Filomat 31:20 (2017), 6419–6429
https://doi.org/10.2298/FIL1720419G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Spectrum of Cayley Graphs Related to the Finite Groups

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Abstract. Let G be a finite group of order pqr where $p > q > r > 2$ are prime numbers. In this paper, we find the spectrum of Cayley graph $Cay(G, S)$ where $S \subseteq G \setminus \{e\}$ is a normal symmetric generating subset.

1. Introduction

Arthur Cayley in 1878 introduced the concept of Cayley graphs in terms of a group to explain the algebraic structures of abstract groups which are described by a set of generators. Recently, this theory has grown into an important branch in algebraic graph theory. The theory of Cayley graphs has some relations with many well-known problems in pure algebra such as classification, isomorphism and enumeration of Cayley graphs, (see for instance [11, 16]) and practical problems which are considered by graph and group theorists. Recently, many authors have studied Cayley graphs and there are a lot of results concerning spectrum of Cayley graphs. Babai was the first mathematician who considered the spectrum of Cayley graphs and in one of his papers [1], he explained how we can determine the eigenvalues of Cayley graphs. This exciting research topic is received increasing attention in recent years, see for example [3, 13]. Babai in that paper employed algebraic graph theory techniques, but computing the eigenvalues of Cayley graphs via the character table of related group is considered by Diaconis et al. in [7], for the first time. Following their method, we compute the spectrum of Cayley graphs of order pqr where $p > q > r > 2$ are prime numbers. In other words, let G be a group of order pqr, this note is concerned the construction of Cayley graph $\Gamma = Cay(G, S)$, where $S \subseteq G \setminus \{1\}$ is a normal symmetric generating subset. We accomplish the computation of eigenvalues of Γ, in three main steps. First, we compute the presentation of groups of order pqr. The second observation is to compute the character table of related groups. In section three, by using Theorem 2.4, we compute the spectrum of these Cayley graphs. Here our notation is standard and mainly taken from the standard books of graph theory and representation theory such as [2, 6, 9, 10, 15] as well as [3, 4, 12].

2. Definitions and Preliminaries

In this section, we introduce some basic notation and terminology used throughout the paper. A Frobenius group of order pq where p is prime and $q|p-1$ has the following presentation:

$$F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u \rangle, \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 05C10; Secondary 05C25, 20B25

Keywords. spectrum of graph, character table, Cayley graph.
Received: 30 April 2016; Accepted: 23 May 2017
Communicated by Francesco Belardo

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where u is an element of order q in multiplicative group $\mathbb{Z}_p^*$.

Let also G and H be two finite groups and $G \times H$ be direct product of G and H. Hölder in [8] introduced the presentation of groups of order pqr. By using his results, we can prove that all groups of order pqr where $p > q > r > 2$ are isomorphic with exactly one of the following structures:

- $G_1 = \mathbb{Z}_{pqr}$,
- $G_2 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
- $G_3 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$,
- $G_4 = F_{p,qr}(qr|p-1)$,
- $G_5 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
- $G_{i+5} = \langle a, b, c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i} \rangle$, where $r|p-1, q-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \leq i \leq r-1)$.

Let Γ be a simple graph with the adjacency matrix $A(\Gamma)$. The characteristic polynomial $\chi(\Gamma, \lambda)$ of $A(\Gamma)$ is defined as $\chi(\Gamma, \lambda) = |\lambda I - A|$ and the roots of this polynomial are called the spectrum of graph Γ, see [5]. By a circulant matrix, we mean a square $n \times n$ matrix whose rows are a cyclic permutation of the first row. A circulant matrix with the first row $[c_0, c_1, c_2, \cdots, c_{n-1}]$ is denoted by $[[c_0, c_1, c_2, \cdots, c_{n-1}]]$. In other words, if

$$\chi(\Gamma, \lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_s)^{m_s},$$

then the spectrum of graph Γ is $Spec(\Gamma) = \{[\lambda_1]^{m_1}, \cdots, [\lambda_s]^{m_s}\}$.

By a circulant graph, we mean a graph whose adjacency matrix is circulant. Since the spectrum of circulant matrices plays a significant role in the study of spectrum of Cayley graphs of order pqr, we recall some definition that will be used in the paper. For $\alpha = e^{\frac{2\pi}{n} i}$ (the n-th root of unity) all eigenvalues of circulant matrix $[[c_0, c_1, c_2, \cdots, c_{n-1}]]$ are given by

$$\lambda_j = c_0 + c_{n-1}\alpha^j + c_{n-2}\alpha^{2j} + \cdots + c_1\alpha^{(n-1)j}, \ \ 0 \leq j \leq n-1. \tag{2}$$

The Cartesian product $\Gamma_1 \Box \Gamma_2$ of two graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices $(u, v), (x, y) \in V(\Gamma_1 \Box \Gamma_2)$ are adjacent if and only if either $u = x$ and $(v, y) \in E(\Gamma_2)$ or $(u, x) \in E(\Gamma_1)$ and $v = y$.

