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Quasi-Isometricity and Equivalent Moduli of Continuity of Planar $1/|\omega|^2$-Harmonic Mappings

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Abstract. In this paper, we prove that $1/|\omega|^2$-harmonic quasiconformal mapping is bi-Lipschitz continuous with respect to quasihyperbolic metric on every proper domain of $\mathbb{C}\backslash\{0\}$. Hence, it is hyperbolic quasi-isometry in every simply connected domain on $\mathbb{C}\backslash\{0\}$, which generalized the result obtained in [14]. Meanwhile, the equivalent moduli of continuity for $1/|\omega|^2$-harmonic quasiregular mapping are discussed as a by-product.

1. Introduction

Let Ω and Ω' be two proper subdomains of the complex plane $\mathbb{C}$ and let $\rho(\omega)|d\omega|^2$ be a conformal metric on Ω'. Then a sense preserving C^2 mapping f of Ω onto Ω' is called ρ-harmonic if it satisfies the Euler-Lagrange equation

$$f_{z\bar{z}}(z) + (\log \rho)_{\omega} \circ f(z) f_z(z) f_{\bar{z}}(z) = 0 \tag{1.1}$$

in Ω. Especially, f is called a Euclidean harmonic mapping when ρ is a positive constant. Euclidean harmonic mapping is a kind of natural generalization of analytic function and plays an important role in the theory of functions. For a survey of Euclidean harmonic mappings, see[13, 15] for more details. When ρ is the hyperbolic metric of Ω', f is called a hyperbolic harmonic mapping.

It is well known that ρ-harmonic mappings can be characterized by their Hopf differentials. That is, $f(z)$ is a ρ-harmonic mapping of Ω onto Ω' if and only if its Hopf differential $\varphi(z)dz^2 := \rho(f) f_z \overline{f_{\bar{z}}}\, dz^2$ is a holomorphic quadratic differential on Ω [3].

A mapping $f : \Omega \to \Omega'$ is called a ρ-harmonic K-quasiregular mapping if it is ρ-harmonic and there is a constant $K > 1$ such that $|f_{\bar{z}}(z)| \le k|f_z(z)|$ for all $z \in \Omega$, where $k = \frac{K-1}{K+1}$. In addition, if $f(z)$ is a homeomorphism on Ω, then $f(z)$ is called a ρ-harmonic K-quasiconformal mapping.

The Lipschitz continuity with respect to varied metrics of harmonic quasiconformal mappings is an important research content in the theory of harmonic mappings. The Euclidean Lipschitz and bi-Lipschitz continuities of Euclidean harmonic quasiconformal mappings are first investigated by Martio in [34]. The

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hyperbolically bi-Lipschitz continuity of harmonic quasiconformal mappings is further studied by Wan [37]. It is proved in [37] that *every Euclidean (hyperbolic) harmonic quasiconformal diffeomorphism of the unit disk* $\mathbb{D}$ *onto itself is quasi-isometric with respect to the hyperbolic metric*.

Later, Knežvić and Mateljević [23] retrieve it by using Ahlfors-Schwarz lemma (see Lemma 2.4). Moreover, they proved that every Euclidean harmonic quasiconformal mapping of the upper half plane onto itself is also a quasi-isometric with respect to the hyperbolic metric in [23, 24].

In 2010, the above result of Wan is improved to convex domain and the sharpness hyperbolically Lipschitz coefficient is given in [8]. Meanwhile, the first author of [8] finds that $1/|\omega|^2$-harmonic quasiconformal mapping also has an analogy bi-Lipschitz continuity as following [14].

Theorem 1.1. *Let A be an angular domain with the origin of the complex plane $\mathbb{C}$ as its vertex. If f is a $1/|\omega|^2$-harmonic K-quasiconformal mapping of the unit disk $\mathbb{D}$ onto A, then f is hyperbolic K-quasi-isometry. Moreover, the hyperbolically Lipschitz coefficient K is sharp.*

From the above analysis, the bi-Lipschitz continuity is closely related to the image domain and the mapping. Recently, many articles [9, 20, 22, 36] studied this issue for different domains or functions. The hyperbolically Lipschitz and bi-Lipschitz properties of general ρ-harmonic mappings are also studied [12].

