar{\alpha}}(n,0),\ n=0,1,2,\ldots,$
consequently, we get $W_{m,\bar{\alpha}}(n,0)=\alpha_0^n\,W_{m,\bar{\alpha}}(0,0)=\alpha_0^n$. $\square$

The following array for $W_{m,\bar{\alpha}}(n,k)$, $0\le n,\ k\le 3$ were computed using the recurrence relation (6).

$$\begin{pmatrix}
1 & 0 & 0 & 0\\
\alpha_0 & 1 & 0 & 0\\
\alpha_0^2 & \alpha_0+\alpha_1+m & 1 & 0\\
\alpha_0^3 & \alpha_0^2+\alpha_0(\alpha_1+m)+(\alpha_1+m)^2 & \alpha_0+\alpha_1+\alpha_2+3m & 1
\end{pmatrix}$$

Theorem 2.3. *The exponential generating function $\phi_k(t)$ of $W_{m,\bar{\alpha}}(n,k)$, is given by*

$$\phi_k(t)=\sum_{n\ge k} W_{m,\bar{\alpha}}(n,k)\frac{t^n}{n!}=\sum_{j=0}^{k}\frac{\exp\Big((\alpha_j+jm)t\Big)}{\prod_{l=0,l\ne j}^{k}\Big[\alpha_j-\alpha_l+(j-l)m\Big]}\ , \tag{8}$$

where $k=1,2,3,\cdots$ and

$$\phi_k(0)=0\ for\ k\ge 1,\ and\ \phi_0(t)=\exp(\alpha_0\,t). \tag{9}$$

Proof. From the definition of the exponential generating function, we have $\phi_k(0)=0$ for $k\ge 1$, and by (7) we get

$$\phi_0(t)=\sum_{n\ge 0} W_{m,\bar{\alpha}}(n,0)\,\frac{t^n}{n!}=\sum_{n\ge 0}\alpha_0^n\,\frac{t^n}{n!}=\exp(\alpha_0\,t).$$

Now, multiplying (6) by $\frac{t^n}{n!}$ and summing over n, we obtain

$$\sum_{n\ge 0} W_{m,\bar{\alpha}}(n+1,k)\frac{t^n}{n!}=\sum_{n\ge 0} W_{m,\bar{\alpha}}(n,k-1)\frac{t^n}{n!}+\sum_{n\ge 0}(\alpha_k+km)W_{m,\bar{\alpha}}(n,k)\frac{t^n}{n!}.$$

This may be rewritten in the form

$$\sum_{n\ge 1} W_{m,\bar{\alpha}}(n,k)\frac{t^{n-1}}{(n-1)!}-\sum_{n\ge 0}(\alpha_k+km)W_{m,\bar{\alpha}}(n,k)\frac{t^n}{n!}=\sum_{n\ge 0} W_{m,\bar{\alpha}}(n,k-1)\frac{t^n}{n!}.$$

This is precisely equivalent to

$$\frac{d}{dt}\phi_k(t) - (\alpha_k + km)\phi_k(t) = \phi_{k-1}(t),\ k \geq 1.$$

Solving this difference-differential equation for $k = 1, 2, 3, \cdots$ with the initial conditions (9), we obtain (8) (Here the computations are long, therefore it has been omitted) □

Theorem 2.4. *The $\bar{\alpha}$-Whitney numbers of the second kind $W_{m,\bar{\alpha}}(n,k)$ have the explicit formula*

$$W_{m,\bar{\alpha}}(n,k) = \sum_{j=0}^{k} \frac{(\alpha_j + jm)^n}{\prod_{l=0,l\neq j}^{k}\left[\alpha_j - \alpha_l + (j-l)m\right]}. \tag{10}$$

Proof. We have for $0 \leq k \leq n$

$$\begin{aligned}\sum_{n=0}^{\infty} W_{m,\bar{\alpha}}(n,k)t^n &= \sum_{j=0}^{k} \frac{\exp\left((\alpha_j + jm)t\right)}{\prod_{l=0,l\neq j}^{k}\left[\alpha_j - \alpha_l + (j-l)m\right]}\\ &= \sum_{j=0}^{k} \frac{1}{\prod_{l=0,l\neq j}^{k}\left[\alpha_j - \alpha_l + (j-l)m\right]} \sum_{n=0}^{\infty} \frac{(\alpha_j + jm)^n\ t^n}{n!}.\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides gives (10). □

