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A New Second-Order Corrector Interior-Point Algorithm for $P_*(\kappa)$-LCP

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Abstract. In this paper, we propose a second-order corrector interior-point algorithm for solving $P_*(\kappa)$-linear complementarity problems. The method generates a sequence of iterates in a wide neighborhood of the central path introduced by Ai and Zhang. In each iteration, the method computes a corrector direction in addition to the Ai-Zhang direction, in an attempt to improve performance. The algorithm does not depend on the handicap κ of the problem, so that it can be used for any $P_*(\kappa)$–linear complementarity problems. It is shown that the iteration complexity bound of the algorithm is $O\big((1+\kappa)^3\sqrt{n}L\big)$. Some numerical results are provided to illustrate the performance of the algorithm.

1. Introduction

The $P_*(\kappa)$-linear complementarity problem (LCP) requires the computation of a vector pair $(x, s) \in R^{2n}$ satisfying

$$-Mx + s = q, \ xs = 0, \quad x, s \geq 0, \tag{1}$$

where $q \in R^n$ and $M \in R^{n\times n}$ is a $P_*(\kappa)$-matrix. The class of P_*-matrices was introduced by Kojima et al. [4] and it contains many types of matrices encountered in practical applications. Let κ be a nonnegative number. A matrix M is called a $P_*(\kappa)$-matrix iff it satisfies the following condition:

$$(1+4\kappa)\sum_{i\in I_+} x_i(Mx)_i + \sum_{i\in I_-} x_i(Mx)_i \geq 0, \ \ \forall x \in R^n,$$

where $I_+ = \{i : x_i(Mx)_i \geq 0\}$ and $I_- = \{i : x_i(Mx)_i < 0\}$ are two index sets. The class of all $P_*(\kappa)$-matrices is denoted by $P_*(\kappa)$, and the class P_* is defined by $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$, i.e., M is a P_*-matrix iff $M \in P_*(\kappa)$ for some $\kappa \geq 0$. Obviously, $P_*(0)$ is the class of positive semidefinite matrices.

LCPs have many applications, e.g., linear and quadratic programming, finding a Nash-equilibrium in bimatrix games and calculating the interval hull of linear systems of interval equations [2, 14]. There are a variety of solution approaches for LCPs which have been studied intensively. Among them, the interior-point methods (IPMs) gained much attention than other methods. IPMs not only have polynomial complexity but also they are the most effective methods for solving large scale optimization problems. Examples of IPMs that are reliable both in theory and in practice include the primal-dual path-following

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methods of Kojima et al. [5] and since then many other algorithms have been developed based on the primal-dual strategy. Mizuno, Todd and Ye [10] proposed the MTY predictor-corrector algorithm. It was the first algorithm for linear optimization (LO) that had both polynomial complexity and superlinear convergence. More precisely, it has $O(\sqrt{n}L)$ iteration complexity and the duality gap of the sequence generated by the MTY algorithm converges to zero quadratically [17]. In 1995 [9] Miao extended the MTY predictor-corrector method for $P_*(\kappa)$-LCP. His algorithm has $O((1+\kappa)\sqrt{n}L)$ iteration complexity and is quadratically convergent for nondegenerate problems. However, the constant κ is explicitly used in the construction of the algorithm, which implies that the algorithm can not be used for sufficient linear complementarity problems. Potra and Sheng [13] generalized the MTY predictor-corrector method for sufficient complementarity problems. Although the algorithms of [13] do not depend on the constant κ, their computational complexity does: if the problem is a $P_*(\kappa)$-LCP they terminate in at most $O((1+\kappa)\sqrt{n}L)$ iterations. The proposed algorithm in [13] has Q-order 2 in nondegenerate case, and 1.25 in the degenerate case. Predictor-corrector algorithms with higher order of convergence for degenerate sufficient LCPs were given in [15]. These algorithms have $O((1+\kappa)\sqrt{n}L)$ iteration complexity for $P_*(\kappa)$–LCPs [16].

Although some of the above mentioned algorithms have optimal iteration complexity, in practice those which use wide neighborhood perform better. This is one of the paradoxes of the interior-point methods because algorithms which use large neighborhoods of the central path are usually more difficult to analyze and, in general, their computational complexity is worse than the corresponding one for algorithms using smaller neighborhoods. Potra and Liu [12] presented two predictor-corrector methods for sufficient linear complementarity problems based on the $\mathcal{N}_\infty^-$ wide neighborhood. The first algorithm depends on the handicap κ while the second does not. Both algorithms are superlinearly convergent even for degenerate problems and have $O((1+\kappa^{1+1/m})\sqrt{n}L)$ iteration complexity. In the next paper, the authors presented a corrector-predictor algorithm acting in the wide neighborhood $\mathcal{N}_\infty^-$ of the central path that does not depend on the handicap κ of the problem, has $O((1+\kappa)\sqrt{n}L)$ iteration complexity, and is superlinearly convergent even for general sufficient linear complementarity problems.

Considering other wide neighborhoods that are different from the classical $\mathcal{N}_\infty^-$ neighborhood could be a choice for improving the complexity of interior-point methods. In [1] Ai and Zhang introduced a new wide neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$ of the central path. Their algorithm decomposes the classical Newton direction into two orthogonal directions using different step-length for each of them. Based on Ai and Zhang idea, Liu and Liu [6] proposed a primal-dual second order corrector IPM for linear programming (LP). The main difference between the method in [1] and [6] lies in that at each iteration, the latter method computes a corrector direction in addition to the Ai-Zhang direction. Later on, the authors generalized the proposed algorithm in [6] to semidefinite optimization (SDO) [7]. Recently, Potra [11] presented three interior-point algorithms for sufficient horizontal linear complementarity problems (HLCP) acting in a wide neighborhood of the central path proposed by Ai and Zhang. Motivated by the above mentioned works, we present a second-order corrector interior-point algorithm for $P_*(\kappa)$-LCP based on the Ai-Zhang wide neighborhood, and we derive the complexity bound for our algorithm. Numerical results show that our algorithm is promising.

The outline of this paper is as follows. In section 2, we introduce $P_*(\kappa)$-LCP and review some basic concepts for IPMs for solving LCPs, such as the central path. In section 3, we state and prove some technical lemmas and then, based on these results, we establish the iteration complexity bound of the proposed algorithm. Numerical results are presented in section 4. Finally, some conclusions and remarks are given in section 5.

