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# An Extension of Pochhammer's Symbol and its Application to Hypergeometric Functions

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**Abstract.** By using a special property of the gamma function, we first define a productive form of gamma and beta functions and study some of their general properties in order to define a new extension of the Pochhammer symbol. We then apply this extended symbol for generalized hypergeometric series and study the convergence problem with some illustrative examples in this sense. Finally, we introduce two new extensions of Gauss and confluent hypergeometric series and obtain some of their general properties.

## 1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ respectively denote the sets of real and complex numbers and $z$ be an arbitrary complex variable. The well known (Euler's) gamma function is defined, for $\mathrm{Re}(z) > 0$, as

$$\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x,$$

and for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$, as

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{k=0}^{n-1} (z+k)} \quad (n \in \mathbb{N}).$$

The limit definition of the gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{\prod_{k=0}^{n} (z+k)}, \tag{1}$$

is valid for all complex numbers except the non-positive integers. An alternative definition is the productive form of the gamma function, i.e.,

$$\Gamma(z) = \frac{1}{z} \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^z \left(1 + \frac{z}{k}\right)^{-1}. \tag{2}$$

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When $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$ , the (Euler's) beta function [4] has a close relationship with the classical gamma function as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x). \tag{3}$$

The generalized binomial coefficient may be defined (for real or complex parameters $a$ and $b$ ) by

$$\binom{a}{n} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \binom{a}{a-b} \qquad (a,b \in \mathbb{C}),$$

which is reduced to the following special case when $b = n$ ($n \in \mathbb{N} \cup \{0\}$) :

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!} = \frac{(-1)^n(-a)_n}{n!},$$

where $(a)_b$ $(a,b \in \mathbb{C})$ denotes the Pochhammer symbol [19] given, in general, by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & (b = 0,\ a \in \mathbb{C}\backslash\{0\}), \\ a(a+1)\cdots(a+n-1) & (b \in \mathbb{N},\ a \in \mathbb{C}). \end{cases}$$

A remarkable property of the gamma function, which is provable via the limit definition (1), is

$$\overline{\Gamma(z)} = \Gamma(\bar{z}) \overset{(z=p+\mathrm{i}q)}{\Rightarrow} \Gamma(p+\mathrm{i}q)\,\Gamma(p-\mathrm{i}q) \in \mathbb{R}\,. \tag{4}$$

In this paper, we exploit the property (4) to introduce an extension of the Pochhammer symbol in order to apply it in the hypergeometric series of any arbitrary order. Then, we study the convergence problem of the involved hypergeometric series with some illustrative examples. Finally, we introduce two new extensions of Gauss and confluent hypergeometric series and obtain some of their general properties. For this purpose, we first define a productive form of the gamma function, by referring to the property (4), as follows

$$\Pi(p,q) = \frac{\Gamma(p+\mathrm{i}q)\,\Gamma(p-\mathrm{i}q)}{\Gamma(p)} \qquad (p > 0,\ q \in \mathbb{R}). \tag{5}$$

For analogous extensions of the gamma function see e.g. [2, 14]. The limit definition of (5) can be derived from (1), so that we have

$$\Pi(p,q) = \frac{1}{\Gamma(p)} \lim_{n\to\infty} \frac{(n!)^2\, n^{2p}}{\prod\limits_{k=0}^{n} ((p+k)^2 + q^2)} = \lim_{n\to\infty} n!\, n^p \prod_{k=0}^{n} \frac{p+k}{(p+k)^2+q^2}. \tag{6}$$

Also, the limit relation (1) implies that relation (6) is written as

$$\Pi(p,q) = \Gamma(p) \prod_{k=0}^{\infty} \frac{(p+k)^2}{(p+k)^2+q^2}\,. \tag{7}$$