Theorem 2.5. *The $\bar{\alpha}$-Whitney numbers of the second kind $W_{m,\bar{\alpha}}(n,k)$ have the explicit formula*

$$\begin{aligned}W_{m,\bar{\alpha}}(n,k) = &\sum_{\substack{i_0+i_1+\cdots+i_{n-1}=n-k\\ i_j\in\{0,\,1\}}} \binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0} + (1-i_0)m}{i_1}\binom{\alpha_{2-i_0-i_1} + (2-i_0-i_1)m}{i_2}\cdots\\ &\cdots\binom{\alpha_{n-1-i_0-i_1-\cdots-i_{n-2}} + (n-1-i_0-i_1-\cdots-i_{n-2})m}{i_{n-1}}\end{aligned}$$

Proof. For $k = 0$, we have $i_0 + i_1 + \cdots + i_{n-1} = n \Leftrightarrow i_0 = i_1 = \cdots = i_{n-1} = 1$ and

$$W_{m,\bar{\alpha}}(n,0) = \underbrace{\alpha_0.\alpha_0\cdots\alpha_0}_{n} = \alpha_0^n,\ for\ n \geq 0.$$

For $i_{n-1} \in \{0, 1\}$ we get

$$\begin{aligned}W_{m,\bar{\alpha}}(n,k) = &\sum_{\substack{i_0+i_1+\cdots+i_{n-2}=n-1-(k-1)\\ i_j\in\{0,\,1\}}} \binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0} + (1-i_0)m}{i_1}\cdots\\ &\cdots\binom{\alpha_{n-2-i_0-\cdots-i_{n-3}} + (n-2-i_0-\cdots-i_{n-3})m}{i_{n-2}} + \sum_{i_0+i_1+\cdots+i_{n-2}=n-1-k} \binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0} + (1-i_0)m}{i_1}\cdots\\ &\cdots\binom{\alpha_{n-2-i_0-\cdots-i_{n-3}} + (n-2-i_0-\cdots-i_{n-3})m}{i_{n-2}}(\alpha_{n-1-i_0-i_1-\cdots-i_{n-2}} + (n-1-i_0-\cdots-i_{n-2})m)\\ &= W_{m,\bar{\alpha}}(n-1,k-1) + (\alpha_{n-1-(n-1-k)} + (n-1-(n-1-k))m)\ W_{m,\bar{\alpha}}(n-1,k)\\ &= W_{m,\bar{\alpha}}(n-1,k-1) + (\alpha_k + km)W_{m,\bar{\alpha}}(n-1,k),\end{aligned}$$

which is the recurrence relation (6). This completes the proof. □

Remark 2.6. *This explicit formula is better than (10) for bigger volumes of k (i.e. for $k > [\frac{n}{2}]$)*

Corollary 2.7. *The following combinatorial identity is valid. For $n \geq k \geq 1$ we have (from Th.2.4 and Th.2.5)*

$$\sum_{\substack{i_0+i_1+\cdots+i_{n-1}=n-k\\ i_j\in\{0,1\}}} \binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0}+(1-i_0)m}{i_1}\cdots\binom{\alpha_{n-1-i_0-i_1-\cdots-i_{n-2}}+(n-1-i_0-i_1-\cdots-i_{n-2})m}{i_{n-1}}$$

$$=\sum_{j=0}^{k}\frac{(\alpha_j+jm)^n}{\prod_{l=0,l\neq j}^{k}\left[\alpha_j-\alpha_l+(j-l)m\right]}.$$

Some special cases:

$$W_{m,\bar{\alpha}}(2,1)=\sum_{i_0+i_1=1}\binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0}+(1-i_0)m}{i_1}\overset{(1,0),(0,1)}{=}\alpha_0+\alpha_1+m$$

$$W_{m,\bar{\alpha}}(3,1)=\sum_{i_0+i_1+i_2=2}\binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0}+(1-i_0)m}{i_1}\binom{\alpha_{2-i_0-i_1}+(2-i_0-i_1)m}{i_2}$$

$$\underset{(1,1,0)}{\overset{(0,1,1),(1,0,1)}{=}}(\alpha_1+m)(\alpha_1+m)+\alpha_0(\alpha_1+m)+\alpha_0(\alpha_0)=\alpha_0^2+\alpha_0(\alpha_1+m)+(\alpha_1+m)^2$$

$$W_{m,\bar{\alpha}}(3,2)=\sum_{i_0+i_1+i_2=1}\binom{\alpha_0}{i_0}\binom{\alpha_{1-i_0}+(1-i_0)m}{i_1}\binom{\alpha_{2-i_0-i_1}+(2-i_0-i_1)m}{i_2}$$

$$\underset{(0,0,1)}{\overset{(0,1,0),(1,0,0)}{=}}\alpha_0+\alpha_1+m+\alpha_2+2m=\alpha_0+\alpha_1+\alpha_2+3m$$