The following notations are used through the paper. R^n denotes the n-dimensional Euclidean space. All the vectors are column vectors and e denotes the vector with all components equal to one. For any vectors x and s, xs denotes componentwise product (Hadamard product) of vectors x and s, and so is true for other operations, e.g., if $x \in R^n_+$, then $\sqrt{x}$ denotes the vector with component $\sqrt{x_i}$ and $(xs)^{-\frac{1}{2}}$ with component $\frac{1}{\sqrt{x_i s_i}}$.The positive and negative parts of a vector $v \in R^n$ are defined by $v^+ = \max\{v, 0\}$ and $v^- = \min\{v, 0\}$, so that $v^+ \geq 0$, $v^- \leq 0$ and $v = v^+ + v^-$.

2. The Central Path and Wide Neighborhood

The concept of the central path plays a critical role in the development of IPMs. Kojima et al. [4] first proved the existence and uniqueness of the central path for $P_*(\kappa)$-LCP. Throughout the paper, we assume that $P_*(\kappa)$-LCP satisfies the interior-point condition (IPC), i.e., there exists a pair $(x^0, s^0) > 0$ such that $s^0 = Mx^0 + q$, which implies the existence of a solution for $P_*(\kappa)$-LCP [4]. The set of feasible interior points is denoted by

$$\mathcal{F}^0 := \Big\{(x,s) \in R^{2n} : s = Mx + q, (x,s) > 0\Big\}.$$

The basic idea of IPMs is to replace the second equation in (1), the so-called complementarity condition for $P_*(\kappa)$-LCP, by the relaxed equation $xs = \mu e$, with $\mu > 0$. Thus, we consider the system

$$-Mx + s = q, \;\; xs = \mu e, \;\; x, s \geq 0. \tag{2}$$

Since M is a $P_*(\kappa)$-matrix and the IPC holds, the system (2) has a unique solution for each $\mu > 0$ (cf. Lemma 4.3 in [4]). This solution is denoted as $(x(\mu), s(\mu))$ and is called the μ-center of $P_*(\kappa)$-LCP. The set of μ-centers with all $\mu > 0$ gives the central path of $P_*(\kappa)$-LCP, i.e.,

$$C := \Big\{(x,s) \in \mathcal{F}^0 : xs = \mu e\Big\}.$$

It has been shown that the limit of the central path (as μ goes to zero) exists and yields a solution for $P_*(\kappa)$-LCP (Theorem 4.4 in [4]).

Applying Newton's method to (2) for a given feasible point (x, s) gives the following linear system of equations

$$M\Delta x - \Delta s = 0, \;\; x\Delta s + s\Delta x = \mu e - xs. \tag{3}$$

Since M is $P_*(\kappa)$-matrix, the system (3) uniquely defines $(\Delta x, \Delta s)$ for any $x > 0$ and $s > 0$. At each iteration, the method would choose a target on the central path and apply the Newton method to move closer to the target, while confining the iterate to stay within a certain neighborhood of the central path.

As usual according, a neighborhood of the central path, the so-called small neighborhood, is defined as

$$\mathcal{N}_2(\beta) := \Big\{(x,s) \in \mathcal{F}^0 : \|xs - \mu e\|_2 \leq \beta\mu\Big\},$$

where $\beta \in (0,1)$ is a given constant and $\mu = \frac{x^T s}{n}$. Alternatively, the so-called wide neighborhood is defined as follows:

$$\mathcal{N}_\infty^-(\alpha) := \Big\{(x,s) \in \mathcal{F}^0 : xs \geq \alpha\mu e\Big\},$$

where $0 < \alpha < 1$. In this paper, we will work with the following neighborhood considered in [1]:

$$\widetilde{\mathcal{N}}_\tau^-(\alpha) = \Big\{(x,s) \in \mathcal{F}^0 : \left\|(xs - \tau\mu e)^-\right\|_2 \leq \alpha\tau\mu\Big\}. \tag{4}$$

It is clear that $\left\|(xs - \tau\mu e)^-\right\|_2 = 0$, for all $(x,s) \in \mathcal{N}_\infty^-(\tau)$, and that for any $(x,s) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$ we have

$$\left\|(xs - \tau\mu e)^-\right\|_2 \leq \alpha\tau\mu \;\text{ and }\; x_i s_i \leq \tau\mu,$$

which imply

$$0 \leq 1 - \frac{x_i s_i}{\tau\mu} \leq \alpha, \;\text{ and or equivalently }\; x_i s_i \geq (1-\alpha)\tau\mu. \tag{5}$$

Therefore,

$$\mathcal{N}_\infty^-(\tau) \subset \widetilde{\mathcal{N}}_\tau^-(\alpha) \subset \mathcal{N}_\infty^-((1-\alpha)\tau) \quad \forall \alpha, \tau \in (0,1). \tag{6}$$

Since $\mathcal{N}_\infty^-(\tau)$ is a wide neighborhood, so is $\widetilde{\mathcal{N}}_\tau^-(\alpha)$. We note that an interior–point method from [1, 6] use the neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$ only for $\alpha \in (0, \frac{1}{2}]$, while in this paper we will construct an interior–point method based on the neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$ for any value of $\alpha \in (0,1)$.

3. A Large Update Path–Following Algorithm

In this section, we present a large update path–following method for solving $P_*(k)$-LCP. Our algorithm generalizes the large update path–following method proposed in [6] for $P_*(k)$-LCP. Similar to Ai and Zhang [1] and Liu et al. [6], we decompose the Newton direction (3), from xs to the target on the central path $\tau\mu e$ (large update), into two separate parts according to the positive and negative parts of $\tau\mu e - xs$. Then, we solve the following two systems:

$$\begin{aligned} M\Delta x_- - \Delta s_- &= 0, \\ s\Delta x_- + x\Delta s_- &= (\tau\mu e - xs)^-, \end{aligned} \tag{7}$$

and

$$\begin{aligned} M\Delta x_+ - \Delta s_+ &= 0, \\ s\Delta x_+ + x\Delta s_+ &= (\tau\mu e - xs)^+. \end{aligned} \tag{8}$$