Theorem 2.1. *[5] Let Γ_1 and Γ_2 be two graphs with eigenvalues $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_m$, respectively. Then for $1 \leq i \leq n$ and $1 \leq j \leq m$, all eigenvalues of $\Gamma_1 \Box \Gamma_2$ are $\lambda_i + \mu_j$.*

A symmetric subset of group G is a subset $S \subseteq G$, where $1 \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G, S)$ on G with respect to S is a graph with vertex set $V(\Gamma) = G$ and two vertices $x, y \in V(\Gamma)$ are adjacent if and only if $y = xs$ for an element $s \in S$. It is a well-known fact that $Cay(G, S)$ is connected if and only if S generates the group G, see [17].

Proposition 2.2. *[5]. Let $\Gamma_1 = Cay(G, \Delta_1)$ and $\Gamma_2 = Cay(H, \Delta_2)$ be two Cayley graphs. Then the Cartesian product $\Gamma_1 \Box \Gamma_2$ is the Cayley graph $Cay(G \times H, S)$, where*

$$S = \{(x, 1), (1, y) : x \in \Delta_1, y \in \Delta_2\} = (\Delta_1, 1) \cup (1, \Delta_2).$$

Let V be a vector space, a general linear group $GL(V)$ of V is the set of all $A \in End(V)$ where A is invertible. A representation of a group G is a homomorphism $\rho : G \to GL(V)$ and the degree of ρ is equal to the dimension of V. The representation $\rho : G \to \mathbb{C}^*$ is trivial if and only if for all $g \in G$, $\rho(g) = 1$. Let $\varphi : G \to GL(V)$ be a representation with $\varphi(g) = \varphi_g$, the character $\chi_\varphi : G \to \mathbb{C}^*$ afforded by φ is defined by setting $\chi_\varphi(g) = tr(\varphi_g)$. An irreducible character is the character of an irreducible representation and the character χ is linear, if $\chi(1) = 1$. The set of all irreducible characters of G is denoted by $Irr(G)$. It is a well-known fact that the number of irreducible characters of G is equal to the number of conjugacy classes of G and the number of linear characters of finite group G is $[G : G^{'}]$ where $G^{'}$ denotes the derivative subgroup of G.

A character table is a matrix whose rows are correspond to the irreducible characters, whereas the columns correspond to the conjugacy classes of G. The study of spectrum of Cayley graphs is closely related to irreducible characters of G. If G is abelian, the eigenvalues of the Cayley graph are easily determined as follows.

Theorem 2.3. *Let G be a finite abelian group and S be a symmetric subset of G. Then the eigenvalues of the adjacency matrix of $\Gamma = Cay(G, S)$ are given by*

$$\lambda_{\varphi} = \sum_{s\in S} \varphi(s)$$

where $\varphi \in Irr(G)$.

Let G be a finite group with symmetric subset S. We recall that S is normal subset if and only if $S^g = g^{-1}Sg = S$, for all $g \in G$. The following theorem is implicitly contained in [7, 14].

Theorem 2.4. *Let G be a finite group with a normal symmetric subset S. Let A be the adjacency matrix of the graph $\Gamma = Cay(G, S)$. Then the eigenvalues of A are given by*

$$\lambda_{\varphi} = \frac{1}{\varphi(1)} \sum_{s\in S} \varphi(s)$$

where $\varphi \in Irr(G)$. Moreover, the multiplicity of λ_{φ} is $\varphi(1)^2$.

Proposition 2.5. *[10] Let G and H be two finite groups with irreducible characters $\varphi_1, \varphi_2, \cdots, \varphi_r$ and $\eta_1, \eta_2, \cdots, \eta_s$, respectively. Let $M(G)$ and $M(H)$ be character tables of G and H, respectively. Then the direct product group $G \times H$ has exactly rs irreducible characters $\varphi_i\eta_j$, where $1 \leq i \leq r$ and $1 \leq j \leq s$. In particular, the character table of group $G \times H$ is*

$$M(G \times H) = M(G) \otimes M(H),$$

where $\otimes$ denotes the Kronecker product.

Before computing the spectrum of Cayley graphs of order pqr, we need to study the spectrum of $Cay(G, S)$ where G is isomorphic to one of the following groups that will serve as basic building blocks in the considered Cayley graphs in Section 3.3: the cyclic group $\mathbb{Z}_n$, Dihedral group D_{2n} and Frobenius group $F_{p,q}$. In what follows, assume that

$$\delta_A(B) = \begin{cases} 1 & A \subseteq B \\ 0 & A \nsubseteq B \end{cases}.$$

For $g \in G$, let g^G denotes the conjugacy class of g in G and $C_g = g^G \cup (g^{-1})^G$. It is clear that every normal subset of G is a union of its conjugacy classes. In other words, if S is a symmetric normal generating subset of G, then $S \subseteq \bigcup_{g\in G} C_g$ and all eigenvalues of Cayley graph $Cay(G, S)$ are as follows:

$$\lambda_{\chi} = \frac{1}{\chi(1)} \sum_{g\in G} \sum_{s\in C_g} \delta_{C_g}(S)|C_g|[\chi(s)],$$

where $\chi \in Irr(G)$.