Theorem 2.8. *The ordinary generating function of $W_{m,\bar{\alpha}}(n,k)$ is given by*

$$Y_k(t)=\sum_{n\geq k}W_{m,\bar{\alpha}}(n,k)\,t^n=\frac{t^k}{\prod_{j=0}^{k}\left[1-(\alpha_j+jm)t\right]}\ ,\tag{11}$$

where $k=1,2,3.\cdots$,

$$Y_k(0)=0\ for\ k\geq 1\ and\ Y_0(t)=(1-\alpha_0 t)^{-1}.\tag{12}$$

Proof. Equation (12) can easily obtained from the definition of generating function and hence

$$Y_0(t)=\sum_{n\geq 0}W_{m,\bar{\alpha}}(n,0)\,t^n=\sum_{n\geq 0}\alpha_0^n t^n=\sum_{n\geq 0}(\alpha_0 t)^n=(1-\alpha_0 t)^{-1}.$$

By virtue of (6), we obtain

$$\sum_{n\geq 0}W_{m,\bar{\alpha}}(n,k)\,t^n=\sum_{n\geq 0}W_{m,\bar{\alpha}}(n-1,k-1)\,t^n+(\alpha_k+km)\sum_{n\geq 0}W_{m,\bar{\alpha}}(n-1,k)\,t^n,$$

hence

$$Y_k(t)=t\,Y_{k-1}+(\alpha_k+km)\,t\,Y_k(t).$$

Thus

$$Y_k(t)=Y_{k-1}(t)\,\frac{t}{[1-(\alpha_k+km)t]}\ ,\ k=1,\ 2,\ 3,\ \cdots,\tag{13}$$

consequently, we get

$$Y_k(t) = Y_0(t)\,\frac{t}{[1-(\alpha_1+m)t]}\,\frac{t}{[1-(\alpha_2+2m)t]}\cdots\frac{t}{[1-(\alpha_k+km)t]} = \frac{t^k}{\prod_{j=0}^{k}\left[1-(\alpha_j+jm)t\right]}.$$

□

The following Corollary shows that $\bar{\alpha}$-Whitney numbers of the second kind $W_{m,\bar{\alpha}}(n,k)$ are the complete symmetric function of the numbers $\alpha_0,\ \alpha_1+m,\ \alpha_2+2m,\ \cdots,\ \alpha_k+km$ of order $n-k$, where the complete symmetric function h_k is defined by

$$h_k(z_1,z_2,\cdots,z_n) = \sum_{1\le j_1\le j_2\le\cdots\le j_k\le n}\prod_{i=1}^{k} z_{j_i},$$

where $h_0=1$ and $h_k=0$ for $k<0$ or $k>n$.

Corollary 2.9. *The $\bar{\alpha}$-Whitney numbers of the second kind satisfy*

$$W_{m,\bar{\alpha}}(n,k) = \sum_{0\le j_1\le\cdots\le j_{n-k}\le k}\prod_{i=1}^{n-k}(\alpha_{j_i}+j_i\,m) = h_{n-k}(\alpha_0,\ \alpha_1+m,\ \alpha_2+2m,\ \cdots,\ \alpha_k+km). \quad (14)$$

Proof. From (11), we get

$$\sum_{n\ge k} W_{m,\bar{\alpha}}(n,k)\,t^{n-k} = \frac{1}{\prod_{j=0}^{k}\left[1-(\alpha_j+jm)t\right]} = \prod_{j=0}^{k}\left[1-(\alpha_j+jm)t\right]^{-1}. \quad (15)$$

For $|t|<\frac{1}{|\alpha_k+km|}$, expanding the right hand side, hence the coefficient of t^{n-k} gives (14). □

3. Special Cases

In the following we show that, the coefficients $W_{m,\bar{\alpha}}(n,k)$ give back the Stirling, r-Stirling, Comtet, Whitney , r-Whitney, Jacobi, and Legendre-Stirling numbers as special cases. For the different cases, we drive only the explicit formula and the exponential generating function. By substituting the particular values of $\bar{\alpha}$ and m in (6), (11), and (14), one easily obtain the other properties (recurrence relations, ordinary generating function, and interpreting as complete symmetric function).