The result (7) shows that for any $p > 0$ and $q \in \mathbb{R}$ we respectively have

$$0 \le \Pi(p,q) \le \Gamma(p),$$

and

$$\lim_{q\to\infty} \Pi(p,q) = \lim_{q\to\infty} \frac{\Gamma(p+\mathrm{i}q)\,\Gamma(p-\mathrm{i}q)}{\Gamma(p)} = 0.$$

In order to obtain an integral representation for $\Pi(p,q)$, we should first study the real function $\Gamma(p+iq)\Gamma(p-iq)$. Hence, we consider the second kind of (Cauchy's) beta function [4], which says that if $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$ and $\mathrm{Re}(c+d) > 1$ then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(a+\mathrm{i}t)^c (b-\mathrm{i}t)^d} = \frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)}\,(a+b)^{1-(c+d)}. \tag{8}$$

One of the consequences of (8) is the definite integral

$$\int_{-\pi/2}^{\pi/2} \mathrm{e}^{st}(\cos t)^r\,\mathrm{d}t = \frac{2^{-r}\Gamma(r+1)\,\pi}{\Gamma\left(1+\frac{r+\mathrm{i}s}{2}\right)\Gamma\left(1+\frac{r-\mathrm{i}s}{2}\right)}, \tag{9}$$

which can be derived from the well-known identity

$$(a-\mathrm{i}t)^{p+\mathrm{i}q}(a+\mathrm{i}t)^{p-\mathrm{i}q} = (a^2+t^2)^p \exp\left(2q\arctan\frac{t}{a}\right).$$

The simplified version of (9) is as

$$\Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q) = \frac{\pi\, 2^{2-2p}\Gamma(2p-1)}{\int_{-\pi/2}^{\pi/2} \mathrm{e}^{2qt}(\cos t)^{2p-2}\,\mathrm{d}t}. \tag{10}$$

On the other hand, since

$$\frac{\Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q)}{\Gamma(2p)} = \mathrm{B}(p+\mathrm{i}q, p-\mathrm{i}q) = \int_0^1 (x-x^2)^{p-1}\left(\frac{x}{1-x}\right)^{\mathrm{i}q}\mathrm{d}x \tag{11}$$

$$= \int_0^1 (x-x^2)^{p-1}\cos\left(q\log\frac{x}{1-x}\right)\mathrm{d}x + \mathrm{i}\int_0^1 (x-x^2)^{p-1}\sin\left(q\log\frac{x}{1-x}\right)\mathrm{d}x, \tag{12}$$

is a real value, for any $p > 0$ and $q \in \mathbb{R}$ we can conclude that

$$\int_0^1 (x-x^2)^{p-1}\sin\left(q\log\frac{x}{1-x}\right)\mathrm{d}x = 0.$$

Therefore, by noting relations (10) and (11), two integral representations of $\Pi(p,q)$ are as

$$\Pi(p,q) = \frac{\Gamma(2p)}{\Gamma(p)}\int_0^1 (x-x^2)^{p-1}\cos\left(q\log\frac{x}{1-x}\right)\mathrm{d}x = \frac{\pi\, 2^{2-2p}\Gamma(2p-1)}{\Gamma(p)\int_{-\pi/2}^{\pi/2} \mathrm{e}^{2qt}(\cos t)^{2p-2}\,\mathrm{d}t}. \tag{13}$$

Note that the definite integral in the second equality of (13) can be computed in terms of a series. In fact, since

$$(\cos t)^a = 2^{-a}(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})^a = 2^{-a}\sum_{k=0}^{\infty}\binom{a}{k}\mathrm{e}^{(a-2k)\mathrm{i}t} = 2^{-a}\sum_{k=0}^{\infty}\binom{a}{k}\cos(a-2k)t,$$

and

$$\int \mathrm{e}^{pt}\cos qt\,\mathrm{d}t = \mathrm{e}^{pt}\frac{p\cos qt + q\sin qt}{p^2+q^2} + c,$$