Example 2.6. *Consider the cyclic group $\mathbb{Z}_n$ in two following cases:*
Case 1. *n is odd, thus $C_0 = \{1\}$ and $C_i = \{x^i, x^{-i}\}$ $(1 \le i \le \frac{n-1}{2})$ are non-trivial symmetric subsets of $\mathbb{Z}_n$, so*

$$S \subseteq \bigcup_{i=1}^{\frac{n-1}{2}} C_i.$$

For $0 \le j \le n-1$, $\chi_j(x^i) = \omega^{ij}$ are all irreducible characters of $\mathbb{Z}_n$, where x is a generator of $\mathbb{Z}_n$ and $\omega = e^{\frac{2\pi}{n}i}$. Hence

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}).$$

Case 2. *n is even, hence all non-trivial symmetric subsets are*

$$C_0 = \{1\},\ C_i = \{x^i, x^{-i}\}\ (1 \le i \le \frac{n}{2} - 2) \text{ and } C_{\frac{n}{2}-1} = \{x^{n/2}\}.$$

Therefore,

$$S \subseteq \bigcup_{i=1}^{\frac{n}{2}-1} C_i.$$

Similar to the last case, we have

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n}{2}-2} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}) + (-1)^j \delta_{C_{\frac{n}{2}-1}}(S).$$

Example 2.7. *Here we determine the spectrum of $Cay(D_{2n}, S)$ where S is normal symmetric subset. In finding the number of conjugacy classes of dihedral group, it is convenient to consider two separately cases:*
Case 1. *n is odd, then D_{2n} has precisely $\frac{1}{2}(n+3)$ conjugacy classes:*

$$1^G = \{1\}, (a^i)^G = \{a^i, a^{-i}\}\ (1 \le i \le (n-1)/2), b^G = \{b, ba, \cdots, ba^{n-1}\}.$$

Hence the non-trivial symmetric subsets of D_{2n} are

$$C_i = (a^i)^G,\ \ (1 \le i \le \frac{n-1}{2}) \text{ and } C_{\frac{n+1}{2}} = b^G.$$

This implies that $S \subseteq \bigcup_{i=1}^{\frac{n+1}{2}} C_i$ and so by using Table 1, we have

$$\begin{aligned}
\lambda_{\chi_1} &= n\delta_{C_{\frac{n+1}{2}}}(S) + 2\sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S), \\
\lambda_{\chi_2} &= -n\delta_{C_{\frac{n+1}{2}}}(S) + 2\sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S), \\
\lambda_{\psi_j} &= \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\epsilon^{ij} + \epsilon^{-ij})\ (1 \le j \le \frac{n-1}{2}),
\end{aligned}$$

where $\epsilon = e^{\frac{2\pi}{n}i}$.

Case 2. *n is even, then D_{2n} has precisely $\frac{n}{2}+3$ conjugacy classes ($0 \le j \le n-1$):*

$$1^G = \{1\}, (a^{\frac{n}{2}})^G, (a^i)^G, (ba^{2j})^G, (ba^{2j+1})^G.$$

So the non-trivial symmetric subsets of D_{2n} are:

$$C_i = (a^i)^G, \ \ (1 \le i \le \frac{n}{2}-1), C_{\frac{n}{2}} = (a^{n/2})^G, C_{\frac{n}{2}+1} = b^G \text{ and } C_{\frac{n}{2}+2} = (ba)^G.$$

Hence $S \subseteq \bigcup_{i=1}^{\frac{n}{2}+2} C_i$ and by using Table 2, we have

$$\begin{aligned}
\lambda_{\chi_1} &= \delta_{C_{\frac{n}{2}}}(S) + \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) + \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S),\\
\lambda_{\chi_2} &= \delta_{C_{\frac{n}{2}}}(S) - \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) + \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S),\\
\lambda_{\chi_3} &= (-1)^{\frac{n}{2}}\delta_{C_{\frac{n}{2}}}(S) + \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) - \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S)(-1)^j,\\
\lambda_{\chi_4} &= (-1)^{\frac{n}{2}}\delta_{C_{\frac{n}{2}}}(S) - \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) - \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S)(-1)^j,\\
\lambda_{\psi_j} &= (-1)^j\delta_{C_{\frac{n}{2}}}(S) + \sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S)(\epsilon^{ij} + \epsilon^{-ij}) \ (1 \le j \le \frac{n}{2}-1).
\end{aligned}$$

As a special case, one of the minimal symmetric normal generating subset of group D_{2n} is

$$S = \begin{cases} b^G \cup \{a, a^{-1}\} & 2|n \\ b^G & 2 \not| n \end{cases}.$$

Hence the spectrum of Cayley graph $\Gamma = Cay(D_{2n}, S)$ when $2 \nmid n$ is $\{[-n]^1, [n]^1, [0]^{2n-2}\}$ and when $2|n$ is as follows:

$$\{[\pm n/2 \pm 2]^1, [0]^{2n-4}\}$$

.

g	1	a^r	b
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$\epsilon^{jr}+\epsilon^{-jr}$	0

Table 1. *The character table of D_{2n} where n is odd and $1 \le r, j \le \frac{n-1}{2}$.*

g	1	$a^{\frac{n}{2}}$	a^r	b	ba
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^{\frac{n}{2}}$	$(-1)^r$	1	-1
χ_4	1	$(-1)^{\frac{n}{2}}$	$(-1)^r$	-1	1
ψ_j	2	$2(-1)^j$	$\epsilon^{jr}+\epsilon^{-jr}$	0	0

Table 2. *The character table of D_{2n} where n is even and $1 \le r, j \le \frac{n}{2}-1$.*

Since all eigenvalues of $\Gamma = Cay(D_{2n}, S)$ are symmetric with respect to the origin, according to [5, Theorem 3.2.3] Γ is bipartite.