Case 1

Setting $m=1$, equation (5) can be rewritten in the form

$$x^n = \sum_{k=0}^{n} W_{1,\bar{\alpha}}(n,k)\,(x;\bar{\alpha}|1)_k = \sum_{k=0}^{n} W_{1,\bar{\alpha}}(n,k)\prod_{j=0}^{k-1}(x-(\alpha_j+j)). \quad (16)$$

Setting $\beta_j=\alpha_j+j,\ j=0,\ 1,\ \cdots,\ n-1$, we get

$$x^n = \sum_{k=0}^{n} W_{1,\bar{\alpha}}(n,k)\prod_{j=0}^{k-1}(x-\beta_j) = \sum_{k=0}^{n} W_{1,\bar{\alpha}}(n,k)(x;\bar{\beta})_k. \quad (17)$$

Thus $W_{1,\bar{\alpha}}(n,k)$, is reduced to Comtet numbers of the second kind $S_{\bar{\beta}}(n,k)$. One easily deduce the properties of Comtet numbers (see [8, 11]) as follows:

The exponential generating function (8) is reduced to

$$\sum_{n\ge k} W_{1,\bar{\alpha}}(n,k)\frac{t^n}{n!} = \sum_{n\ge k} S_{\bar{\beta}}(n,k)\frac{t^n}{n!} = \sum_{j=0}^{k}\frac{\exp\left(\beta_j\,t\right)}{(\beta_j)_k}, \quad (18)$$

where $(\beta_j)_k = \prod_{l=0,l\neq j}^{k}(\beta_j - \beta_l)$, $j \leq k$ with $(\beta_0)_0 = 1$ (notice that $\beta_0 \neq \beta_1 \neq \cdots$). From (10), we get the explicit formula of Comtet numbers

$$S_{\bar{\beta}}(n,k) = \sum_{j=0}^{k} \frac{\beta_j^n}{(\beta_j)_k}.$$

Case 2
When $\bar{\alpha} = (1, 1, \cdots, 1) := \mathbf{1}$, hence

$$x^n = \sum_{k=0}^{n} W_{m,\mathbf{1}}(n,k)\,(x;\mathbf{1}|m)_k\,. \tag{19}$$

Setting $mx + 1$ in place of x, yields

$$(mx+1)^n = \sum_{k=0}^{n} m^k\, W_{m,\mathbf{1}}(n,k)(x)_k.$$

Hence $W_{m,\mathbf{1}}(n,k) = W_m(n,k)$, the Whitney numbers of the second kind. The exponential generating function of $W_m(n,k)$ can easily deduced from (8)

$$\begin{aligned}\sum_{n\geq k} W_m(n,k)\frac{t^n}{n!} &= \sum_{j=0}^{k} \frac{\exp\left((1+jm)t\right)}{\prod_{l=0,l\neq j}^{k}(j-l)m} \\ &= \frac{1}{m^k\,k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\exp((1+jm)t) = \frac{\exp(t)}{m^k\,k!}\left(\exp(mt)-1\right)^k.\end{aligned}$$

From (10), we get the explicit formula of $W_m(n,k)$,

$$W_m(n,k) = \sum_{j=0}^{k} \frac{(1+jm)^n}{\prod_{l=0,l\neq j}^{k}(j-l)m} = \frac{1}{m^k\,k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1+jm)^n,$$

this is in agreement with Benoumhani [4].
Case 3
When $\bar{\alpha} = (r, r, \cdots, r) := \mathbf{r}$, we get

$$x^n = \sum_{k=0}^{n} W_{m,\mathbf{r}}(n,k)\,(x;\mathbf{r}|m)_k\,. \tag{20}$$

Setting $mx + r$ in place of x , we get

$$(mx+r)^n = \sum_{k=0}^{n} m^k\, W_{m,\mathbf{r}}(n,k)(x)_k.$$

Hence the $W_{m,\mathbf{r}}(n,k)$ is reduced to r-Whitney numbers of second kind $W_{m,r}(n,k)$. The exponential generating function (8) is reduced to

$$\begin{aligned}\sum_{n\geq k} W_{m,r}(n,k)\frac{t^n}{n!} &= \sum_{j=0}^{k} \frac{\exp\left((r+jm)t\right)}{\prod_{l=0,l\neq j}^{k}(j-l)m} \\ &= \frac{1}{m^k\,k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\exp\left((r+jm)t\right) = \frac{\exp(rt)}{m^k\,k!}\left(\exp(mt)-1\right)^k.\end{aligned}$$

Also from (10), we get the explicit formula of $W_{m,r}(n,k)$,

$$W_{m,r}(n,k)=\sum_{j=0}^{k}\frac{(r+jm)^n}{\prod_{l=0,l\neq j}^{k}(j-l)m}=\frac{1}{m^k\,k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(r+jm)^n,$$

this is in agreement with Mezö [16].