Based on the direction from (7), we compute the corrector direction by

$$\begin{aligned} M\Delta x_-^c - \Delta s_-^c &= 0, \\ s\Delta x_-^c + x\Delta s_-^c &= -\Delta x_- \Delta s_-. \end{aligned} \tag{9}$$

Finally, the new iterate is given by

$$\begin{aligned} (x(\theta), s(\theta)) &:= (x, s) + (\Delta x(\theta), \Delta s(\theta)) \\ &:= (x, s) + \theta_1(\Delta x_-, \Delta s_-) + \theta_2(\Delta x_+, \Delta s_+) + \theta_1^2(\Delta x_-^c, \Delta s_-^c), \end{aligned} \tag{10}$$

where $\theta = (\theta_1, \theta_2), 0 \leq \theta_1, \theta_2 \leq 1$ is the step length vector. To get the best step length for three of the directions, we expect to solve the following subproblem

$$\begin{aligned} \min_{\theta \in [0,1]\times[0,1]} \quad & \mu(\theta) \\ \text{s.t.} \quad & (x(\theta), s(\theta)) \in \widetilde{\mathcal{N}}_\tau^-(\alpha). \end{aligned} \tag{11}$$

We are now in the position to describe our second-order corrector algorithm for $P_*(k)$-LCPs.

Second − Order Corrector Algorithm

Input :
- accuracy parameter $\epsilon > 0$;
- neighborhood parameter α, $0 < \alpha < 1$;
- centering parameter τ, $0 < \tau \leq 0.5$;
- an initial point $(x, s) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$, $\mu_0 = (x^0)^T s^0/n$;
- set k := 0.

begin

while $(x^k)^T s^k > \epsilon$ **do**

compute the directions $(\Delta x_-^k, \Delta s_-^k)$ by (7) and $(\Delta x_+^k, \Delta s_+^k)$ by (8).
compute the directions $(\Delta x_-^{c,k}, \Delta s_-^{c,k})$ by (9). Find a step length vector $\theta^k = (\theta_1^k, \theta_2^k)$ giving a sufficirent reduction of μ_k and assuring $(x(\theta^k), s(\theta^k)) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$.
let $(x^{k+1}, s^{k+1}) := (x(\theta^k), s(\theta^k))$, $\mu_{k+1} = (x^{k+1})^T s^{k+1}/n$.
set $k := k + 1$.

end

Figure 1 : The algorithm

Here, we give two technical lemmas that will be used during the analysis. Their proofs are the same to the proofs of the lemmas 3.1 and 3.4 in [6].

Lemma 3.1. *Let* $(x,s) \in \mathcal{F}^0$, $(\Delta x_-, \Delta s_-)$ *be the solution of* (7). *Note* $\mathcal{I}_+ = \{i \in \mathcal{I} : (\Delta x_-)_i(\Delta s_-)_i) > 0\}$ *and* $\mathcal{I}_- = \mathcal{I} \backslash \mathcal{I}_+$ *Then*

$$(\Delta x_-)_i(\Delta s_-)_i \leq \frac{x_i s_i}{4}, \ \ \forall i \in \mathcal{I}_+.$$

Lemma 3.2. *Suppose that* $(x,s) \in \mathcal{F}^0$ *and* $a + 2xs \geq 0$. *Let* (u,v) *be the solution of*

$$\begin{aligned} &Mu - v = 0, \\ &su + xv = a. \end{aligned} \tag{12}$$

If $(x + t_0 u)(s + t_0 v) > 0$ *for some* $0 < t_0 \leq 1$, *then* $(x + tu)(s + tv) > 0$ *for all* $0 \leq t \leq t_0$.

It is easy to verify that $(\Delta x(\theta), \Delta s(\theta))$ as defined in (10) satisfies the following system:

$$\begin{aligned} &M\Delta x(\theta) - \Delta s(\theta) = 0, \\ &s\Delta x(\theta) + x\Delta s(\theta) = \theta_1(\tau\mu e - xs)^- + \theta_2(\tau\mu e - xs)^+ - \theta_1^2 \Delta x_- \Delta s_-. \end{aligned} \tag{13}$$

The term $a + 2xs$ is sometimes called the target to be tracked, and it is naturally nonnegative for most IPMs. In particular, for Algorithm in Fig.1, this property is given in the next lemma.

Lemma 3.3. *Let* $(x,s) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$ *and let* $(x(\theta), s(\theta))$ *be defined by* (10). *Then* $(x(\theta), s(\theta)) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$ *if and only if* $\mu(\theta) = \dfrac{(x(\theta))^T s(\theta)}{n} > 0$ *and* $\|(x(\theta)s(\theta) - \tau\mu(\theta)e)^-\|_2 \leq \alpha\tau\mu(\theta)$.

Proof. Let us assume that $\mu(\theta) > 0$ and $\|(x(\theta)s(\theta) - \tau\mu(\theta)e)^-\|_2 \leq \alpha\tau\mu(\theta)$. Since $M\Delta x(\theta) - \Delta s(\theta) = 0$, we obtain $s(\theta) = Mx(\theta) + q$. In order to complete the proof of lemma, it suffices to show that $x(\theta) > 0$ and $s(\theta) > 0$. From (5) we deduce that $x(\theta)s(\theta) > 0$. According to Lemma 3.2, it is sufficient to prove that $s\Delta x(\theta) + x\Delta s(\theta) + 2xs \geq 0$. From (13) it follows that the left-hand side of this inequality is equal to

$$2xs + \theta_1(\tau\mu e - xs)^- + \theta_2(\tau\mu e - xs)^+ - \theta_1^2 \Delta x_- \Delta s_-. \tag{14}$$

We consider the following cases:
Case 1: Let $i \in \mathcal{I}_-$. In this case, either $x_i s_i \leq \tau\mu$ or $x_i s_i \geq \tau\mu$, the relation (14) becomes positive.
Case 2: Let $i \in \mathcal{I}_+$ and $x_i s_i \leq \tau\mu$. In this case, the ith component of (14) becomes

$$2x_i s_i + \theta_2(\tau\mu - x_i s_i) - \theta_1^2(\Delta x_-)_i(\Delta s_-)_i \geq \Big(2 - \theta_2 - \frac{\theta_1^2}{4}\Big)x_i s_i + \theta_2\tau\mu > 0.$$