so we have

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{2qt}(\cos t)^{2p-2}\,\mathrm{d}t &= 2^{-2p+2}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{2qt}\sum_{k=0}^{\infty}\binom{2p-2}{k}\cos(2p-2-2k)t\,\mathrm{d}t\\ &= 2^{-2p+2}\sum_{k=0}^{\infty}\binom{2p-2}{k}\int_{-\pi/2}^{\pi/2}\mathrm{e}^{2qt}\cos(2p-2-2k)t\,\mathrm{d}t\\ &= 2^{-2p+2}\sum_{k=0}^{\infty}(-1)^k\binom{2p-2}{k}\frac{q\sinh(q\pi)\cos\left((p-1)\pi\right)+(p-1-k)\cosh(q\pi)\sin\left((p-1)\pi\right)}{q^2+(p-1-k)^2}\,.\end{aligned}$$

**Remark 1.1.** By noting the well-known identity $\Gamma(z+1)=z\,\Gamma(z)$, since

$$\Gamma(p+1+\mathrm{i}q)=(p+\mathrm{i}q)\,\Gamma(p+\mathrm{i}q)\quad\text{and}\quad\Gamma(p+1-\mathrm{i}q)=(p-\mathrm{i}q)\,\Gamma(p-\mathrm{i}q),$$

so

$$\Pi(p+1,q)=\frac{p^2+q^2}{p}\,\Pi(p,q). \tag{14}$$

Similarly, the approach (14) can be followed for e.g. the Legendre duplication formula [5, 15]

$$(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz)=\Gamma(z)\,\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right),$$

(when $m=2$) so that we can eventually obtain

$$\Pi(2p,2q)=\frac{2^{2p-1}}{\sqrt{\pi}}\,\Pi(p,q)\,\Pi\left(p+\frac{1}{2},q\right).$$

**Remark 1.2.** When $m\in\mathbb{N}$, to compute $\Pi(m+1,q)$ we can again use the recurrence relation $\Gamma(z+1)=z\,\Gamma(z)$ to finally obtain

$$\begin{aligned}\Pi(m+1,q) &= \frac{(m+\mathrm{i}q)(m-1+\mathrm{i}q)\cdots(1+\mathrm{i}q)\,\Gamma(1+\mathrm{i}q)\,(m-\mathrm{i}q)(m-1-\mathrm{i}q)\cdots(1-\mathrm{i}q)\,\Gamma(1-\mathrm{i}q)}{m!}\\ &= \frac{q\pi}{m!\,\sinh(q\pi)}\prod_{k=0}^{m-1}\left((m-k)^2+q^2\right).\end{aligned}$$

One of the other applications of (4) is to define the productive form of the beta function as follows

$$\mathrm{B}(r,s;q)=\frac{\mathrm{B}(r+\mathrm{i}q,s-\mathrm{i}q)\,\mathrm{B}(r-\mathrm{i}q,s+\mathrm{i}q)}{\mathrm{B}(r,s)}=\frac{\Pi(r,q)\,\Pi(s,q)}{\Gamma(r+s)}. \tag{15}$$

For analogous extensions of the family of beta functions see e.g. [6, 13]. By referring to relation (7), the productive form of (15) can be obtained as

$$\mathrm{B}(r,s;q)=\mathrm{B}(r,s)\,\prod_{k=0}^{\infty}\frac{(r+k)^2(s+k)^2}{((r+k)^2+q^2)((s+k)^2+q^2)}\,. \tag{16}$$