Case 4

When $m=1$ and $\bar{\boldsymbol{\alpha}}=(0,0,\cdots,0):=\mathbf{0}$, in (5), we get

$$x^n=\sum_{k=0}^{n}W_{1,\mathbf{0}}(n,k)\,(x;\mathbf{0}|1)_k=\sum_{k=0}^{n}W_{1,\mathbf{0}}(n,k)(x)_k.$$

Therefore $W_{1,\mathbf{0}}(n,k)=S(n,k)$, the Stirling numbers of the second kind, see for example [9, Ch. 5] and hence (10) is reduced to

$$W_{1,\mathbf{0}}(n,k)=\sum_{j=0}^{k}\frac{(j)^n}{\prod_{l=0,l\neq j}^{k}(j-l)}=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j)^n=S(n,k).$$

The exponential generating function (8) is reduced to

$$\begin{aligned}\sum_{n\geq k}S(n,k)\frac{t^n}{n!}&=\sum_{j=0}^{k}\frac{\exp(jt)}{\prod_{l=0,l\neq j}^{k}(j-l)}\\&=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\exp(jt)=\frac{1}{k!}\left(\exp(t)-1\right)^k.\end{aligned}$$

Case 5

When $m=1$ and $\bar{\boldsymbol{\alpha}}=(r,r,\cdots,r):=\mathbf{r}$, the $W_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ is reduced to r-Stirling numbers of the second kind, (see Broder [6]), as follows

$$x^n=\sum_{k=0}^{n}W_{1,\mathbf{r}}(n,k)\,(x;\mathbf{r}|1)_k\,. \tag{21}$$

Setting $x+r$ in place of x , we get

$$(x+r)^n=\sum_{k=0}^{n}W_{1,\mathbf{r}}(n,k)(x)_k.$$

Hence $W_{1,\mathbf{r}}(n,k)=S_r(n+r,k+r)$. The exponential generating function (8) is reduced to

$$\begin{aligned}\sum_{n\geq k}S_r(n+r,k+r)\frac{t^n}{n!}&=\sum_{j=0}^{k}\frac{\exp\left((r+j)t\right)}{\prod_{l=0,l\neq j}^{k}(j-l)}\\&=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\exp\left((r+j)t\right)=\frac{\exp(rt)}{k!}\left(\exp(t)-1\right)^k.\end{aligned}$$

Also (10) is reduced to

$$S_r(n+r,k+r)=\sum_{j=0}^{k}\frac{(r+j)^n}{\prod_{l=0,l\neq j}^{k}(j-l)}=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(r+j)^n.$$

Notice that this case is equivalent to the case $\bar{\boldsymbol{\alpha}} = (-a, -a, \cdots, -a) := -\mathbf{a}$ and $m = 1$, hence $W_{1,-\mathbf{a}}(n,k) = S_a(n,k)$, the non-central Stirling numbers of the second kind, see Koutras [15].

Case 6

The Jacobi-Stirling numbers of the second kind were introduced by Everitt [12], denoted by $P^{(\alpha,\beta)}S_n^k$, where $\alpha,\ \beta > -1$ are fixed constant parameters. For more properties and combinatorial interpretations of these numbers, see [1, 12, 14]. Everitt [12] obtained an explicit summation formula for $P^{(\alpha,\beta)}S_n^k$, showing that these numbers depend only on one parameter $z = \alpha + \beta + 1,\ z > -1$. So Gelineau [14] defined the Jacobi-Stirling numbers of the second kind as follows:

$$x^n = \sum_{k=0}^{n} JS_n^k(z) \prod_{j=0}^{k-1}(x - j(j+z)), \tag{22}$$

where $JS_n^k(z) = P^{(\alpha,\beta)}S_n^k$ in the notation of [12]. Comparing (22) to (5), we see that when $\bar{\boldsymbol{\alpha}} = (0^2, 1^2, 2^2, \cdots, (n-1)^2)$, and $z = m$, we get $JS_n^k(z) = W_{z,\bar{\boldsymbol{\alpha}}}(n,k)$. The most properties of the Jacobi-Stirling numbers of the second kind can be easily deduced from those of $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the second kind by taking $\alpha_j = j^2$, $j = 0,\ 1,\ 2,\ \cdots,\ n-1$, and $z = m$ (note that in our case $z = m \geq 1$). The explicit formula of $JS_n^k(z)$ is given by