Case 3: Let $i \in \mathcal{I}_+$ and $x_i s_i \geq \tau\mu$. In this case the ith component of (14) reduces to

$$2x_i s_i + \theta_1(\tau\mu - x_i s_i) - \theta_1^2(\Delta x_-)_i(\Delta s_-)_i \geq \Big(2 - \theta_1 - \frac{\theta_1^2}{4}\Big)x_i s_i + \theta_1\tau\mu > 0.$$

Note that in case 2 and case 3 the first inequalities follow by Lemma 3.1. These complete the proof. □

It follows that the optimization problem (11) is equivalent to

$$\begin{aligned} \min_{\theta \in [0,1]\times[0,1]} \quad & \mu(\theta) \\ \text{s.t. } & \mu(\theta) > 0, \\ & \|(x(\theta)s(\theta) - \tau\mu(\theta)e)^-\|_2 \leq \alpha\tau\mu(\theta). \end{aligned} \tag{15}$$

If $\theta^+ = (\theta_1^+, \theta_2^+)$ is the solution of the above minimization problem, then according to lemma 3.3 the point $(x(\theta^+), s(\theta^+))$ belongs to the wide neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$ and the process can be repeated.

Lemma 3.4. (cf. Lemma 2 in [3]) *If LCP is* $P_*(k)$, *then for any* $(x,s) \in \mathbb{R}^{2n}_{++}$ *and any* $a \in \mathbb{R}^n$ *the linear system (12) has a unique solution* (u,v) *for which the following estimates hold*

$$\|uv\|_2 \leq \Big(\frac{1}{\sqrt{8}} + \kappa\Big)\|\widetilde{a}\|_2^2, \ -\kappa\|\widetilde{a}\|_2^2 \leq u^T v \leq \frac{1}{4}\|\widetilde{a}\|_2^2,$$

where $\widetilde{a} = (xs)^{-1/2}a$.

Lemma 3.5. *The solutions of* (7) *and* (13) *satisfy the following inequalities:*

$$\|\Delta x_-\Delta s_-\|_2 \leq \Big(\frac{1}{\sqrt{8}}+\kappa\Big)n\mu, \quad -\kappa n\mu \leq \Delta x_-^T\Delta s_- \leq \frac{n\mu}{4}, \tag{16}$$

$$\|\Delta x(\theta)\Delta s(\theta)\|_2 \leq \mu\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\Bigg(\sqrt{n\theta_1^2+\frac{\theta_2^2\alpha^2\tau}{1-\alpha}}+\frac{n\theta_1^2}{\sqrt{(1-\alpha)\tau}}\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\Bigg)^2, \tag{17}$$

$$\Delta x(\theta)^T\Delta s(\theta) \geq -\kappa\mu\Bigg(\sqrt{n\theta_1^2+\frac{\theta_2^2\alpha^2\tau}{1-\alpha}}+\frac{n\theta_1^2}{\sqrt{(1-\alpha)\tau}}\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\Bigg)^2, \tag{18}$$

$$\Delta x(\theta)^T\Delta s(\theta) \leq \frac{\mu}{4}\Bigg(\sqrt{n\theta_1^2+\frac{\theta_2^2\alpha^2\tau}{1-\alpha}}+\frac{n\theta_1^2}{\sqrt{(1-\alpha)\tau}}\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\Bigg)^2. \tag{19}$$

Proof. For any $(x,s)\in\mathbb{R}^{2n}_{++}$ there holds

$$\|(xs)^{-1/2}(\tau\mu e-xs)^-\|_2^2 = \|(xs)^{-1/2}(xs-\tau\mu e)^+\|_2^2 \leq \|(xs)^{1/2}\|_2^2 = n\mu.$$

By considering $(u,v)=(\Delta x_-,\Delta s_-)$ and $a=(\tau\mu e-xs)^-$in system (12), using Lemma 3.4 and the above inequality, we obtain

$$\|\Delta x_-\Delta s_-\|_2 \leq \Big(\frac{1}{\sqrt{8}}+\kappa\Big)\|(xs)^{-1/2}(\tau\mu e-xs)^-\|_2^2 \leq \Big(\frac{1}{\sqrt{8}}+\kappa\Big)n\mu,$$

and

$$-\kappa n\mu \leq -\kappa\|(xs)^{-1/2}(\tau\mu e-xs)^-\|_2^2 \leq \Delta x_-^T\Delta s_- \leq \frac{1}{4}\|(xs)^{-1/2}(\tau\mu e-xs)^-\|_2^2 \leq \frac{n\mu}{4}.$$

The above inequalities prove (16). If $(x,s)\in\widetilde{\mathcal{N}}_\tau^-(\alpha)$, then according to (5) we can write

$$\|(xs)^{-1/2}(\tau\mu e-xs)^+\|_2^2 \leq \frac{\|(\tau\mu e-xs)^+\|_2^2}{(1-\alpha)\tau\mu} \leq \frac{\tau\mu\alpha^2}{1-\alpha}.$$

Therefore, using the orthogonality of $(\tau\mu e-xs)^-$ and $(\tau\mu e-xs)^+$ we have

$$\begin{aligned}&\Big\|(xs)^{-1/2}\Big(\theta_1(\tau\mu e-xs)^- + \theta_2(\tau\mu e-xs)^+\Big)\Big\|_2^2\\ &\qquad = \theta_1^2\Big\|(xs)^{-1/2}(\tau\mu e-xs)^-\Big\|_2^2+\theta_2^2\Big\|(xs)^{-1/2}(\tau\mu e-xs)^+\Big\|_2^2\\ &\qquad \leq n\mu\theta_1^2+\frac{\tau\mu\alpha^2\theta_2^2}{1-\alpha}.\end{aligned}$$

Further, from (5) and (16) we get

$$\Big\|(xs)^{-1/2}\Big(\Delta x_-\Delta s_-\Big)\Big\|_2^2 \leq \frac{\|\Delta x_-\Delta s_-\|_2^2}{(1-\alpha)\tau\mu} \leq \frac{n^2\mu^2}{(1-\alpha)\tau\mu}\Big(\frac{1}{\sqrt{8}}+\kappa\Big)^2.$$