More recently, Mateljević [32] further generalizes Wan's result to any simply connected proper subdomain in $\mathbb{C}$ for Euclidean harmonic quasiconformal mappings by using the following conclusion. (See [31–33])

Theorem 1.2. *Suppose D and D' are proper domains in $\mathbb{R}^2$. If $f : D \to D'$ is a Euclidean harmonic K-quasiconformal mapping, then it is bi-Lipschitz with respect to quasihyperbolic metrics of D and D'.*

In this paper, we mainly investigate the hyperbolically Lipschitz continuity and equivalent modulus of continuity of $1/|\omega|^2$- harmonic quasiconformal (quasiregular) mapping. In the first part of this paper, we prove that Theorem 1.2 is also true for $1/|\omega|^2$-harmonic K-quasiconformal mapping. Thus, Theorem 1.1 can be generalized to any simply connected proper subdomain of $\mathbb{C}\backslash\{0\}$ (See Theorem 3.1 and Theorem 3.2), although the sharp hyperbolically Lipschitz coefficient is not obtained.

Equivalent modulus of continuity has been investigated by some scholars, such as Dyakonov, Pavlović, Abaob, Arsenović, Mateljević, Ponnusamy and others. Recently, Chen Shaolin and his colleagues obtain three equivalent conditions about Equivalent modulus of continuity of Euclidean harmonic mapping. As a by-product of the first part, We find that those equivalent conditions also hold for $1/|\omega|^2$-harmonic mapping. The detail research backgrounds and our results are given in Section 4.

2. Preliminary

In this section, we introduce and prove some lemmas, which are needed in the following discussion.

For the sake of comprehending deeply the family of $1/|\omega|^2$-harmonic mapping, a class of logharmonic mapping should be introduced at first[1]. A sense preserving mapping f on Ω is called logharmonic if $f(z) \neq 0$ in Ω and there is an analytic function $a(z)$ in Ω with $|a(z)| < 1$ such that $f(z)$ is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}(z)} = a(z)\frac{\overline{f(z)}}{f(z)} f_z(z). \tag{2.0}$$

Then we prove the following equivalent relation of logharmonic mappings and $1/|\omega|^2$-harmonic mappings, which plays an important role in the proofs of Theorem 3.1 and Theorem 3.2.

Proposition 2.1. *Let f be a sense preserving mapping defined on $\Omega \subset \mathbb{C}$ with $\omega = f(z) \neq 0$ for all $z \in \Omega$. Then f is a logharmonic mapping if and only if f is a $1/|\omega|^2$-harmonic mapping.*

Proof. The necessary part of this statement is proved in [14]. Now, we mainly want to prove the sufficiency.
Let f be a $1/|\omega|^2$-harmonic mapping on Ω with $\omega = f(z) \neq 0$ for $z \in \Omega$. Then by (1.1), we get

$$f f_{z\bar{z}} = f_z f_{\bar{z}}, \;\; z \in \Omega,$$

which directly implies

$$\left(\frac{f_z}{f}\right)_{\bar{z}} = 0 \;\text{ and }\; \left(\frac{f_{\bar{z}}}{f}\right)_z = 0, \;\; \forall z \in \Omega. \tag{2.1}$$

Let

$$h(z) = \frac{f_z(z)}{f(z)} \;\text{ and }\; g(z) = \frac{f_{\bar{z}}(z)}{f(z)}, \;\; z \in \Omega.$$

Then from (2.1), $h(z)$ and $\overline{g(z)}$ are analytic function in Ω. Moreover, $h(z) \neq 0$ in Ω, Since f is sense preserving and $f \neq 0$ in Ω. So there exists an analytic function

$$a(z) := \frac{\overline{g(z)}}{h(z)} = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} \frac{f(z)}{\overline{f(z)}}, \;\; z \in \Omega \tag{2.2}$$

which satisfies $|a(z)| < 1$ in Ω by the sense preserving property of f. Therefore, $f(z)$ satisfies (2.0) with analytic function $a(z)$ defined in (2.2) and $f(z)$ is logharmonic in Ω conseqently. The proof of Proposition 2.1 is completed. □

Next, in order to prove the quasi-isometry of $1/|\omega|^2$-harmonic quasiconformal mapping in Section 3, the following three conclusions are needed.

Lemma 2.2. (Astala-Gehring) [6] *Suppose that D and D' are domains in $\mathbb{R}^n$ $(n \geq 2)$, if $f : D \to D'$ is a K-quasiconformal mapping, then there exists a positive constant $c := c(K, n)$ such that*

$$\frac{1}{c}\frac{d(f(z), \partial D')}{d(z, \partial D)} \leq \alpha_{f,D}(z) \leq c\frac{d(f(z), \partial D')}{d(z, \partial D)}.$$

where

$$\alpha_{f,D}(x) = \exp\left\{\frac{1}{n|\mathrm{B}_x|}\int_{\mathrm{B}_x} \log J_f(z) dz\right\} \tag{2.3}$$

for all $x \in D$, here $\mathrm{B}_x := \mathrm{B}(x, d(x, \partial D))$ is a ball and $|\mathrm{B}_x|$ stands for the volume of the ball B_x.