$$JS_n^k(z) = \sum_{j=0}^{k} \frac{(j^2 + jz)^n}{\prod_{l=0,l\neq j}^{k}[j^2 - l^2 + (j-l)z]} = \sum_{j=1}^{k}(-1)^{k+j}\,\frac{(j^2+jz)^n}{j!\,(k-j)!\,\langle z+j\rangle_j\,\langle z+2j+1\rangle_{k-j}}\,, \tag{23}$$

where $\langle z\rangle_n = z(z+1)\cdots(z+n-1)$. We deduce the exponential generating function of the Jacobi-Stirling numbers of the second kind from (8),

$$\sum_{n=0}^{\infty} JS_n^k(z)\frac{t^n}{n!} = \sum_{j=0}^{k} \frac{\exp\left((j^2+jz)t\right)}{\prod_{l=0,l\neq j}^{k}[j^2 - l^2 + (j-l)z]} = \sum_{j=0}^{k}(-1)^{k+j}\,\frac{\exp\left((j^2+jz)t\right)}{j!\,(k-j)!\,\langle z+j\rangle_j\,\langle z+2j+1\rangle_{k-j}}\,. \tag{24}$$

Case 7

It is well known that when $z = 1$, the Jacobi-Stirling numbers become the Legendre-Stirling numbers introduced by Everitt [13]. So when $\alpha_j = j^2,\ j = 0,\ 1,\ 2,\ \cdots,\ n-1$, and $m = 1$, the $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the Second kind become the Legendre- Stirling numbers of the second kind denoted by PS_n^k. For more properties of these numbers, see [2, 3]. The explicit formula of PS_n^k is given by

$$PS_n^k = \sum_{j=0}^{k} \frac{(j^2+j)^n}{\prod_{l=0,l\neq j}^{k}[j^2 - l^2 + (j-l)]} = \sum_{j=1}^{k}(-1)^{k+j}\,\frac{(2j+1)\,(j^2+j)^n}{(j+k+1)!\,(k-j)!}\,. \tag{25}$$

The exponential generating function of PS_n^k can be deduced from (8),

$$\sum_{n=0}^{\infty} PS_n^k\,\frac{t^n}{n!} = \sum_{j=0}^{k} \frac{\exp\left((j^2+j)t\right)}{\prod_{l=0,l\neq j}^{k}[j^2 - l^2 + (j-l)]} = \sum_{j=0}^{k}(-1)^{k+j}\,\frac{(2j+1)\,\exp\left((j^2+j)t\right)}{(j+k+1)!\,(k-j)!}\,. \tag{26}$$

4. $\bar{\boldsymbol{\alpha}}$-Whitney Numbers of the First Kind

Definition 4.1. *The $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the first kind, $w_{m,\bar{\boldsymbol{\alpha}}}(n,k)$, are defined by*

$$(x;\bar{\boldsymbol{\alpha}}|m)_n = \sum_{k=0}^{n} w_{m,\bar{\boldsymbol{\alpha}}}(n,k)x^k, \tag{27}$$

where $w_{m,\bar{\boldsymbol{\alpha}}}(0,0) = 1$ and $w_{m,\bar{\boldsymbol{\alpha}}}(n,k) = 0$ for $k > n$ or $k < 0$.

Theorem 4.2. *The $\bar{\alpha}$-Whitney numbers of the first kind satisfy the recurrence relation*

$$w_{m,\bar{\alpha}}(n+1,k) = w_{m,\bar{\alpha}}(n,k-1) - (\alpha_n + nm)w_{m,\bar{\alpha}}(n,k), \tag{28}$$

where $n \geq k \geq 1$, for $k = 0$ we have

$$w_{m,\bar{\alpha}}(n,0) = (-1)^n \prod_{i=0}^{n-1}(\alpha_i + im) \tag{29}$$

Proof. Since

$$(x;\bar{\alpha}|m)_{n+1} = (x;\bar{\alpha}|m)_n\,(x - \alpha_n - nm),$$

hence using (27), we get

$$\sum_{k=0}^{n+1} w_{m,\bar{\alpha}}(n+1,k)x^k = (x - \alpha_n - nm)\sum_{k=0}^{n} w_{m,\bar{\alpha}}(n,k)x^k$$
$$= \sum_{k=0}^{n} w_{m,\bar{\alpha}}(n,k)x^{k+1} - (\alpha_n + nm)\sum_{k=0}^{n} w_{m,\bar{\alpha}}(n,k)x^k$$
$$= \sum_{k=1}^{n+1} w_{m,\bar{\alpha}}(n,k-1)x^k - (\alpha_n + nm)\sum_{k=0}^{n} w_{m,\bar{\alpha}}(n,k)x^k.$$