Now, comparing system (12) with the system (13) and considering $(u,v)=(\Delta x(\theta),\Delta s(\theta))$ and $a=\theta_1(\tau\mu e-xs)^-+\theta_2(\tau\mu e-xs)^+-\theta_1^2\Delta x_-\Delta s_-$, we have

$$\begin{aligned}\|\widehat{a}\|_2 &= \Big\|(xs)^{-1/2}\Big(\theta_1(\tau\mu e-xs)^-+\theta_2(\tau\mu e-xs)^+-\theta_1^2\Delta x_-\Delta s_-\Big)\Big\|_2\\ &\leq \Big\|(xs)^{-1/2}\Big(\theta_1(\tau\mu e-xs)^-+\theta_2(\tau\mu e-xs)^+\Big)\Big\|_2+\theta_1^2\Big\|(xs)^{-1/2}\Big(\Delta x_-\Delta s_-\Big)\Big\|_2\\ &\leq \Bigg(\sqrt{n\theta_1^2+\frac{\tau\alpha^2\theta_2^2}{1-\alpha}}+\frac{n\theta_1^2}{\sqrt{(1-\alpha)\tau}}\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\Bigg)\sqrt{\mu}.\end{aligned} \tag{20}$$

The inequalities (17), (18) and (19) can be derived from (20) and Lemma 3.4. The proof of lemma is complete. □

In the sequel, we denote

$$\psi(\theta) := \psi(\theta_1, \theta_2) := \left(\sqrt{n\theta_1^2 + \frac{\tau\alpha^2\theta_2^2}{1-\alpha}} + \frac{n\theta_1^2}{\sqrt{(1-\alpha)\tau}}\Big(\frac{1}{\sqrt{8}} + \kappa\Big)\right)^2. \tag{21}$$

In the next lemma we give upper and lower bounds for $\mu(\theta)$. We first note that, from (10) and the second equation of (13), we have

$$\begin{aligned} x(\theta)s(\theta) &= (x + \Delta x(\theta))(s + \Delta s(\theta)) = xs + s\Delta x(\theta) + x\Delta s(\theta) + \Delta x(\theta)\Delta s(\theta) \\ &= xs + \theta_1(\tau\mu e - xs)^- + \theta_2(\tau\mu e - xs)^+ - \theta_1^2\Delta x_-\Delta s_- + \Delta x(\theta)\Delta s(\theta). \end{aligned} \tag{22}$$

Moreover,

$$\begin{aligned} \mu(\theta) &= \frac{x(\theta)^T s(\theta)}{n} = \frac{e^T(x(\theta)s(\theta))}{n} \\ &= \mu + \theta_1\frac{e^T(\tau\mu e - xs)^-}{n} + \theta_2\frac{e^T(\tau\mu e - xs)^+}{n} - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n}. \end{aligned} \tag{23}$$

Since

$$n(1-\tau)\mu = e^T(xs - \tau\mu e) = e^T(xs - \tau\mu e)^+ + e^T(xs - \tau\mu e)^-, \tag{24}$$

by applying the Cauchy-Schwartz inequality and the definition of the neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$, we get

$$|e^T(xs - \tau\mu e)^-| \le \sqrt{n}\|(xs - \tau\mu e)^-\|_2 \le \tau\alpha\mu\sqrt{n}. \tag{25}$$

It follows from (24) and (25) that

$$n(1-\tau)\mu \le e^T(xs - \tau\mu e)^+ \le \big(n(1-\tau) + \tau\alpha\sqrt{n}\big)\mu. \tag{26}$$

Lemma 3.6. *For any $(x, s) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$ and $\theta = (\theta_1, \theta_2)$ with $0 \le \theta_1 \le \theta_2 \le 1$, we have*

$$\mu(\theta) \le \left(1 - \Big(1 - \tau + \frac{\tau\alpha}{\sqrt{n}}\Big)\theta_1 + \frac{\tau\alpha\theta_2}{\sqrt{n}} + \kappa\theta_1^2 + \frac{\psi(\theta)}{4n}\right)\mu, \tag{27}$$

$$\mu(\theta) \ge \left(1 - \theta_1(1-\tau) - \frac{1}{4}\theta_1^2 - \frac{\kappa\psi(\theta)}{n}\right)\mu. \tag{28}$$

Proof. By using (23), the equality $(\tau\mu e - xs)^- = -(xs - \tau\mu e)^+$, (24), (25) and (26) we obtain

$$\begin{aligned} \mu(\theta) &= \mu - \theta_1\frac{e^T(xs - \tau\mu e)^+}{n} - \theta_2\frac{e^T(xs - \tau\mu e)^-}{n} - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n} \\ &= \mu - \theta_1\frac{n(1-\tau)\mu - e^T(xs - \tau\mu e)^-}{n} - \theta_2\frac{e^T(xs - \tau\mu e)^-}{n} \\ &\qquad - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n} \\ &= \mu - \theta_1(1-\tau)\mu - (\theta_2 - \theta_1)\frac{e^T(xs - \tau\mu e)^-}{n} - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n} \\ &\le \mu - \theta_1(1-\tau)\mu + (\theta_2 - \theta_1)\frac{\tau\alpha\mu}{\sqrt{n}} - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n} \\ &= \mu - \theta_1(1 - \tau + \frac{\tau\alpha}{\sqrt{n}})\mu + \theta_2\frac{\tau\alpha\mu}{\sqrt{n}} - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n} \\ &\le \Big(1 - \Big(1 - \tau + \frac{\tau\alpha}{\sqrt{n}}\Big)\theta_1 + \frac{\tau\alpha\theta_2}{\sqrt{n}} + \kappa\theta_1^2 + \frac{\psi(\theta)}{4n}\Big)\mu, \end{aligned} \tag{29}$$

where the last inequality derives from (16) and (19), which proves the inequality (27). Similarly, we get

$$\begin{aligned}\mu(\theta) &= \mu - \theta_1(1-\tau)\mu - (\theta_2-\theta_1)\frac{e^T(xs-\tau\mu e)^-}{n} - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n}\\ &\geq \mu - \theta_1(1-\tau)\mu - \theta_1^2\frac{\Delta x_-^T\Delta s_-}{n} + \frac{\Delta x(\theta)^T\Delta s(\theta)}{n}\\ &\geq \Big(1-\theta_1(1-\tau) - \frac{1}{4}\theta_1^2 - \frac{\kappa\psi(\theta)}{n}\Big)\mu,\end{aligned}$$

where the last inequality holds due to (16) and (18). This completes the proof of (28). □

To follow the central path, we need to make sure that the iterates remain in the prescribed neighborhood of the central path. So in the next lemma we give an upper bound for 2-norm of $(x(\theta)s(\theta) - \tau\mu(\theta)e)^-$.