Proposition 2.3. *Let f be a logharmonic mapping defined on the domain $\Omega \subset \mathbb{C}$. Then $\log\left|f_z(z)\right|$ is a real-valued Euclidean harmonic function on Ω.*

Proof. Since f is a logharmonic mapping, then f satisfies (2.0) in Ω for some analytic function $a(z)$ with $|a(z)| < 1$ in Ω. Meanwhile, f also is a $1/|\omega|^2$-harmonic mapping by Proposition 2.1. From the definition of $1/|\omega|^2$-harmonic mapping, we get

$$f_{z\bar{z}}(z) f(z) = f_z(z) f_{\bar{z}}(z), \;\; z \in \Omega. \tag{2.4}$$

Moreover, the Hopf differential

$$\varphi(z) := \frac{1}{|f(z)|^2} f_z(z) \overline{f_{\bar{z}}(z)} \tag{2.5}$$

of f is holomorphic in Ω.

Let $A = \{z \in \Omega : f_{\bar{z}}(z) = 0\}$. By (2.0) and (2.5), A is the set of zero points of the analytic function $a(z)$ and A is that of $\varphi(z)$ also. Consequently, A is countable and discrete. Furthermore, by (2.0) and (2.5) again,

$$\frac{\varphi(z)}{a(z)} = \frac{f_z^2(z)}{f^2(z)}, \quad z \in \Omega \setminus A. \tag{2.6}$$

Thus, when z tends to every $\zeta \in A$, the limit of $\dfrac{\varphi(z)}{a(z)}$ exists. So $\dfrac{\varphi(z)}{a(z)}$ is a non-vanishing analytic function in Ω and (2.6) holds for all $z \in \Omega$. Hence,

$$\log \frac{|f_z(z)|^2}{|f(z)|^2} = \log \left| \frac{\varphi(z)}{a(z)} \right|$$

is harmonic in Ω, and

$$\Delta \log \frac{1}{|f(z)|^2} + \Delta \log(|f_z(z)|^2) = 0, \quad z \in \Omega \tag{2.7}$$

consequently.

By simple computation, we have

$$\Delta \log \frac{1}{|f(z)|^2} = -4 \frac{\left(f_{z\bar{z}}\overline{f} + f\overline{f_{z\bar{z}}}\right)|f|^2 - \left(f_z f_{\bar{z}} \overline{f^2} + \overline{f_z f_{\bar{z}}} f^2\right)}{|f(z)|^4}, \quad z \in \Omega.$$

So from (2.4) we conclude that

$$\Delta \log \frac{1}{|f(z)|^2} = -4 \frac{\left(f_{z\bar{z}}\overline{f} + \overline{f_{z\bar{z}}} f\right)|f(z)|^2 - \left(f_{z\bar{z}}\overline{f} + \overline{f_{z\bar{z}}} f\right)|f(z)|^2}{|f(z)|^4} = 0, \quad z \in \Omega.$$

Thus by (2.7) we have

$$\Delta \log |f_z(z)| = 0, \quad z \in \Omega.$$

That is, $\log |f_z(z)|$ is Euclidean harmonic in Ω. The proof of Proposition 2.3 is finished. □

For simply connected domains, the following classic theorem reveal the link between the hyperbolic density and the quasihyperbolic density [7, 18].

Lemma 2.4. *Let $\Omega \subset \mathbb{C}$ be a simply connected hyperbolic domain and and let $\lambda_\Omega(z)|dz|$ be its hyperbolic metric. Then*

$$\frac{1}{4} \frac{1}{d(z, \partial\Omega)} \le \lambda_\Omega(z) \le \frac{1}{d(z, \partial\Omega)}, \quad z \in \Omega,$$

where

$$d(z, \partial\Omega) := \inf \{|z - \omega| : \omega \in \partial\Omega\}, \quad z \in \Omega.$$

The right equality holds if and only if Ω is a disk with center z, and the left equality holds if and only if Ω is to a slit plane. Moreover, if Ω is a convex domain, then

$$\frac{1}{2} \frac{1}{d(z, \partial\Omega)} \le \lambda_\Omega(z) \le 1, \quad \textit{for all } z \in \Omega.$$

The left equality holds if and only if Ω is a half plane.

Lemma 2.4 can be roughly state as

$$C_2\kappa_\Omega(z_1,z_2) \le d_h(z_1,z_2) \le C_1\kappa_\Omega(z_1,z_2), \quad \text{for all } z_1,z_2 \in \Omega, \tag{2.8}$$

where C_1 and C_2 are two universal constants and κ_Ω is the quasihyperbolic metric of Ω which is defined as

$$\kappa_\Omega(z_1,z_2) := \inf_\gamma \int_\gamma \frac{1}{d(z,\partial\Omega)}|dz|, \quad \text{for all } z_1,z_2 \in \Omega. \tag{2.9}$$

Here the infimum is taken over all rectifiable curves γ in Ω joining z_1 and z_2.