Equating the coefficients of x^k on both sides, we obtain (28).
For $k = 0$, we find

$$w_{m,\bar{\alpha}}(n+1,0) = -w_{m,\bar{\alpha}}(n,0)\,(\alpha_n + nm),\; n = 0,1,2,\ldots,$$

consequently, we get

$$w_{m,\bar{\alpha}}(n,0) = (-1)^n\, w_{m,\bar{\alpha}}(0,0)\,\alpha_0\,(\alpha_1 + m)\cdots(\alpha_{n-1} + (n-1)m).$$

□

The following array $w_{m,\bar{\alpha}}(n,k),\; 0 \leq n,\, k \leq 3$ were computed using the recurrence relation (28).

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha_0 & 1 & 0 & 0 \\ \alpha_0(\alpha_1 + m) & -\alpha_0 - \alpha_1 - m & 1 & 0 \\ -\alpha_0(\alpha_1 + m)(\alpha_2 + 2m) & \alpha_0(\alpha_1 + m) + (\alpha_0 + \alpha_1 + m)(\alpha_2 + 2m) & -\alpha_0 - \alpha_1 - 3m - \alpha_2 & 1 \end{pmatrix}$$

Notice that by setting $k = n$ in the recurrence relations (28) and (6), we get

$$W_{m,\bar{\alpha}}(n+1,n) + w_{m,\bar{\alpha}}(n+1,n) = 0,\; \forall n.$$

Theorem 4.3. *The $\bar{\alpha}$-Whitney numbers of the first kind $w_{m,\bar{\alpha}}(n,k)$ have the following explicit expression:*

$$w_{m,\bar{\alpha}}(n,k) = \sum_{\substack{i_1+i_2+\cdots+i_{n-1}=k \\ i_j\in\{0,\,1\}}} \binom{i_1 - \alpha_0}{1 - i_1}\binom{i_2 - \alpha_1 - m}{1 - i_2}\cdots\binom{i_n - \alpha_{n-1} - (n-1)m}{1 - i_n}$$

Proof. For $k = 0$ we have $w_{m,\bar{\alpha}}(n, 0) = (-\alpha_0)(-\alpha_1 - m) \cdots (-\alpha_{n-1} - (n-1)m) = (-1)^n \prod_{i=0}^{n-1}(\alpha_i + im)$, i.e. (29). For $i_n \in \{0, 1\}$ we get

$$\begin{aligned} w_{m,\bar{\alpha}}(n,k) = & \sum_{\substack{i_1+i_2+\cdots+i_{n-1}=k-0 \\ i_j\in\{0,1\}}} \binom{i_1-\alpha_0}{1-i_1}\binom{i_2-\alpha_1-m}{1-i_2}\cdots\binom{i_n-\alpha_{n-2}-(n-2)m}{1-i_{n-1}}.(-\alpha_{n-1}-(n-1)m) \\ & + \sum_{\substack{i_1+i_2+\cdots+i_{n-1}=k-1 \\ i_j\in\{0,1\}}} \binom{i_1-\alpha_0}{1-i_1}\binom{i_2-\alpha_1-m}{1-i_2}\cdots\binom{i_n-\alpha_{n-2}-(n-2)m}{1-i_{n-1}}.1 \\ & = w_{m,\bar{\alpha}}(n-1,k-1) - (\alpha_{n-1}+(n-1)m)w_{m,\bar{\alpha}}(n-1,k). \end{aligned}$$

This is the recurrence relation for the Whitney numbers of the first kind (28). □

Remark 4.4. *By the definition of* $w_{m,\bar{\alpha}}(n,k)$ *in (27), we have the ordinary generating function of* $w_{m,\bar{\alpha}}(n,k)$*,*

$$\sum_{k=0}^{n} w_{m,\bar{\alpha}}(n,k)x^k = \prod_{j=0}^{n-1}\Big(x-(\alpha_j+jm)\Big), \tag{30}$$

Now we show that $\bar{\alpha}$-Whitney numbers of the first kind $w_{m,\bar{\alpha}}(n,k)$ are the elementary symmetric functions of the numbers $\alpha_0,\ \alpha_1+m,\ \alpha_2+2m,\ \cdots,\ \alpha_{n-1}+(n-1)m$ of order $n-k$, where the elementary symmetric function σ_k is defined by

$$\sigma_k(z_1,z_2,\cdots,z_n) = \sum_{1\le j_1<j_2<\cdots<j_k\le n}\ \prod_{i=1}^{k} z_{j_i},$$

where $\sigma_0 = 1$ and $\sigma_k = 0$ for $k < 0$ or $k > n$.