Lemma 3.7. *For any* $(x, s) \in \widetilde{\mathcal{N}}_\tau^-(\alpha)$ *and* $\theta = (\theta_1, \theta_2)$ *with*

$$\frac{\tau\alpha\theta_2}{(1-\tau)\sqrt{n}+\alpha\tau} \leq \theta_1 \leq \theta_2 \leq 1, \tag{30}$$

we have

$$\left\|(x(\theta)s(\theta) - \tau\mu(\theta)e)^-\right\|_2 \leq \Big((1-\theta_2)\tau\alpha + \theta_1^2\Big(\frac{1}{\sqrt{8}}+\kappa\Big)n + \Big(\frac{1}{\sqrt{8}}+\kappa\Big)\psi(\theta)\Big)\mu.$$

Proof. By subtracting and adding $\tau\mu e$ to the right-hand side of (22) we obtain

$$\begin{aligned}x(\theta)s(\theta) &= \tau\mu e + (xs-\tau\mu e) - \theta_1(xs-\tau\mu e)^+ - \theta_2(xs-\tau\mu e)^- - \theta_1^2\Delta x_-\Delta s_- + \Delta x(\theta)\Delta s(\theta)\\ &= \tau\mu e + (1-\theta_1)(xs-\tau\mu e)^+ + (1-\theta_2)(xs-\tau\mu e)^- - \theta_1^2\Delta x_-\Delta s_- + \Delta x(\theta)\Delta s(\theta).\end{aligned}$$

From the above equality and (29) we deduce that

$$\begin{aligned}x(\theta)s(\theta) - \tau\mu(\theta)e &= (1-\theta_1)(xs-\tau\mu e)^+ + (1-\theta_2)(xs-\tau\mu e)^-\\ &\quad + \tau\theta_1\frac{e^T(xs-\tau\mu e)^+}{n}e + \tau\theta_2\frac{e^T(xs-\tau\mu e)^-}{n}e - \theta_1^2\Delta x_-\Delta s_-\\ &\quad + \Delta x(\theta)\Delta s(\theta) + \tau\theta_1^2\frac{\Delta x_-^T\Delta s_-}{n}e - \tau\frac{\Delta x(\theta)^T\Delta s(\theta)}{n}e.\end{aligned}$$

On the other hand, by (24), (25) and (30), we have

$$\begin{aligned}\theta_1\frac{e^T(xs-\tau\mu e)^+}{n} + \theta_2\frac{e^T(xs-\tau\mu e)^-}{n} &= \theta_1\frac{e^T(xs-\tau\mu e)}{n} + (\theta_2-\theta_1)\frac{e^T(xs-\tau\mu e)^-}{n}\\ &\geq (1-\tau)\mu\theta_1 - (\theta_2-\theta_1)\frac{\tau\alpha\mu}{\sqrt{n}}\\ &= \Big(1-\tau+\frac{\tau\alpha}{\sqrt{n}}\Big)\mu\theta_1 - \frac{\tau\alpha\mu}{\sqrt{n}}\theta_2 \geq 0.\end{aligned}$$

Therefore, we obtain the following inequality

$$x(\theta)s(\theta) - \tau\mu(\theta)e \geq (1-\theta_2)(xs-\tau\mu e)^- + \theta_1^2\Big(\frac{\tau\Delta x_-^T\Delta s_-}{n}e - \Delta x_-\Delta s_-\Big)^- + \Big(\Delta x(\theta)\Delta s(\theta) - \frac{\tau\Delta x(\theta)^T\Delta s(\theta)}{n}e\Big)^-,$$

which implies, by $v \geq u \Rightarrow \|v^-\|_2 \leq \|u^-\|_2$ and $\|u^-\|_2 \leq \|u\|_2$,

$$\begin{aligned}\left\|\left(x(\theta)s(\theta)-\tau\mu(\theta)e\right)^-\right\|_2 &\leq (1-\theta_2)\left\|(xs-\tau\mu e)^-\right\|_2+\theta_1^2\left\|\Delta x_-\Delta s_- - \frac{\tau\Delta x_-^T\Delta s_-}{n}e\right\|_2\\ &\quad+\left\|\Delta x(\theta)\Delta s(\theta)-\frac{\tau\Delta x(\theta)^T\Delta s(\theta)}{n}e\right\|_2\\ &\leq (1-\theta_2)\tau\alpha\mu+\theta_1^2\left\|\Delta x_-\Delta s_-\right\|_2+\left\|\Delta x(\theta)\Delta s(\theta)\right\|_2\\ &\leq \Big((1-\theta_2)\tau\alpha+\theta_1^2\Big(\frac{1}{\sqrt{8}}+\kappa\Big)n+\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\psi(\theta)\Big)\mu,\end{aligned}$$

where the last inequality follows from (16) and (17), and the proof of lemma is complete. □

Lemma 3.8. *If $0<\tau\leq\frac{1}{2}$, then the point $(x(\theta),s(\theta))$ defined in* (10) *belongs to $\widetilde{\mathcal{N}}_\tau^-(\alpha)$ for any $\theta=(\theta_1,\theta_2)$ satisfying*

$$\theta_1=\frac{\sqrt{\tau\alpha}\theta_2}{(1-\tau)\sqrt{n}},\qquad n\geq 2,\qquad 0<\theta_2\leq\widetilde{\theta}_2:=\frac{(1-\alpha)(1-\tau)^4}{\left(5\sqrt{2}\kappa+4\right)\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}. \tag{31}$$

Proof. We first note that the first equation in (31) implies that

$$(1-\tau)\theta_1=\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}\leq\frac{1}{2}. \tag{32}$$