Clearly the latter relation (16) shows that if $r,s>0$ and $q\in\mathbb{R}$ then

$$\left|\mathrm{B}(r,s;q)\right|\le\mathrm{B}(r,s).$$

## 2. An Extension of Pochhammer's Symbol and its Application to Hypergeometric Functions

The generalized hypergeometric series appear in a wide variety of applied mathematics and engineering sciences [1, 3, 12, 18]. For instance, there is a large set of hypergeometic-type polynomials whose variable is located in one or more of the parameters of the corresponding hypergeometric functions [8–10]. These polynomials are of great importance in mathematics as well as in many areas of physics. A few examples of their applications are discussed by Nikiforov, Suslov and Uvarov [16]. See also [5, 15]. Hence, it seems that any change in hypergeometric series, especially in Gauss and confluent hypergeometric functions, can be notable in various branches of mathematics. In recent years, some new extensions are given in this direction, e.g. [7, 17]. A main reason for introducing and developing the generalized hypergeometric series is that many special functions [4, 9, 11] can be represented in terms of them and therefore their initial properties can be directly found via the initial properties of hypergeometric functions. Also, they appear as solutions of many important ordinary differential equations [9, 11, 15]. The generalized hypergeometric function

$$
{}_pF_q\left(\begin{array}{c} a_1,\ a_2,\ \ldots,\ a_p \\ b_1,\ b_2,\ \ldots,\ b_q \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{17}
$$

in which $(r)_k = \prod_{j=0}^{k-1}(r+j)$ denotes the same as Pochhammer symbol and $z$ may be a complex variable is indeed a Taylor series expansion for a function, say $f$, as $\sum_{k=0}^{\infty} c_k^* z^k$ with $c_k^* = f^{(k)}(0)/k!$ for which the ratio of successive terms can be written as

$$
\frac{c_{k+1}^*}{c_k^*} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}.
$$

According to the ratio test [4], the series (17) is convergent for any $p \le q+1$. In fact, it converges in $|z|<1$ for $p = q+1$, converges everywhere for $p < q+1$ and converges nowhere ($z \neq 0$) for $p > q+1$. Moreover, for $p = q+1$ it absolutely converges for $|z| = 1$ if the condition

$$
A^* = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) > 0, \tag{18}
$$

holds and is conditionally convergent for $|z| = 1$ and $z \neq 1$ if $-1 < A^* \le 0$ and is finally divergent for $|z| = 1$ and $z \neq 1$ if $A^* \le -1$.

There are two important cases of the series (16) arising in many physical problems [3, 8, 12, 15]. The first case is the Gauss hypergeometric function convergent in $|z| \le 1$ that is denoted by

$$
y = {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},
$$

and satisfies the differential equation

$$
z\,(z-1)\,y'' + ((a+b+1)z - c)\,y' + ab\,y = 0. \tag{19}
$$

Particular choices of the parameters in the linearly independent solutions of the differential equation (19) yield 24 special cases. The ${}_2F_1$ can be given an integral representation as

$$
{}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t \quad \Big(\mathrm{Re}\,c > \mathrm{Re}\,b > 0;\ |\arg(1-z)| < \pi\Big). \tag{20}
$$

By using a series expansion of $(1-tz)^{-a}$ in (20), one can also write the ${}_2F_1$ in terms of the beta function as

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}. \tag{21} $$

The second case, which converges everywhere, is the confluent hypergeometric function

$$ y = {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!}, $$

as a basis solution of the differential equation

$$ z\,y'' + (c-z)\,y' - b\,y = 0, $$

which is a degenerate form of equation (19) where two of the three regular singularities merge into an irregular singularity. The ${}_1F_1$ has an integral form as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \mathrm{e}^{zt}\,\mathrm{d}t \qquad \Big(\operatorname{Re} c > \operatorname{Re} b > 0;\ |\arg(1-z)| < \pi\Big), $$

and can be represented in terms of the beta function as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{\mathrm{B}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}. \tag{22} $$

Now, we can introduce an extension of the Pochhammer symbol in order to apply it in the generalized hypergeometric series of any arbitrary order. Let us reconsider the gamma form of the Pochhammer symbol

$$ (s)_k = \frac{\Gamma(s+k)}{\Gamma(s)}. \tag{23} $$

By noting (5), a real extension of (23) may be defined as

$$ [s;q]_k = \frac{\Pi(s+k,q)}{\Pi(s,q)} = \frac{(s+\mathrm{i}q)_k(s-\mathrm{i}q)_k}{(s)_k} = \prod_{j=0}^{k-1} \frac{(s+j)^2+q^2}{s+j}. $$