Example 2.8. *For the Frobenius group $F_{p,q}$ introduced in section one, let L be the subgroup of $\mathbb{Z}_p^*$ consisting of the powers of u. Write $t = (p-1)/q$, and choose coset representatives $v_1, \cdots, v_t$ for L in $\mathbb{Z}_p^*$. By applying [11, Proposition 25.9], the conjugacy classes of $F_{p,q}$ are*

$$\{1\},$$
$$(a^{v_i})^G = \{a^{v_i l} : l \in L\}\ (1 \le i \le t)$$
$$(b^n)^G = \{a^m b^n :\ 0 \le m \le p-1\}(1 \le n \le q-1).$$

It follows that the Frobenius group $F_{p,q}$ has precisely

1. *q linear characters $\chi_n (0 \le n \le q-1)$, where $\chi_n(a^x b^y) = \omega^{ny}$, $\omega = e^{2\pi i/q}, 0 \le x \le p-1$ and $0 \le y \le q-1$.*
2. *t characters of degree q given by*

$$\begin{aligned} \varphi_j(a^x b^y) &= 0, \qquad 1 \le y \le q-1 \\ \varphi_j(a^x) &= \sum_{l \in L} \alpha^{v_j l x}, \quad 1 \le x \le p-1 \end{aligned}$$

where $\alpha = e^{\frac{2\pi}{p} i}$, $1 \le j \le t$, $1 \le x \le p-1$ and $v_1 L, \cdots, v_t L$ are the cosets in $\mathbb{Z}_p^*$ of the subgroup L.

The non-trivial symmetric subsets of $F_{p,q}$ are

$$C_n = (b^n)^G \cup (b^{-n})^G,\ (1 \le n \le (q-1)/2) \text{ and } C_{n+i} = (a^{v_i})^G \cup (a^{-v_i})^G,\ (1 \le i \le t).$$

Hence $S \subseteq \bigcup_{i=1}^{(q+t-1)/2} C_i$ and so all eigenvalues of $\Gamma = Cay(F_{p,q}, S)$ are as follows:

$$\begin{aligned} \lambda_{\chi_m} &= p \sum_{n=1}^{(q-1)/2} \delta_{C_n}(S)(\omega^{nm} + \omega^{-nm}) + 2q \sum_{i=1}^{t/2} \delta_{C_{n+i}}(S)\ (0 \le m \le q-1), \\ \lambda_{\varphi_j} &= \sum_{i=1}^{t/2} \delta_{C_{n+i}}(S) \sum_{l \in L} (\alpha^{v_i v_j l} + \alpha^{-v_i v_j l})\ (1 \le j \le t). \end{aligned}$$

Here we determine a minimal normal symmetric generating subset S of $F_{p,q}$ such that $F_{p,q} = \langle S \rangle$. Since $b^G = \{a^m b : 0 \le m \le p-1\}$, by putting $m = 0$, it follows that $b \in \langle b^G \rangle$ and consequently $a \in \langle b^G \rangle$. Hence $F_{p,q} = \langle a, b \rangle \subseteq \langle b^G \rangle$ and thus $F_{p,q} = \langle b^G \rangle$. Since $S^{-1} = S$, then necessarily $S = b^G \cup (b^{-1})^G$. According to Theorem 2.4, we have:

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s) = \frac{|b^G|\chi(b) + |(b^{-1})^G|\chi(b^{-1})}{\chi(1)} = \frac{p(\chi(b) + \chi(b^{-1}))}{\chi(1)}$$

for all $\chi \in Irr(F_{p,q})$. Hence the spectrum of $F_{p,q}$ is $\{[p(\omega^j + \omega^{-j}]^1, [0]^{tq^2}\}$ where $(0 \le j \le q-1)$.

3. Main Results and Discussions

Following Example 2.8, the aim of this section is to compute the spectrum of Cayley graphs of order pqr where $p > q > r > 2$ are prime numbers. To do this, at first we determine the character tables of all groups of order pqr.

3.1. Character table of groups $G_1 - G_5$

Let G be a cyclic group of order n, then all irreducible characters of G are linear and for $1 \le i, j \le n$, we have $\chi_i : G \to C$ with $\chi_i(a^j) = \epsilon^{ij}$ where $\epsilon = e^{\frac{2\pi}{n} i}$. This implies that in this case, all irreducible characters of G_1 can be computed by putting $n = pqr$. By using Proposition 2.5 and Example 2.6, the character table of groups $\mathbb{Z}_r \times F_{p,q}$, $\mathbb{Z}_q \times F_{p,r}$ and $\mathbb{Z}_p \times F_{q,r}$ are $CT(\mathbb{Z}_r) \otimes CT(F_{p,q})$, $CT(\mathbb{Z}_q) \otimes CT(F_{p,rp})$ and $CT(\mathbb{Z}_p) \otimes CT(F_{q,r})$, respectively. Finally, the character table of G_4 can be computed directly from Example 2.8.