The Gauss curvature of a Riemannian metric $\rho(z)|dz|^2$ of a domain Ω is

$$K(\rho)(z) := -\frac{1}{2}\frac{\Delta\log\rho(z)}{\rho(z)}, \quad z \in \Omega.$$

The following is the famous Ahlfors-Schwarz lemma [25].

Lemma 2.5. *Let $\rho(z)$ is the density of a Riemannian metric of the unit disk $\mathbb{D}$ with Gaussian curvature $K(\rho)(z) \le -1$. Then $\rho(z) \le \lambda(z)$ for all $z \in \mathbb{D}$, where*

$$\lambda(z) = \frac{4}{(1-|z|^2)^2}$$

is the hyperbolic density of $\mathbb{D}$.

Applying Proposition 2.3 and Lemma 2.5 to $1/|\omega|^2$-harmonic K-quasiregular mappings, we get the following Schwarz lemma which will be used in the discussion of section 4.

Proposition 2.6. *Let $f : \mathbb{D} \to \mathbb{D}$ be a $1/|\omega|^2$-harmonic K-quasiregular mapping. Then*

$$\Lambda_f(z) := |f_z(z)| + |f_{\bar z}(z)| \le K\frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{2.10}$$

Proof. Let

$$\sigma(z) = (1-k)^2\,\lambda(f(z))|f_z(z)|^2, \quad z \in \mathbb{D}, \tag{2.11}$$

where $k = (K-1)/(K+1)$. By Proposition 2.1, $f(z)$ is a logharmonic mapping. So $f(z)$ is sense preserving and $|f_z(z)| \ne 0$ in $\mathbb{D}$, consequently. Thus, $\sigma(z) > 0$ for $z \in \mathbb{D}$.

From (2.11) and Proposition 2.3,

$$\Delta\log\sigma(z) = \Delta\log\lambda(f(z)).$$

By simple computation on $\Delta\log\lambda(f(z))$, we get

$$\Delta\log\sigma(z) = 8\frac{(f_{z\bar z}\overline{f} + f\overline{f_{z\bar z}})(1-|f|^2) + (|f_z|^2 + |f_{\bar z}|^2) + (f_z f_{\bar z}\overline{f^2} + \overline{f_z f_{\bar z}}f^2)}{(1-|f|^2)^2}. \tag{2.12}$$

Since f is K-quasiregular mapping, we have $|f_{\bar z}(z)| \le k|f_z(z)|$ and consequently,

$$|a(z)| \le k < 1, \quad z \in \mathbb{D}, \tag{2.13}$$

where $a(z)$ is the analytic function in (2.0) associated to the logharmonic mapping f. Thus, by (2.0), (2.4), (2.12) and (2.13), we have

$$\begin{aligned}
\Delta\log\sigma(z) &= 8\frac{|f_z(z)|^2 + |f_{\bar z}(z)|^2 + f_{z\bar z}(z)\overline{f(z)} + \overline{f_{z\bar z}(z)}f(z)}{(1-|f(z)|^2)^2} \\
&= 8\frac{|f_z(z)|^2}{(1-|f(z)|^2)^2}\left[1 + |a(z)|^2 + a(z) + \overline{a(z)}\right] \\
&\ge 8\frac{|f_z(z)|^2}{(1-|f(z)|^2)^2}(1-k)^2,
\end{aligned}$$

which implies that

$$K(\sigma)(z) = -\frac{1}{2}\frac{\Delta \log \sigma(z)}{\sigma(z)} \leq -1.$$

Therefore, by Lemma 2.5, $\sigma(z) \leq \lambda(z)$ for $z \in \mathbb{D}$, which implies

$$|f_z(z)| \leq \frac{1}{1-k}\frac{1-|f(z)|^2}{1-|z|^2}, \;\; z \in \mathbb{D}$$

and then (2.10) directly. The proof of this Proposition is completed. □

3. Quasi-Isometricity of $1/|\omega|^2$-Harmonic Mappings with Respect to Quasihyperbolic Metrics

In this section we prove that $1/|\omega|^2$-harmonic mappings are of quasi-isometricity with respect to quasi-hyperbolic metrics. As a corollary we obtained that they are of quasi-isometricity with respect to hyperbolic metrics also if their domains are simply connected hyperbolic domains and their image domains not containing 0, which generalizes Theorem 1.1.