Subsequently, a real extension of the hypergeometric functions may be defined as

$$ {}_pF_q\left(\begin{array}{c} [a_1;\alpha_1 r],\ [a_2;\alpha_2 r],\ \ldots,\ [a_p;\alpha_p r] \\ [b_1;\beta_1 r],\ [b_2;\beta_2 r],\ \ldots,\ [b_q;\beta_q r] \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{[a_1;\alpha_1 r]_k[a_2;\alpha_2 r]_k \cdots [a_p;\alpha_p r]_k}{[b_1;\beta_1 r]_k[b_2;\beta_2 r]_k \cdots [b_q;\beta_q r]_k} \frac{z^k}{k!}, \tag{24} $$

where $\{a_k, \alpha_k\}_{k=1}^p, \{b_k, \beta_k\}_{k=1}^q \in \mathbb{R}$ and $r \in \mathbb{R}$. On the other hand, the definition

$$ [s;q]_k = \frac{(s+\mathrm{i}q)_k(s-\mathrm{i}q)_k}{(s)_k}, $$

implies that the fraction term of (24) is expanded as

$$ \frac{[a_1;\alpha_1 r]_k[a_2;\alpha_2 r]_k \cdots [a_p;\alpha_p r]_k}{[b_1;\beta_1 r]_k[b_2;\beta_2 r]_k \cdots [b_q;\beta_q r]_k} = \prod_{j=1}^{p} \frac{(a_j+\mathrm{i}\alpha_j r)_k(a_j-\mathrm{i}\alpha_j r)_k}{(a_j)_k} \prod_{j=1}^{q} \frac{(b_j)_k}{(b_j+\mathrm{i}\beta_j r)_k(b_j-\mathrm{i}\beta_j r)_k}. $$

This means that the real series (24) can be transformed to a standard hypergeometric function as follows

$$
{}_pF_q\left(\begin{array}{c} [a_1;\alpha_1 r],\,[a_2;\alpha_2 r],\,\ldots\,,\,[a_p;\alpha_p r] \\ [b_1;\beta_1 r],\,[b_2;\beta_2 r],\,\ldots\,,\,[b_q;\beta_q r] \end{array}\middle|\, z\right)
$$

$$
= {}_{2p+q}F_{2q+p}\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ \ldots\,,\ a_p+\mathrm{i}\alpha_p r,\ a_p+\mathrm{i}\alpha_p r,\ b_1,\,b_2,\,\ldots,\,b_q \\ b_1+\mathrm{i}\beta_1 r,\ b_1-\mathrm{i}\beta_1 r,\ \ldots\,,\ b_q+\mathrm{i}\beta_q r,\ b_q-\mathrm{i}\beta_q r,\ a_1,\,a_2,\,\ldots,\,a_p \end{array}\middle|\, z\right). \tag{25}
$$

Hence, the convergence radius of (24) would directly depend on the convergence radius of ${}_{2p+q}F_{2q+p}$ in (25) as the following illustrative examples show.

**Example 2.1.** Let $(p, q) = (2, 1)$. In this case, (24) is reduced to

$$
{}_2F_1\left(\begin{array}{c} [a_1;\alpha_1 r],\ [a_2;\alpha_2 r] \\ [b_1;\beta_1 r] \end{array}\middle|\, z\right) = {}_5F_4\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ a_2+\mathrm{i}\alpha_2 r,\ a_2-\mathrm{i}\alpha_2 r,\ b_1 \\ b_1+\mathrm{i}\beta_1 r,\quad b_1-\mathrm{i}\beta_1 r,\ \ a_1,\ \ a_2 \end{array}\middle|\, z\right), \tag{26}
$$