3.2. Character table of $G_{i+5}(1 \leq i \leq r-1)$

Here we compute the character table of group G_6 and the others can be computed similarly. Let $G = G_6$, first we compute the conjugacy classes of G. Let $U = \langle u \rangle$ and $V = \langle v \rangle$ be the subgroups of order r of $\mathbb{Z}_q^*$ and $\mathbb{Z}_p^*$, respectively.

Lemma 3.1. *The conjugacy classes of G are*

$$\{1\}, (a^{v_i})^G, (b^{u_i})^G, (c^i)^G, (b^{u_i'}a^{v_i'})^G$$

where u_i is a coset representative of U in $\mathbb{Z}_q^*$, v_i is a coset representative of V in $\mathbb{Z}_p^*$ and (u_i', v_i') is a coset representative of $\langle (u,v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$.

Proof. It is easy to see that for $1 \leq k \leq r-1$, $c^{-k}bc^k = b^{u^k}$ and so b^{u^i}'s are conjugate and so $|b^G| \geq r$. On the other hand, $\langle ba \rangle \leq C_G(b)$ and hence $|C_G(b)| \geq pq$. This implies that $|b^G| \leq r$ and thus $|b^G| = r$. Further, one can prove that $(b^{u_i})^G(1 \leq i \leq \frac{q-1}{r})$ and $(a^{v_j})^G(1 \leq j \leq \frac{p-1}{r})$ are conjugacy classes of G. We can prove that

$$\begin{aligned} c^G &= \{cb^ia^j |\, 0 \leq i \leq q-1, 0 \leq j \leq p-1\}, \\ &\vdots \\ (c^{r-1})^G &= \{c^{r-1}b^ia^j |\, 0 \leq i \leq q-1, 0 \leq j \leq p-1\}, \\ (b^{u_i'}a^{v_i'})^G &= \{b^{u_i'}a^{v_i'}, b^{u_i'u}a^{v_i'v}, \cdots, b^{u_i'u^{r-1}}a^{v_i'v^{r-1}}\} \end{aligned}$$

where (u_i', v_i') is a coset representative of $\langle (u,v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$ and $|\langle (u,v) \rangle| = r$.

It follows from Lemma 3.1 that G has $\frac{p-1}{r} + \frac{q-1}{r} + \frac{(p-1)(q-1)}{r} + r$ conjugacy classes and then the same number of irreducible characters. On the other hand, $G/G' \cong \langle c | c^r = 1 \rangle \cong \mathbb{Z}_r$. Hence G has r linear characters lifted from linear characters of G/G'. These characters are as $\tilde{\chi}_n : G/G' \to \mathbb{C}^*$ with $\tilde{\chi}_n(c^mG') = \epsilon^{mn}$ where $\epsilon = e^{\frac{2\pi i}{r}}$ and $m, n \in \{0, 1, \cdots, r-1\}$.

According to [11, Theorem 17.11], all linear characters of G are as $\chi_n : G \to \mathbb{C}^*$ with $\chi_n(g) = \tilde{\chi}_n(gG')$. Hence

$$\begin{aligned} \chi_n(a^w) &= \tilde{\chi}_n(a^wG') = \tilde{\chi}_n(G') = \chi_n(1) = 1, \\ \chi_n(b^v) &= \tilde{\chi}_n(b^vG') = \tilde{\chi}_n(G') = \chi_n(1) = 1, \\ \chi_n(b^{v_0}a^{w_0}) &= \tilde{\chi}_n(b^{v_0}a^{w_0}G') = \tilde{\chi}_n(G') = \chi_n(1) = 1, \\ \chi_n(c^t) &= \tilde{\chi}_n(c^tG') = \epsilon^{tn}(0 \leq n \leq r-1 \; and \; 1 \leq t \leq r-1), \end{aligned}$$

where (v_0, w_0) is a coset representative of $\langle (u,v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$.

Here we determine all non-linear irreducible characters of G. First notice that $H = \langle a \rangle$ is a normal subgroup of G and if $u^r \equiv 1 (mod\ q)$, then

$$G/H \cong \langle b, c |\, b^q = c^r = 1, c^{-1}bc = b^u \rangle \cong F_{q,r}.$$

According to [11, Theorem 25.10], the Frobenius group $F_{q,r}$ has r linear characters and $\frac{q-1}{r}$ irreducible characters of degree r. Let us denote the non-linear characters by $\tilde{\varphi}_m$. Then we have:

$$\begin{aligned} \tilde{\varphi}_m(H) &= r, \\ \tilde{\varphi}_m(b^xH) &= \sum_{i=0}^{r-1} \lambda^{u_m x u^i} (1 \leq m \leq \frac{q-1}{r}, 1 \leq x \leq q-1), \\ \tilde{\varphi}_m(b^xc^yH) &= 0 \; (1 \leq y \leq r-1), \end{aligned}$$

where $\lambda = e^{\frac{2\pi i}{q}}$ and $u_1, \cdots, u_m$ are distinct coset representative of $U = \langle u \rangle$ in $\mathbb{Z}_q^*$. By lifting these characters, we can compute $\frac{q-1}{r}$ irreducible characters of G of degree r denoted by $\varphi_m(1 \leq m \leq \frac{q-1}{r})$, e.g.