Corollary 4.5. *The* $\bar{\alpha}$*-Whitney numbers of the first kind satisfy*

$$w_{m,\bar{\alpha}}(n,k) = (-1)^{n-k}\,\sigma_{n-k}(\alpha_0,\ \alpha_1+m,\ \cdots,\ \alpha_{n-1}+(n-1)m) = (-1)^{n-k} \sum_{0\le j_1<\cdots<j_{n-k}\le n-1}\ \prod_{i=1}^{n-k}(\alpha_{j_i}+j_i\,m). \tag{31}$$

Proof. Setting t^{-1} in place of x in (30), then multiplying both sides by t^n, we obtain the identity

$$\sum_{k=0}^{n} w_{m,\bar{\alpha}}(n,k)t^{n-k} = \prod_{j=0}^{n-1}\Big(1-(\alpha_j+jm)t\Big), \tag{32}$$

Replacing t by $-t$, we obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\, w_{m,\bar{\alpha}}(n,k)t^{n-k} = \prod_{j=0}^{n-1}\Big(1+(\alpha_j+jm)t\Big), \tag{33}$$

now use the fact that the right hand side of (33) is equal to

$$\sum_{k}\sigma_{n-k}(\alpha_0,\ \alpha_1+m,\ \cdots,\ \alpha_{n-1}+(n-1)m)\ t^{n-k}.$$

□

5. Matrix Relations for $\bar{\alpha}$-Whitney Numbers

By substituting (5) into the right-hand side of (27) (or vice-versa) and changing the order of summation, one easily get the following orthogonality relation

$$\sum_{k=i}^{n} w_{m,\bar{\alpha}}(n,k)\, W_{m,\bar{\alpha}}(k,i) = \sum_{k=i}^{n} W_{m,\bar{\alpha}}(n,k)\, w_{m,\bar{\alpha}}(k,i) = \delta_{n,i}, \tag{34}$$

where $\delta_{n,i}$ denotes the Kronecker delta.
Let **w** and **W** be infinite lower triangular matrices whose entries are the numbers $w_{m,\bar{\alpha}}(n,k)$ and $W_{m,\bar{\alpha}}(n,k)$, respectively, where $n,\ k \geq 0$. Then equation (34) is equivalent to matrix equation $\mathbf{w}.\mathbf{W} = \mathbf{I}$, with **I** is the unit infinite matrix. Thus $\mathbf{w} = \mathbf{W}^{-1}$ is the inverse of **W** and vise versa.

Remark 5.1. *Some special cases can be introduced as follows:*
1- If $m = 1$ and $\alpha_j + j = \beta_j$, for $j = 0, 1, \cdots, n-1$, then $w_{1,\bar{\alpha}}(n,k) = s_{\bar{\beta}}(n,k)$ Comtet- numbers of the first kind.
2- If $\bar{\boldsymbol{\alpha}} = (1, 1, \cdots, 1) := \mathbf{1}$, hence $w_{m,\mathbf{1}}(n,k) = w_m(n,k)$, the Whitney numbers of the first kind.
3-If $\bar{\boldsymbol{\alpha}} = (r, r, \cdots, r) := \mathbf{r}$, hence $w_{m,\mathbf{r}}(n,k) = w_{m,r}(n,k)$, the r-Whitney numbers of the first kind.
4-If $m = 1$ and $\bar{\boldsymbol{\alpha}} = (0, 0, \cdots, 0) := \mathbf{0}$, hence $w_{1,\mathbf{0}}(n,k) = s(n,k)$, the signed Stirling numbers of the first kind.
5- If $m = 1$ and $\bar{\boldsymbol{\alpha}} = (r, r, \cdots, r) := \mathbf{r}$, then $w_{1,\mathbf{r}}(n,k) = s_r(n+r, k+r)$, the singed r-Stirling numbers of the first kind.
6- If $m = z$ and $\bar{\boldsymbol{\alpha}} = (0, 1^2, \cdots, (n-1)^2)$, then $w_{z,\bar{\alpha}}(n,k) = Js_n^k(z)$, the Jacobi-Stirling numbers of the first kind.
7- If $m = 1$ and $\bar{\boldsymbol{\alpha}} = (0, 1^2, \cdots, (n-1)^2)$, then $w_{1,\bar{\alpha}}(n,k) = Ps_n^k$, the Legendre-Stirling numbers of the first kind.

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