Theorem 3.1. *Let Ω and Ω' be two proper domains of $\mathbb{C}$ with $0 \notin \Omega'$. Then every $1/|\omega|^2$-harmonic K-quasiconformal mapping $f : \Omega \to \Omega'$ of Ω onto Ω' is bi-Lipschitz continuous with respect to quasihyperbolic metrics on Ω and Ω'. That is to say, there exists a positive constant c such that*

$$\frac{1}{c}\,\kappa_\Omega(z_1, z_2) \leq \kappa_{\Omega'}(f(z_1), f(z_2)) \leq c\,\kappa_\Omega(z_1, z_2), \;\; \forall z_1, z_2 \in \Omega, \tag{3.1}$$

where κ_Ω and $\kappa_{\Omega'}$ are the quasihyperbolic metrics of Ω and Ω' respectively.

Proof. As f is a $1/|\omega|^2$-harmonic mapping in Ω, by Proposition 2.1, there exists an analytic function $a(z)$ with $|a(z)| < 1$ on Ω such that (2.0) holds in Ω. So the Jacobian $J_f(z)$ of $f(z)$ can be represented as

$$J_f(z) = \Lambda_f(z)\lambda_f(z) = (1 - a(z))f_z(z), \; z \in \Omega,$$

where

$$\Lambda_f(z) = |f_z(z)| + |f_{\bar{z}}(z)|, \quad \lambda_f(z) = |f_z(z)| - |f_{\bar{z}}(z)|.$$

Thus, the quantity $\alpha_{f,\Omega}(z)$ defined in Lemma 2.2 has the following form

$$\alpha_{f,\Omega}(z) = \exp\left\{\frac{1}{2|\mathrm{B}(z,r)|}\int_{\mathrm{B}(z,r)}\left[\log\left(1-|a(\xi)|^2\right) + \log|f_\xi(\xi)|^2\right]dxdy\right\}, \tag{3.2}$$

where $\xi = x + iy$.

Since $\log|f_z(z)|$ is a real-valued harmonic function on Ω from Proposition 2.3, Then by mean value theorem,

$$\log|f_z(z)| = \frac{1}{2|\mathrm{B}(z,r)|}\int_{\mathrm{B}(z,r)} \log|f_z(\xi)|^2 dxdy, \quad \xi = x + iy \in \mathrm{B}(z,r) \tag{3.3}$$

holds for every $z \in \Omega$ and every disk $\mathrm{B}(z,r) \subset \Omega$ centered at z with radius r. Therefore, by (3.2) and (3.3),

$$\alpha_{f,\Omega}(z) = |f_z(z)|\exp\left\{\frac{1}{2|\mathrm{B}(z,r)|}\int_{\mathrm{B}(z,r)} \log\left(1-|a(\xi)|^2\right)dxdy\right\}, \quad z \in \Omega. \tag{3.4}$$

As f is a K-quasiconformal mapping, we have

$$|a(z)| \leq k = \frac{K-1}{K+1} < 1, \quad z \in \Omega. \tag{3.5}$$

Then from (3.4) and (3.5),

$$\sqrt{1-k^2}|f_z(z)| \le \alpha_{f,\Omega}(z) \le |f_z(z)|, \quad z \in \Omega,$$

which implies

$$\frac{1}{\sqrt{K}}\Lambda_f(z) \le \alpha_{f,\Omega}(z) \le \frac{K+1}{2}\lambda_f(z), \quad z \in \Omega \tag{3.6}$$

directly.

Applying Lemma 2.2, (3.6) implies

$$C_2\Lambda_f(z) \le \frac{d(f(z),\ \partial\Omega')}{d(z,\ \partial\Omega)} \le C_1\lambda_f(z), \quad z \in \Omega, \tag{3.7}$$

where $C_1 := \frac{K+1}{2}c,\ C_2 := \frac{1}{\sqrt{K}}c$ and $c = c(K, 2)$ is the constant appeared in Lemma 2.2.

For any z_1 and $z_2 \in \Omega$. By a result of [17], there exists a quasihyperbolic geodesic γ_0 in Ω connecting z_1 and z_2. So

$$\kappa_\Omega(z_1, z_2) = \int_{\gamma_0} \frac{1}{d(z, \partial\Omega)}|dz| \ge \int_{\gamma_0} \frac{1}{d(z, \partial\Omega)} \frac{1}{\Lambda_f(z)} \left| f_z dz + f_{\bar{z}} d\bar{z} \right|. \tag{3.8}$$

Thus, from the left inequality of (3.7) and (3.8),

$$\kappa_\Omega(z_1, z_2) \ge \int_{f(\gamma_0)} \frac{C_2}{d(w, \partial\Omega')}|dw|,$$

and consequently

$$\kappa_\Omega(z_1, z_2) \ge C_2\kappa_{\Omega'}(f(z_1), f(z_2)), \quad z_1, z_2 \in \Omega, \tag{3.9}$$

since $f(\gamma_0)$ is curve in Ω' joining $f(z_1)$ and $f(z_2)$.