$$\begin{aligned}
\varphi_m(a^x) &= r \ (0 \leq x \leq p-1), \\
\varphi_m(b^y a^x) &= \sum_{i=0}^{r-1} \lambda^{u_m y u^i} (0 \leq x \leq p-1, 1 \leq y \leq q-1), \\
\varphi_m(c^k) &= 0 \ (1 \leq k \leq r-1).
\end{aligned}$$

Similarly, for the normal subgroup $K = \langle b \rangle$ of G, we have:

$$G/K \cong \langle a, c | \, a^p = c^r = 1, c^{-1}ac = a^v \rangle \cong F_{p,r}.$$

Consequently, this group has $\frac{p-1}{r}$ irreducible characters of degree r denoted by $\tilde{\theta}_l(1 \leq l \leq \frac{p-1}{r})$. Similar to the last discussion, the irreducible characters of G lifted from $\tilde{\theta}_l$ are as follows:

$$\begin{aligned}
\theta_l(a^x) &= \theta_l(b^y a^x) = \sum_{i=0}^{r-1} \gamma^{v_l x v^i}, \\
\theta_l(b^y) &= r, \\
\theta_l(c^k) &= 0 \ (1 \leq k \leq r-1),
\end{aligned}$$

where $\gamma = e^{\frac{2\pi i}{p}}$ and $v_1, \cdots, v_l$ are distinct coset representative of $V = \langle v \rangle$ in $\mathbb{Z}_p^*$.

Finally, by considering subgroup $G' = \langle ba \rangle \cong \mathbb{Z}_q \times \mathbb{Z}_p$, its irreducible characters are of the form $\psi_i \xi_j (0 \leq i \leq q-1, 0 \leq j \leq p-1)$ and

$$\psi_i(b^y) = \lambda^{iy}, \xi_j(a^x) = \gamma^{jx}.$$

This leads us to conclude that

$$\psi_i \xi_j (b^y a^x) = \psi_i(b^y) \xi_j(a^x) = \lambda^{iy} \gamma^{jx}.$$

Let now $m \in \mathbb{Z}_q^*$ and $n \in \mathbb{Z}_p^*$, then

$$(\psi_m \xi_n \uparrow G)(1) = \frac{|G|}{|\langle ba \rangle|} (\psi_m \xi_n)(1) = \frac{pqr}{pq} = r.$$

On the other hand,

$$\begin{aligned}
|C_G(b^y)| &= |C_{G'}(b^y)| = |C_G(a^x)| = |C_{G'}(a^x)| = |C_G(b^y a^x)| \\
&= |C_{G'}(b^y a^x)| = pq
\end{aligned}$$

and so

$$\begin{aligned}
(\psi_m \xi_n \uparrow G)(a^x) &= \sum_{i=0}^{r-1} \xi_n(a^{xv^i}) = \sum_{i=0}^{r-1} \gamma^{nxv^i}, \\
(\psi_m \xi_n \uparrow G)(b^y) &= \sum_{i=0}^{r-1} \psi_m(b^{yu^i}) = \sum_{i=0}^{r-1} \lambda^{myu^i}, \\
(\psi_m \xi_n \uparrow G)(b^y a^x) &= \sum_{i=0}^{r-1} \psi_m(b^{yu^i}) \xi_n(a^{xv^i}) = \sum_{i=0}^{r-1} \lambda^{myu^i} \gamma^{nxv^i} \\
(\psi_m \xi_n \uparrow G)(c^k) &= 0 \ (k = 1, \cdots, r-1).
\end{aligned}$$

Since

$$\psi_m \xi_n \uparrow G = \psi_{mu^i} \xi_{nv^i} \uparrow G$$

we get $z = \frac{(p-1)(q-1)}{r}$ irreducible characters of G. There still remains the question as to whether such characters are distinct irreducible. Assume $(u_i^{'}, v_i^{'})$ be a coset representative of subgroup $\{(1,1),(u,v),\cdots,(u^{r-1},v^{r-1})\}$ of $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$ and $\eta_j = \psi_{u_j^{'}}\xi_{v_j^{'}} \uparrow G$. According to Frobenius Reciprocity Theorem, for $H = G^{'} = \langle ba \rangle$ we verify:

$$\begin{aligned}\langle \eta_j \downarrow H, \psi_{u_j^{'}u^i}\xi_{v_j^{'}v^i}\rangle_H &= \langle \eta_j, \psi_{u_j^{'}}\xi_{v_j^{'}} \uparrow G\rangle_G \\ &= \langle \eta_j, \eta_j\rangle_G.\end{aligned}$$

Therefore, we can observe that

$$\eta_j \downarrow H = \langle \eta_j, \eta_j\rangle_G(\sum_{i=0}^{r-1} \psi_{u_j^{'}u^i}\xi_{v_j^{'}v^i}) + \chi$$

where $\chi = 0$ or it is a character of H. Hence $\eta_j(1) \geq r\langle \eta_j, \eta_j\rangle_G$. Finally, $\eta_j(1) = r$ implies that $\langle \eta_j, \eta_j\rangle_G = 1$ and so η_j is irreducible. On the other hand, for $(u_j^{'}, v_j^{'}) \in \mathbb{Z}_q^* \times \mathbb{Z}_p^*$, all $\psi_{u_j^{'}}\xi_{v_j^{'}}$'s are linearly independent and thus all $\eta_j \downarrow H$ $(1 \leq j \leq \frac{(p-1)(q-1)}{r})$ are distinct. Consequently, the irreducible characters $\eta_1, \cdots \eta_z$ are distinct. We summarize the character table of G in the following theorem.

