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Weak Solutions for a Second Order Impulsive Boundary Value Problem

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Abstract. In this paper we use topological degree theory and critical point theory to investigate the existence of weak solutions for the second order impulsive boundary value problem

$$\begin{cases} -x''(t) - \lambda x(t) = f(t), t \neq t_j, t \in (0, \pi), \\ \Delta x'(t_j) = x'(t_j^+) - x'(t_j^-) = I_j(x(t_j)), j = 1, 2, \dots, p, \\ x(0) = x(\pi) = 0, \end{cases}$$

where λ is a positive parameter, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = \pi$, $f \in L^2(0, \pi)$ is a given function and $I_j \in C(\mathbb{R}, \mathbb{R})$ for $j = 1, 2, \dots, p$.

1. Introduction

Consider the second order impulsive boundary value problem

$$\begin{cases} -x''(t) - \lambda x(t) = f(t), t \neq t_j, t \in (0, \pi), \\ \Delta x'(t_j) = x'(t_j^+) - x'(t_j^-) = I_j(x(t_j)), j = 1, 2, \dots, p, \\ x(0) = x(\pi) = 0, \end{cases} \quad (1)$$

where λ is a positive parameter, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = \pi$, $f \in L^2(0, \pi)$ is a given function and $I_j \in C(\mathbb{R}, \mathbb{R})$ for $j = 1, 2, \dots, p$.

Variational methods and critical point theory were used by many authors to study the existence and subsequent qualitative properties of solutions for differential equations; see for example [1-9] and the references therein.

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In [1], Zhang and Dai studied impulsive differential equations with periodic boundary conditions

$$\begin{cases} -u''(t) + cu(t) = \lambda f(t, u(t)), t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), j = 1, 2, \dots, p-1, \\ u(0) = u(T) = 0, u'(0^+) = u'(T^-), \end{cases} \tag{2}$$

where the nonlinearity f and the impulsive functions I_j are superlinear. Using a $\mathbf{Z}_2$ version of the mountain pass theorem, the authors obtained some existence results on infinitely many solutions for (2).

In [2], Xu et al. studied the p-Laplacian Dirichlet boundary value problem with impulses

$$\begin{cases} -(|u'|^{p-2}u')' = f(t, u), \text{ in } \Omega, \\ \Delta |u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)), j = 1, 2, \dots, n, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where $\Omega = (0, 1) \backslash \{t_1, \dots, t_n\}$. Using (S_+)-type topological degree theory the existence of a weak solution for (3) for the nonresonance case was obtained.

In [3], P. Drábek and M. Langerová studied the Dirichlet boundary value problem for the one-dimensional p-Laplacian

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' - \lambda |u(x)|^{p-2}u(x) = f(x), \text{ for a.e. } x \in (0, 1), \\ \Delta_p u'(x_j) = I_j(u(x_j)), j = 1, 2, \dots, r, \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

Using a linking theorem, the authors obtained the existence of a solution for (4) for the resonance case using the Landesman-Lazer condition (for example [3,(5),(6)], [7,(2.1)], [8,(16)]).

In this paper we use topological degree theory and critical point theory to investigate the existence of weak solutions for (1). We assume the following condition for f and I_j:

(H) $f \in L^2(0, \pi)$ is a given function and for $j = 1, 2, \dots, p$, $I_j \in C(\mathbb{R}, \mathbb{R})$ are strictly decreasing, and have finite limits $\lim_{s\to\infty} I_j(s)$ (which we call $I_j(\infty)$), $\lim_{s\to-\infty} I_j(s)$ (which we call $I_j(-\infty)$) such that

$$I_j(+\infty) < I_j(s) < I_j(-\infty), \forall s \in \mathbb{R}, j = 1, 2, \dots, p.$$

In the future it would be of interest to continue this line of research and discuss qualitative properties of weak solutions of (1).