Similarly, there exists a geodesic γ_0' in Ω' joining $f(z_1)$ and $f(z_2)$, and thus

$$\kappa_{\Omega'}(f(z_1), f(z_2)) = \int_{\gamma_0'} \frac{1}{d(w, \partial\Omega')}|dw| \ge \int_{f^{-1}(\gamma_0')} \frac{1}{d(f(z),\ \partial\Omega')}\lambda_f(z)|dz|. \tag{3.10}$$

From the second inequality in (3.7) and (3.10),

$$\kappa_{\Omega'}(f(z_1), f(z_2)) \ge \int_{f^{-1}(\gamma_0')} \frac{1}{C_1 d(z,\ \partial\Omega)}|dz| \ge \frac{1}{C_1}\kappa_\Omega(z_1, z_2), \quad z_1, z_2 \in \Omega. \tag{3.11}$$

Therefore, (3.1) comes from (3.9) and (3.11), and the proof of Theorem 3.1 is completed. □

By Theorem 3.1 and Lemma 2.4 or the inequality (2.8), we obtain the following theorem.

Theorem 3.2. *Suppose that Ω and Ω' are two simply connected proper subdomains of $\mathbb{C}$. Let $f : \Omega \to \Omega'$ be a $1/|\omega|^2$-harmonic K-quasiconformal mapping with $\omega = f(z)$ and $0 \notin \Omega'$. Then there exists a positive constant C such that*

$$\frac{1}{C}\, d_h(z_1, z_2) \le d_h(f(z_1), f(z_2)) \le C\, d_h(z_1, z_2), \tag{3.11}$$

holds for all $z_1, z_2 \in \Omega$.

Remark Theorem 3.2 is a generalization of Theorem 1.1 to arbitrary simply connected hyperbolic domains, though the sharp estimate of Lipschitz constant is not obtained here.

4. Equivalent Moduli of Continuity for $1/|\omega|^2$-Harmonic Mapping

In order to illustrate the modulus of continuity, some terminologies and notations should be introduced.

A continuous increasing function $\sigma : [0, +\infty) \to [0, +\infty)$ with $\sigma(0) = 0$ is called a *majorant* if $\sigma(t)/t$ is decreasing for $t > 0$. A majorant σ is said to be *regular* if there are constants $\delta_0 > 0$ and $C > 0$ such that

$$\int_0^{\delta} \frac{\sigma(t)}{t} dt \le C\sigma(\delta), \;\; 0 < \delta < \delta_0, \tag{4.1}$$

and

$$\delta \int_{\delta}^{+\infty} \frac{\sigma(t)}{t^2} dt \le C\sigma(\delta), \;\; 0 < \delta < \delta_0. \tag{4.2}$$

Given a subset $\Omega \subset \mathbb{C}$ and a majorant σ, the *Lipschitz space* $\Lambda_\sigma(\Omega)$ is the set of all mappings $f : \Omega \to \mathbb{C}$ satisfying for some constant $C = C_f > 0$

$$|f(z) - f(\xi)| \le C\, \sigma(|z - \xi|) \tag{4.3}$$

whenever z and $\xi \in \Omega$. The *local Lipschitz space* $\Lambda_\sigma^{loc}(\Omega)$ is the set of all mappings $f : \Omega \to \mathbb{C}$ such that (4.3) holds for some constant $C = C_f > 0$ whenever z and ξ in any open disk contained in Ω. A domain Ω is called a Λ_σ-extension domain if $\Lambda_\sigma(\Omega) = \Lambda_\sigma^{loc}(\Omega)$.

Λ_σ-extension domain is an important concept in complex analysis. Gehring and Matrio first give a geometric characterization of Λ_σ-extension domains in [16] with the important special majorant $\sigma(t) = t^\alpha$ $(0 < \alpha \le 1)$. Lappalainen [26] extends their results to a general case and proves that

Ω is a Λ_σ-extension domain if and only if there is a constant $C = C(\Omega, \sigma) > 0$ such that

$$\int_\gamma \frac{\sigma\left(d\left(z, \partial\Omega\right)\right)}{d\left(z, \partial\Omega\right)} |dz| \le C\sigma\left(|z_1 - z_2|\right), \tag{4.4}$$

holds for all z_1 and $z_2 \in \Omega$ and all rectifiable curves γ in Ω joining them. Moreover, Λ_σ-extension domains exist only for majorants σ satisfying the inequality (4.1).