whose convergence radius is $|z| < 1$. Moreover, according to (18), if $a_1 + a_2 < b_1$ in (26), then the convergence radius is extended to $|z| \le 1$.
As a particular case of (26), taking $r = 0$ gives the same as classical ${}_2F_1$ and $(\alpha_1, \alpha_2) = (0, 0)$ with $\beta_1 r = q$ yields

$$
{}_2F_1\left(\begin{array}{c} [a_1;0],\ [a_2;0] \\ [b_1;q] \end{array}\middle|\, z\right) = {}_3F_2\left(\begin{array}{c} a_1,\quad a_2,\quad b_1 \\ b_1+\mathrm{i}q,\ \ b_1-\mathrm{i}q \end{array}\middle|\, z\right),
$$

which is convergent in $|z| \le 1$ if $a_1 + a_2 < b_1$. Finally, if $\beta_1 = 0$, (26) is reduced to

$$
{}_2F_1\left(\begin{array}{c} [a_1;\alpha_1 r],\ [a_2;\alpha_2 r] \\ [b_1;0] \end{array}\middle|\, z\right) = {}_4F_3\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ a_2+\mathrm{i}\alpha_2 r,\ a_2-\mathrm{i}\alpha_2 r \\ b_1,\ \ a_1,\ \ a_2 \end{array}\middle|\, z\right),
$$

convergent in $|z| \le 1$ when $a_1 + a_2 < b_1$.

**Example 2.2.** Let $(p, q) = (1, 1)$. Then (24) changes to

$$
{}_1F_1\left(\begin{array}{c} [a_1;\alpha_1 r] \\ [b_1;\beta_1 r] \end{array}\middle|\, z\right) = {}_3F_3\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ b_1 \\ b_1+\mathrm{i}\beta_1 r,\ b_1-\mathrm{i}\beta_1 r,\ a_1 \end{array}\middle|\, z\right), \tag{27}
$$

which is convergent everywhere. For instance, if $\alpha_1 = 0$ and $\beta_1 r = q$ in (27), then the following real series converges everywhere

$$
{}_1F_1\left(\begin{array}{c} [a_1;0] \\ [b_1;q] \end{array}\middle|\, z\right) = {}_2F_2\left(\begin{array}{c} a_1,\ \ b_1 \\ b_1+\mathrm{i}q,\ b_1-\mathrm{i}q \end{array}\middle|\, z\right).
$$

**Example 2.3.** An interesting case of (24) is when $(p, q) = (1, 0)$, because the real series

$$
y = {}_1F_0\left(\begin{array}{c} [a;q] \\ - \end{array}\middle|\, z\right) = {}_2F_1\left(\begin{array}{c} a+\mathrm{i}q,\ a-\mathrm{i}q \\ a \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty}\left(\prod_{j=0}^{k-1}(a+j)^2+q^2\right)\frac{z^k}{(a)_k k!}, \tag{28}
$$

satisfies the second order differential equation

$$
z\,(z-1)\,y'' + ((2a+1)z-a)\,y' + (a^2+q^2)\,y = 0.
$$

Note that the more general case of (28) is indeed the real series

$$
y = {}_2F_1\left(\begin{array}{c} a+\mathrm{i}q,\ a-\mathrm{i}q \\ b \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty}\left(\prod_{j=0}^{k-1}(a+j)^2+q^2\right)\frac{z^k}{(b)_k k!},
$$

which satisfies the differential equation

$$
z\,(z-1)\,y'' + ((2a+1)z-b)\,y' + (a^2+q^2)\,y = 0.
$$

### 2.1. A new extension of Gauss and confluent hypergeometric functions

Since many special functions of mathematical physics can be represented in terms of ${}_2F_1$ or ${}_1F_1$ by special choices of the parameters, they play a unifying role in the theory of special functions. Hence, any significant generalization of them may be useful. In this section, we apply the generalized beta function (15) for two relations (21) and (22) to respectively extend the functions ${}_2F_1$ and ${}_1F_1$. First, by noting two relations (15) and (21), the proposed extension of ${}_2F_1$ may be considered as