Theorem 3.2. *Let $p > q > r > 2$ be prime numbers, $l_1 = \frac{(p-1)(q-1)}{r}, l_2 = \frac{p-1}{r}, l_3 = \frac{q-1}{r}$ and $\epsilon = e^{\frac{2\pi i}{r}}$. Then the group G has $l_1 + l_2 + l_3 + r$ irreducible characters as reported in Table 3:*

g	1	a^{v_i} $1 \leq i \leq l_1$	b^{u_i} $1 \leq i \leq l_2$	$b^{u_i^{'}}a^{v_i^{'}}$ $1 \leq i \leq l_3$	c^k $1 \leq k \leq r-1$
χ_n $0 \leq n \leq r-1$	1	1	1	1	ϵ^{kn}
η_s $1 \leq s \leq l_3$	r	E	F	G	0
θ_l $1 \leq 1 \leq l_1$	r	C	r	D	0
φ_m $1 \leq m \leq l_2$	r	r	A	B	0

Table 3. *The character table of group G.*

where $\lambda = e^{\frac{2\pi i}{q}}, u_1, \cdots, u_{l_1}$ are distinct coset representative of $U = \langle u \rangle$ in $\mathbb{Z}_q^*$, $v_1, \cdots, v_{l_2}$ are distinct coset representative of $V = \langle v \rangle$ in $\mathbb{Z}_p^*$, $(u_i^{'}, v_i^{'})$ are coset representative of $\langle (u,v) \rangle$ in $\mathbb{Z}_q^* \times \mathbb{Z}_p^*$ and

$$\begin{aligned} A &= \sum_{j=1}^{r} \lambda^{u_m u_i u^j}, B = \sum_{j=1}^{r} \lambda^{u_m u_i^{'} u^j}, C = \sum_{j=1}^{r} \gamma^{v_l v_i v^j}, D = \sum_{j=1}^{r} \gamma^{v_l v_i^{'} v^j}, \\ E &= \sum_{j=1}^{r} \gamma^{v_s^{'} v_i v^j}, F = \sum_{j=1}^{r} \lambda^{u_s^{'} u_i u^j}, G = \sum_{j=1}^{r} \lambda^{u_s^{'} u_i^{'} u^j} \gamma^{v_s^{'} v_i^{'} v^j}. \end{aligned}$$

and $1 \leq l \leq l_1$, $1 \leq m \leq l_2$, $1 \leq s \leq l_3$, $1 \leq n \leq r-1$.

3.3. Spectrum of Cayley graphs via their character tables

In this section, we introduce one of the major applications of Tables 1-3: computing the spectrum of Cayley graphs on groups of orders pqr. First, we compute the normal symmetric generating subset of G and then, by applying Theorem 2.4, we compute the spectrum of $Cay(G,S)$ in terms of minimal normal symmetric generating subset S. Let $l_1 = \frac{p-1}{r}$, $l_2 = \frac{q-1}{r}$, $l_3 = \frac{(p-1)(q-1)}{r}$ and $l = l_1 + l_2 + l_3$, then the non-trivial symmetric subsets of $G = G_6$ are

$$\begin{aligned} C_i &= (a^{v_i})^G \cup (a^{-v_i})^G \ (1 \leq i \leq l_1/2),\ C_{l_1+j} = (b^{u_j})^G \cup (b^{-u_j})^G \ (1 \leq j \leq l_2/2), \\ C_{l_1+l_2+k} &= (b^{u_k^{'}a^{v_k^{'}}})^G \cup (b^{-u_k^{'}a^{v_k^{'}}})^G, (1 \leq k \leq l_3/2), \\ C_{l+t} &= (c^t)^G \cup (c^{-t})^G \ (1 \leq t \leq \frac{r-1}{2}). \end{aligned}$$

Hence $S \subseteq \bigcup_{i=1}^{l} C_i$ and so we have

$$\begin{aligned}
\lambda_{\chi_n} &= 2r\sum_{i=1}^{l/2}\delta_{C_i}(S) + pq\sum_{i=0}^{(r-1)/2}\delta_{C_{l+i}}(S)(\epsilon^{ni}+\epsilon^{-ni}) \quad (0 \le n \le r-1),\\
\lambda_{\varphi_m} &= 2r\sum_{i=1}^{l_1/2}\delta_{C_i}(S) + \sum_{j=1}^{l_2/2}\delta_{C_{l_1+j}}(S)(A+\bar{A}) + \sum_{j=1}^{l_3/2}\delta_{C_{l_1+l_2+j}}(S)(B+\bar{B}) \quad (1 \le m \le l_2),\\
\lambda_{\theta_k} &= \sum_{i=1}^{l_1/2}\delta_{C_i}(S)(C+\bar{C}) + 2r\sum_{j=1}^{l_2/2}\delta_{C_{l_1+j}}(S) + \sum_{i=1}^{l_3/2}\delta_{C_{l_1+l_2+j}}(S)(D+\bar{D}) \quad (1 \le k \le l_1),\\
\lambda_{\eta_s} &= \sum_{i=1}^{l_1/2}\delta_{C_i}(S)(E+\bar{E}) + \sum_{j=1}^{l_2/2}\delta_{C_{l_1+j}}(S)(F+\bar{F}) + \sum_{j=1}^{l_3/2}\delta_{C_{l_1+l_2+j}}(S)(G+\bar{G}),
\end{aligned}$$

where $\alpha = e^{\frac{2\pi}{p}i}$, $T = \langle r \rangle$ and $v_1T, \cdots, v_tT$ are coset representatives in $\mathbb{Z}_p^*$.