Based on the above characterization, the equivalent modulus of continuity $|f| \in \Lambda_\sigma(\Omega)$ is discussed by many authors. For example, [4, 10, 21, 22, 35] and the references therein. Dyakonov [21] first characterizes the equivalent modulus of the holomorphic functions. Then Pavlović [35] uses a relatively simple method to prove the results of Dyakonov. Also simple proofs are given using Bloch's theorem by Mateljevic, see [27–30, 32] and the Remark in this paper. Recently, Chen, Ponnusamy and Wang generalize this topic to other functions, such as planar harmonic mappings, plurharmonic mappings in $\mathbb{B}^n$(see [10, 11]). Two main theorems [10] of them are as follows.

Theorem 4.1. *Let σ be a majorant with (4.1), and let Ω be a Λ_σ-extension domain. If f is a planar K-quasiregular Euclidean harmonic mapping of Ω and continuous up to the boundary $\partial\Omega$, then*

$$f \in \Lambda_\sigma(\Omega) \Longleftrightarrow |f| \in \Lambda_\sigma(\Omega) \Longleftrightarrow |f| \in \Lambda_\sigma(\Omega,\ \partial\Omega),$$

where $\Lambda_\sigma(\Omega, \partial\Omega)$ is the set of $f \in C^2(\Omega)$ satisfying (4.3) with some positive constant C for $z \in \Omega$ and $\xi \in \partial\Omega$.

Theorem 4.2. *Let σ be a majorant with (4.1). If f is a Euclidean harmonic K- quasiregular mapping in $\Omega \subset \mathbb{C}$, then*

$$f \in \Lambda_{\sigma,\inf}(\Omega) \Longleftrightarrow |f| \in \Lambda_{\sigma,\inf}(\Omega),$$

where $\Lambda_{\sigma,\inf}(\Omega)$ is the set of $f \in C^2(\Omega)$ which satisfying for some constant $C > 0$

$$\left|f(z_1) - f(z_2)\right| \le C d_{\sigma,\Omega}(z_1, z_2), \tag{4.5}$$

for $z_1, z_2 \in \Omega$. *Here*

$$d_{\sigma,\Omega}(z_1, z_2) := \inf_{\gamma} \int_{\gamma} \frac{\sigma\left(d(z, \partial\Omega)\right)}{d\left(z, \partial\Omega\right)} |dz|$$

and the infimum is taken over all rectifiable curves γ *in* Ω *joining* z_1 *and* z_2.

In addition, there are many investigations about Lipschitz moduli of Euclidean harmonic quasiregular mapping in $\mathbb{R}^n$, see [2, 5] for more detail. At the end of [10], authors quoted the question which is posed by referee: *Does Theorem 4.1 still hold if the hypothesis mapping being harmonic is dropped*? In 2014, Miodrag Mateljević [32] partly answered this question when $\sigma(t) = t^{\alpha}$ and found that it also true when $\Omega \subset \mathbb{R}^n$ in Theorem 24 and Theorem 40. But the general case is still unsolved. In this section, we prove that Theorem 4.1 and Theorem 4.2 also set up for planar $1/|\omega|^2$-harmonic K-quasiregular mappings.

Theorem 4.3. *Let* σ *be a majorant with (4.1), and let* Ω *be a* Λ_σ*-extension domain. If* f *is a* $1/|\omega|^2$*-harmonic* K*-quasiregular mapping of* Ω *and continuous up to the boundary* $\partial\Omega$, *then*

$$f \in \Lambda_\sigma(\Omega) \Longleftrightarrow |f| \in \Lambda_\sigma(\Omega) \Longleftrightarrow |f| \in \Lambda_\sigma(\Omega,\ \partial\Omega).$$

Proof. It is trivial that $f \in \Lambda_\sigma(\Omega) \Rightarrow |f| \in \Lambda_\sigma(\Omega) \Rightarrow |f| \in \Lambda_\sigma(\Omega, \partial\Omega)$. Thus we only need to prove $|f| \in \Lambda_\sigma(\Omega, \partial\Omega) \Rightarrow f \in \Lambda_\sigma(\Omega)$.

For a fixed point $z \in \Omega$, let

$$F(t) = \frac{1}{M_z} f(z + td(z, \partial\Omega)), \ \ t \in \mathbb{D},$$

where

$$M_z = \sup\left\{|f(\xi)| : \ |\xi - z| < d(z, \partial\Omega)\right\}.$$

Then $|F(t)| < 1$ for $t \in \mathbb{D}$.