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z;q\right) = \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}\,(b+k, c-b; q)}{\mathrm{B}\,(b, c-b)} \frac{z^k}{k!}, \tag{29} $$

which reduces to the same as ${}_2F_1$ for $q = 0$. Since $\left|\mathrm{B}(r, s; q)\right| \le \mathrm{B}(r, s)$ for any $r, s > 0$ and $q \in \mathbb{R}$, the extended series (29) converges in $|z| \le 1$ if $c > b > 0$ and $c$ is not a negative integer or zero. Now, the integral representation of (29) can be derived by (13) and (15) as follows

$$ \begin{aligned} {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z;q\right) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{(a)_k\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Pi(c-b, q)}{\Gamma(c)\,(c)_k} \frac{\Gamma(2b+2k)}{\Gamma(b+k)} \int_0^1 (x-x^2)^{b+k-1} \cos\left(q \log \frac{x}{1-x}\right) \mathrm{d}x \\ &= \frac{\Gamma(c-b+\mathrm{i}q)\Gamma(c-b-\mathrm{i}q)\Gamma(2b)}{\Gamma^2(b)\Gamma^2(c-b)} \int_0^1 (x-x^2)^{b-1} \cos\left(q \log \frac{x}{1-x}\right) \left( \sum_{k=0}^{\infty} \frac{(a)_k (b+1/2)_k}{(c)_k} \frac{(4zx(1-x))^k}{k!} \right) \mathrm{d}x \\ &= \frac{\Gamma(c-b+\mathrm{i}q)\Gamma(c-b-\mathrm{i}q)\Gamma(2b)}{\Gamma^2(b)\Gamma^2(c-b)} \int_0^1 (x-x^2)^{b-1} \cos\left(q \log \frac{x}{1-x}\right) {}_2F_1\left(\begin{array}{c} a,\ b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. \end{aligned} \tag{30} $$

Note that $q = 0$ in (30) gives a new representation for ${}_2F_1$ so that we have

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z\right) = \frac{\Gamma(2b)}{\Gamma^2(b)} \int_0^1 (x-x^2)^{b-1} {}_2F_1\left(\begin{array}{c} a,\ b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. $$

Similarly, for the extension of ${}_1F_1$ we can define

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array}\middle|\, z;q\right) = \sum_{k=0}^{\infty} \frac{\mathrm{B}\,(b+k, c-b; q)}{\mathrm{B}\,(b, c-b)} \frac{z^k}{k!}, \tag{31} $$

which reduces to the same as ${}_1F_1$ for $q = 0$. Again since $\left|\mathrm{B}(r, s; q)\right| \le \mathrm{B}(r, s)$ for any $r, s > 0$ and $q \in \mathbb{R}$, the generalized series (31) converges everywhere if $c > b > 0$ and $c$ is not a negative integer or zero. Also, the integral representation of (31) is derived in a similar way as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array}\middle|\, z;q\right) = \frac{\Gamma(c-b+\mathrm{i}q)\Gamma(c-b-\mathrm{i}q)\Gamma(2b)}{\Gamma^2(b)\Gamma^2(c-b)} \int_0^1 (x-x^2)^{b-1} \cos\left(q \log \frac{x}{1-x}\right) {}_1F_1\left(\begin{array}{c} b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. \tag{32} $$

Finally for $q = 0$, (32) reduces to a new representation for the series ${}_1F_1$ as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array}\middle|\, z\right) = \frac{\Gamma(2b)}{\Gamma^2(b)} \int_0^1 (x-x^2)^{b-1} {}_1F_1\left(\begin{array}{c} b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. $$

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