Now, by using (21) and the first equation of (32), we obtain

$$\begin{aligned}\psi(\theta)&\leq\left(\sqrt{\frac{\alpha}{(1-\tau)^2}+\frac{\alpha^2}{1-\alpha}}+\frac{\alpha}{\sqrt{1-\alpha}(1-\tau)^2}\Big(\frac{1}{\sqrt{8}}+\kappa\Big)\right)^2\tau\theta_2^2\\ &=\left(\frac{\sqrt{\alpha(1-\alpha)+\alpha^2(1-\tau)^2}}{\sqrt{1-\alpha}(1-\tau)}+\frac{\alpha\Big(\frac{1}{\sqrt{8}}+\kappa\Big)}{\sqrt{1-\alpha}(1-\tau)^2}\right)^2\tau\theta_2^2\\ &=\left(\frac{(1-\tau)\sqrt{\alpha(1-\alpha)+\alpha^2(1-\tau)^2}+\alpha\Big(\frac{1}{\sqrt{8}}+\kappa\Big)}{\sqrt{1-\alpha}(1-\tau)^2}\right)^2\tau\theta_2^2\\ &\leq\frac{\tau\alpha\Big(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\Big)^2\theta_2^2}{(1-\alpha)(1-\tau)^4},\end{aligned} \tag{33}$$

where the last inequality follows from $\frac{1}{2}\leq 1-\tau<1$ and $\alpha\leq\sqrt{\alpha}$. It follows from (28) and (33) that

$$\begin{aligned}\frac{\mu(\theta)}{\mu}&\geq 1-\theta_1(1-\tau)-\frac{1}{4}\theta_1^2-\frac{\kappa\tau\alpha\Big(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\Big)^2\theta_2^2}{n(1-\alpha)(1-\tau)^4}\\ &\geq\frac{1}{2}-\frac{1}{4}-\frac{\kappa\tau\alpha\Big(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\Big)^2\theta_2}{n(1-\alpha)(1-\tau)^4}\\ &\geq\frac{1}{4}-\frac{\kappa\tau\alpha}{n(5\sqrt{2}\kappa+4)}\\ &\geq\frac{1}{4}\Big(1-\frac{\kappa}{5\sqrt{2}\kappa+4}\Big)>0,\end{aligned}$$

where the second inequality follows from the inequality of (32) and $\theta_1 \leq 1$ and the third inequality follows from (31). Now, by using Lemma 3.7, (33) and (28) we deduce that

$$\begin{aligned}
&\left\|\left(x(\theta)s(\theta)-\tau\mu(\theta)\right)^-\right\|_2-\tau\alpha\mu(\theta)\\
&\leq\left((1-\theta_2)\tau\alpha+\theta_1^2(\frac{1}{\sqrt{8}}+\kappa)n+(\frac{1}{\sqrt{8}}+\kappa)\frac{\tau\alpha\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\theta_2^2}{(1-\alpha)(1-\tau)^4}\right)\mu\\
&\quad-\tau\alpha\left(1-\theta_1(1-\tau)-\frac{1}{4}\theta_1^2-\frac{\kappa}{n}\frac{\tau\alpha\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\theta_2^2}{(1-\alpha)(1-\tau)^4}\right)\mu\\
&=\left(n\theta_1^2\left(\left(\frac{1}{\sqrt{8}}+\kappa\right)+\frac{\tau\alpha}{4n}\right)+\frac{\tau\alpha\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\theta_2^2}{(1-\alpha)(1-\tau)^4}\left(\frac{1}{\sqrt{8}}+\kappa+\frac{\tau\alpha\kappa}{n}\right)\right.\\
&\qquad\left.-\left(\theta_2-(1-\tau)\theta_1\right)\tau\alpha\right)\mu\\
&\leq\left(\left(\frac{5}{4}\kappa+\frac{8+\sqrt{2}}{16\sqrt{2}}\right)\left(\frac{1}{(1-\tau)^2}+\frac{\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}{(1-\alpha)(1-\tau)^4}\right)\theta_2-\left(1-\frac{\sqrt{\tau\alpha}}{\sqrt{n}}\right)\right)\tau\alpha\theta_2\mu\\
&\leq\left(\left(\frac{5\sqrt{2}\kappa+4}{4\sqrt{2}}\right)\left(\frac{1}{(1-\tau)^2}+\frac{\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}{(1-\alpha)(1-\tau)^4}\right)\theta_2-\left(1-\frac{\sqrt{\tau\alpha}}{\sqrt{n}}\right)\right)\tau\alpha\theta_2\mu\\
&\leq\left(\frac{1}{4\sqrt{2}}\left(\frac{(1-\alpha)(1-\tau)^2}{\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}+1\right)-1+\frac{\sqrt{\tau\alpha}}{\sqrt{n}}\right)\tau\alpha\theta_2\mu\\
&\leq\left(\frac{1}{2\sqrt{2}}-\frac{1}{2}\right)\tau\alpha\theta_2\mu<0,
\end{aligned}$$

where the second inequality follows from the first equation of (32) and the following inequalities

$$\frac{1}{\sqrt{8}}+\kappa+\frac{\tau\alpha}{4n}\leq\frac{5}{4}\kappa+\frac{8+\sqrt{2}}{16\sqrt{2}},\qquad \frac{1}{\sqrt{8}}+\kappa+\frac{\tau\alpha\kappa}{n}\leq\frac{5}{4}\kappa+\frac{8+\sqrt{2}}{16\sqrt{2}}.$$

Thus under the hypothesis of our lemma, we showed that $\|(x(\theta)s(\theta)-\tau\mu(\theta)e)^-\|_2 \leq \alpha\tau\mu(\theta)$ and $\mu(\theta) > 0$. This completes the proof of lemma according to Lemma 3.3. □

In the reminder of this paper, we will use the notations

$$\widetilde{\theta}_2 := \frac{(1-\alpha)(1-\tau)^4}{\left(5\sqrt{2}\kappa+4\right)\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2},\quad \widetilde{\theta}_1 := \frac{\sqrt{\tau\alpha}\widetilde{\theta}_2}{(1-\tau)\sqrt{n}},\quad \widetilde{\theta} := (\widetilde{\theta}_1,\widetilde{\theta}_2). \tag{34}$$