Since f is a $1/|\omega|^2$-harmonic K-quasiregular mapping with $\omega = f(z)$, then

$$\left|\frac{F_{\bar{t}}(t)}{F_t(t)}\right| = \left|\frac{f_{\bar{\xi}}(\xi)}{f_{\xi}(\xi)}\right| \leq k := \frac{K-1}{K+1}, \ \ \xi = z + td(z, \partial\Omega)$$

and there exists an analytic function $a(z)$ satisfies $|a(z)| < 1$ in Ω such that

$$\overline{F_{\bar{\xi}}(\xi)} = a(\xi) \frac{\overline{F(\xi)}}{F(\xi)} F_{\xi}(\xi),$$

by Proposition 2.1, that is, F is a $1/|\omega|^2$-harmonic K-quasiregular mapping of $\mathbb{D}$ into itself. Therefore, by Proposition 2.6,

$$|F_{\xi}(\xi)| + |F_{\bar{\xi}}(\xi)| \leq K \frac{1 - |F(\xi)|^2}{1 - |\xi|^2}, \ \ \xi \in \mathbb{D}.$$

Especially,

$$|F_{\xi}(0)| + |F_{\bar{\xi}}(0)| \leq K\left(1 - |F(0)|^2\right) \leq 2K(1 - |F(0)|),$$

that is,

$$d(z, \partial\Omega)\left(|f_z(z)| + |f_{\bar{z}}(z)|\right) \leq 2K(M_z - |f(z)|). \tag{4.6}$$

In what follows, we always use C to denote positive constants, although the C's may not be the same in different places.

Let such that $\tau \in \partial\Omega$ with $|z-\tau| = d(z,\partial\Omega)$. As $|f| \in \Lambda_\sigma(G,\partial G)$, thus for any $\xi \in D(z, d(z,\partial\Omega)) := \{\xi : |\xi - z| < d(z,\partial\Omega)\}$, we get that

$$\begin{aligned}||f(\xi)| - |f(z)|| \leq & ||f(\xi)| - |f(\tau)|| + ||f(\tau)| - |f(z)|| \\ \leq & C\sigma(|\tau - \xi|) + C\sigma(|\tau - z|),\end{aligned}$$

Since $|\tau - \xi| \leq 2d(z,\partial\Omega)$, so by the definition of majorant,

$$||f(\xi)| - |f(z)|| \leq C\sigma(d(z,\partial\Omega)),$$

which implies

$$M_z - |f(z)| \leq C\sigma\left(d(z,\partial\Omega)\right), \ \ z \in \Omega. \tag{4.7}$$

Hence, by (4.6) and (4.7), we get

$$|f_z(z)| + |f_{\bar{z}}(z)| \leq C\frac{\sigma(d(z,\partial\Omega))}{d(z,\partial\Omega)} \ \ z \in \Omega. \tag{4.8}$$

As Ω is a Λ_σ-extension domain, then by (4.4) and (4.8),

$$\begin{aligned}|f(z_1) - f(z_2)| \leq & \int_\gamma \left(|f_z(z)| + |f_{\bar{z}}(z)|\right)|dz| \\ \leq & C\int_\gamma \frac{\sigma(d(z,\ \partial\Omega))}{d(z,\ \partial\Omega)}|dz| \\ \leq & C\sigma\left(|z_1 - z_2|\right)\end{aligned}$$

holds for all z_1 and $z_2 \in \Omega$. Here γ is any rectifiable curve in Ω joining z_1 and z_2.

Therefore we have proved that

$$|f| \in \Lambda_\sigma(G,\partial G) \Rightarrow f \in \Lambda_\sigma(G)$$

and the proof of Theorem 4.3 is finished. □

By the method used in the proof of Theorem 4.3 and the definition of $\Lambda_{\sigma,inf}(\Omega)$, the following outcome is obtained also. We omit the proof of this theorem here.

Theorem 4.4. *Let σ be a majorant satisfying (4.1). If f is a $1/|\omega|^2$-harmonic K-quasiregular mapping in $\Omega \subset \mathbb{C}$ with $\omega = f(z)$, then*

$$f \in \Lambda_{\sigma,\mathrm{inf}}(\Omega) \Longleftrightarrow |f| \in \Lambda_{\sigma,\mathrm{inf}}(\Omega).$$

Remark The methods used in the proof of Theorem 4.1 and Theorem 4.3 are closely dependent on the harmonicity of functions and the Schwarz lemma plays an important role which also illustrates that harmonicity can not be dropped there. Comparatively speaking, Miodrag Mateljević's method is relatively simple [32]. Furthermore, applying his method, Theorem 24 is very likely be extend to all majorant since the time of publication of the first works. Besides, the subject is closely related to other topics such as versions of Bloch's and Koebe's theorems, see [27–30] for more details.

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