Theorem 3.3. *The minimal normal symmetric generating subset of groups G_{5+i} $(1 \le i \le r-1)$ is $S = c^G \cup (c^{-1})^G$.*

Proof. By using Lemma 3.1, it is easy to see that $(a^{v_i})^G$, $(b^{u_i})^G$ and $(b^{u_i'}, a^{v_i'})^G$ do not generate G_6. We show $S = c^G \cup (c^{-1})^G$ satisfies in conditions of the theorem. Since $c, cb \in S$, then $b \in \langle S \rangle$ and then $a \in \langle S \rangle$. This implies that S is a generating set. On the other hand, S is the union of two conjugacy classes and so it is normal. Also, $c^{-1} \in S$ implies that S is symmetric. This completes the proof.

Corollary 3.4. *Let $\Gamma_i = Cay(G_i, S_i)(1 \le i \le r+4)$ $G_1, \cdots, G_{r+4}$ introduced in Section 1 and S_i be a minimal normal symmetric generating subset of G_i. Then*

1. *All eigenvalues of Γ_1 are*
$$\{[\omega^j + \omega^{-j}]^1\}$$
where $\omega = e^{\frac{2\pi}{pqr}i}$ and $0 \le j \le pqr-1$.
2. *All eigenvalues of Γ_2 are*
$$\{[\zeta^i + p(\alpha^j + \alpha^{-j})]^1, [\zeta^i]^{tq^2}\}$$
where $t = (p-1)/q, \alpha = e^{\frac{2\pi i}{q}}, \zeta = e^{\frac{2\pi i}{r}}, 0 \le j \le q-1$ and $0 \le i \le r-1$.
3. *All eigenvalues of Γ_3 are*
$$\{[\xi^i + p(\alpha^j + \alpha^{-j})]^1, [\xi^i]^{tr^2}\}$$
where $t = (p-1)/r, \alpha = e^{\frac{2\pi i}{r}}, \xi = e^{\frac{2\pi i}{q}}, 0 \le j \le r-1$ and $0 \le i \le q-1$.
4. *All eigenvalues of Γ_4 are*
$$\{[p(\alpha^j + \alpha^{-j})]^1, [0]^{tr^2q^2}\},$$
where $t = (p-1)/rq, \alpha = e^{\frac{2\pi i}{rq}}$ and $0 \le j \le rq-1$.
5. *All eigenvalues of Γ_5 are*
$$\{[\varsigma^i + q(\alpha^j + \alpha^{-j})]^1, [\varsigma^i]^{tr^2}\}$$
where $t = (q-1)/r, \alpha = e^{\frac{2\pi i}{r}}, \varsigma = e^{\frac{2\pi i}{p}}, 0 \le j \le r-1$ and $0 \le i \le p-1$.
6. *For $0 \le m \le (q-1)/r$, the spectrum of graphs $\Gamma_{5+i}(1 \le i \le r-1)$ are as follows:*
$$\{[pq(\epsilon^n + \epsilon^{-n})]^1, [0]^{tr^2}\},$$
where $\epsilon = e^{2\pi i/r}$, $t = \frac{pq-1}{r}$ and $0 \le n \le r-1$.

Proof. Let $\mathbb{Z}_{pqr} = \langle x \rangle$ and $S = \{x, x^{-1}\}$ where $(o(x), pqr) = 1$. Clearly, S is a minimal normal symmetric generating subset and so $\Gamma_1 = Cay(\mathbb{Z}_{pqr}, S)$. One can easily prove that $\Gamma_1 \cong C_{pqr}$ where C_n denotes a cycle on n vertices. This implies that the adjacency matrix of Γ_1 is a circulant matrix with first row $[0, 1, 0, \cdots, 0, 1]$. Now all eigenvalues of Γ_1 can be computed directly from Eq.(2). By using Theorem 2.1 and Example 2.6, we can compute eigenvalues of group Γ_2, Γ_3 and Γ_5. By using Example 2.8, the eigenvalues of Γ_4 is computed. Finally, by using Proposition 2.2 and Theorems 2.4 and 3.2 all eigenvalues of Γ_6 are as computed.

Corollary 3.5. *Let $p > q > r > 2$ are prime numbers. There are infinite family of co-spectral Cayley graphs of order pqr by the following spectrum*

$$\{pq(\epsilon^n + \epsilon^{-n})]^1, [0]^{tr^2}\}$$

where $\epsilon = e^{2\pi i/r}$, $t = \frac{pq-1}{r}$ and $0 \leq n \leq r-1$.

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