Theorem 3.9. *If LCP is $P_*(\kappa)$, then the Algorithm 1 is well defined, produces a sequence of points (x^k, s^k) belonging to the neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$, and*

$$\mu_{k+1} \leq \left(1-\frac{c(\alpha,\tau)}{(5\sqrt{2}\kappa+4)\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\sqrt{n}}\right)\mu_k,\quad k=0,1,\ldots \tag{35}$$

where

$$c(\alpha,\tau)=\frac{\sqrt{\tau\alpha}(1-\alpha)(1-\tau)^4}{4}. \tag{36}$$

Proof. The first part of the theorem follows from the definition of Algorithm 1 and Lemma 3.8. Let us denote

$$\phi(\theta)=\phi(\theta_1,\theta_2)=1-\left(1-\tau+\frac{\tau\alpha}{\sqrt{n}}\right)\theta_1+\frac{\tau\alpha\theta_2}{\sqrt{n}}+\kappa\theta_1^2+\frac{\tau\alpha\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\theta_2^2}{4n(1-\alpha)(1-\tau)^4},$$

so that according to (27) we have $\mu(\theta)\le\phi(\theta)\mu$. It is easy to verify that

$$\begin{aligned}
\phi\left(\frac{\sqrt{\tau\alpha}\theta_2}{(1-\tau)\sqrt{n}},\theta_2\right)&=1-\frac{\left(1-\tau+\frac{\tau\alpha}{\sqrt{n}}\right)\sqrt{\tau\alpha}\theta_2}{(1-\tau)\sqrt{n}}+\frac{\tau\alpha\theta_2}{\sqrt{n}}+\left(\frac{\kappa}{(1-\tau)^2}+\frac{\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}{4(1-\alpha)(1-\tau)^4}\right)\frac{\tau\alpha\theta_2^2}{n}\\
&\le 1-\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}+\frac{\tau\alpha\theta_2}{\sqrt{n}}+\left(\frac{\kappa}{(1-\tau)^2}+\frac{\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}{4(1-\alpha)(1-\tau)^4}\right)\frac{\tau\alpha\theta_2^2}{n}\\
&\le 1-\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}\left(1-\sqrt{\tau\alpha}-\left(\frac{\kappa}{(1-\tau)^2}+\frac{\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2}{4(1-\alpha)(1-\tau)^4}\right)\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}\right)\\
&\le 1-\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}\left(1-\sqrt{\tau\alpha}-\frac{\kappa\sqrt{\tau\alpha}(1-\alpha)(1-\tau)^2}{(5\sqrt{2}\kappa+4)\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\sqrt{n}}-\frac{\sqrt{\tau\alpha}}{4\sqrt{n}(5\sqrt{2}\kappa+4)}\right)\\
&\le 1-\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}\left(1-\frac{\sqrt{2}}{2}-\frac{141}{17290}-\frac{1}{32}\right)=1-\frac{507}{2000}\frac{\sqrt{\tau\alpha}\theta_2}{\sqrt{n}}\le 1-\frac{\sqrt{\tau\alpha}\theta_2}{4\sqrt{n}}.
\end{aligned}$$

With the notation from (34), it follows that

$$\mu(\theta^+)\le\mu(\tilde{\theta})\le\left(1-\frac{c(\alpha,\tau)}{(5\sqrt{2}\kappa+4)\left(\frac{\sqrt{8}+1}{\sqrt{8}}+\kappa\right)^2\sqrt{n}}\right)\mu.$$

This completes the proof. □

Corollary 3.10. *Under the hypothesis of Theorem 3.9, Algorithm 1 produces a point $(x,s)\in\widetilde{\mathcal{N}}_\tau^-(\alpha)$ with $\mu(x,s)\le\epsilon$ in at most*

$$O\left((1+\kappa)^3\sqrt{n}L\right)$$

iterations, where $L=L_\epsilon=\log\left(\frac{\mu(x^0,s^0)}{\epsilon}\right)$.

4. Numerical Results

In this section, we present some numerical results for the test problems (LCP), in order to get a feel of how the algorithm might perform in practice. Numerical results were obtained by using MATLAB R2009a, version 7.8.0.347, on an 32-bit system. We choose the parameters $\alpha=0.5$ and $\tau=0.001$. Furthermore, we take $q = e - Me$ to obtain an LCP problem with the starting interior point $(x^0, s^0) = (e, e)$. The algorithm terminates if the relative duality gap (Relgap) satisfies

$$\frac{x^T s}{1 + (x^0)^T s^0} \leq 10^{-8}.$$

The set of testing LCP problems are generated as follows. After one inputs any positive integer n, MATLAB generates an $n \times n$ matrix $A = \text{rand}(n)$ randomly. Then, we take $M = A^T A$ and $q = e - Me$. The numerical results presented in Table 1. To test the influence from the skewness of matrix M, we also test the LCPs generated as follows: $A = \text{rand}(n), B = \text{rand}(n), M = A^T A + (B - B^T)$ and $q = e - Me$. The numerical results of this set of problems are showed in Table 2. It turns out that the number of iterations (Iter.) and CPU time (seconds) is better than those required when M is purely positive semidefinite. Our preliminary implementations show that this algorithm is promising.

Table 1

n	Iter.	CPU	Relgap
100	7	0.1481	2.7653E-11
200	10	0.4998	1.2059E-13
250	7	0.6149	2.0529E-11
300	8	1.1113	1.8348E-12
500	8	3.9769	1.1422E-12
700	10	10.7753	1.9856E-12
900	8	16.1180	4.3346E-12
1000	9	24.6277	2.1069E-13
1300	9	48.1333	5.7763E-12

Table 2

n	Iter.	CPU	Relgap
100	4	0.0338	1.5245E-11
200	4	0.2617	5.9142E-12
250	4	0.3364	3.9534E-12
300	4	0.6456	3.9545E-12
500	4	1.9302	1.9822E-12
700	4	4.4852	1.9804E-12
900	4	8.1951	1.9779E-12
1000	4	10.8264	1.0021E-12
1300	4	21.3097	1.0014E-12

5. Conclusion

In this paper, we have extended the recently proposed second-order corrector interior-point algorithm of Liu et al. for LP to $P_*(\kappa)$-LCP and derived the iteration bound for the algorithm, namely, $O\Big((1+\kappa)^3\sqrt{n}\log(\frac{(x^0)^T s^0}{\epsilon})\Big)$. Moreover, we use the neighborhood $\widetilde{\mathcal{N}}_\tau^-(\alpha)$ for any value of $\alpha \in (0,1)$, while in [6] this neighborhood has been used only for $\alpha \in \Big(0, \frac{1}{2}\Big)$. Our algorithm does not use explicitly the handicap κ of the problem, and it can solve any LCPs. Furthermore, our preliminary numerical experiments show that the new algorithm may also perform well in practice.

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