2. Preliminary Results

Let us recall some basic concepts. In the Sobolev space $H := H_0^1(0, \pi)$, consider the inner product

$$(x, y) = \int_0^\pi x'(t)y'(t)\mathrm{d}t, \forall x, y \in H. \tag{5}$$

Consequently, the corresponding norm is

$$||x|| = \left(\int_0^\pi |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}, \forall x \in H. \tag{6}$$

It is easy to prove that, if $\lambda > 0$, the eigenvalue problem

$$\begin{cases} -x''(t) = \lambda x(t), \\ x(0) = x(\pi) = 0 \end{cases} \tag{7}$$

has nontrivial solutions, which can be written in the form $x(t) = c_1 \cos \sqrt{\lambda} t + c_2 \sin \sqrt{\lambda} t,$ for some $c_i \in \mathbb{R}, i = 1, 2$. Note that the boundary conditions, $x(0) = 0$ implies $c_1 = 0$, and then $c_2 \neq 0$. Hence, $x(\pi) = 0$ implies $\sin \sqrt{\lambda}\pi = 0$, and $\lambda = n^2, n = 1, 2, \ldots$.

Consequently, the eigenvalues of (7) are numbered by $1 = \lambda_1 < 4 = \lambda_2 < \cdots < n^2 = \lambda_n < \cdots \to +\infty$ (counted with their multiplicities) and a corresponding system of eigenfunctions $\{\sin nt\}$ forms the completely orthogonal basis of H. Let $Y = \text{span}\{\sin t\}, Z = Y^{\perp}$. Then $Z = \text{span}\{\sin 2t, \ldots, \sin nt, \ldots\}$ and

$$\int_0^{\pi} |z(t)|^2 \mathrm{d}t \leq \frac{1}{4} \int_0^{\pi} |z'(t)|^2 \mathrm{d}t, \forall z \in Z. \tag{8}$$

Next, we give a simple proof for this inequality. For $z \in Z$, there exist $a_k \in \mathbb{R} (k = 2, 3, \ldots)$ such that

$$z(t) = \sum_{k=2}^{\infty} a_k \sin kt, \text{ and } \int_0^{\pi} |z(t)|^2 \mathrm{d}t = \frac{\pi}{2} \sum_{k=2}^{\infty} a_k^2.$$

From this, we obtain that

$$z'(t) = \sum_{k=2}^{\infty} k a_k \cos kt, \text{ and } \int_0^{\pi} |z'(t)|^2 \mathrm{d}t = \frac{\pi}{2} \sum_{k=2}^{\infty} k^2 a_k^2.$$

As a result,

$$\int_0^{\pi} |z'(t)|^2 \mathrm{d}t \geq 4 \times \frac{\pi}{2} \sum_{k=2}^{\infty} a_k^2 = 4 \int_0^{\pi} |z(t)|^2 \mathrm{d}t.$$

In what follows, we will establish the energy functional of (1). For any $y \in H$, multiplying (1) by y and integrating from 0 to π, we obtain

$$\int_0^{\pi} -x''(t) y(t) \mathrm{d}t - \lambda \int_0^{\pi} x(t) y(t) \mathrm{d}t = \int_0^{\pi} f(t) y(t) \mathrm{d}t.$$

Note the impulsive effects, so we have

$$\begin{aligned}
\int_0^{\pi} -x''(t) y(t) \mathrm{d}t &= \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} -x''(t) y(t) \mathrm{d}t = \sum_{j=0}^{p} \left[-x'(t) y(t) \Big|_{t_j^+}^{t_{j+1}^-} + \int_{t_j}^{t_{j+1}} x'(t) y'(t) \mathrm{d}t \right] \\
&= x'(0) y(0) - x'(\pi) y(\pi) + \sum_{j=1}^{p} \Delta x'(t_j) y(t_j) + \int_0^{\pi} x'(t) y'(t) \mathrm{d}t \\
&= \sum_{j=1}^{p} I_j(x(t_j)) y(t_j) + \int_0^{\pi} x'(t) y'(t) \mathrm{d}t.
\end{aligned}$$

Hence, we have

$$\sum_{j=1}^{p} I_j(x(t_j)) y(t_j) + \int_0^{\pi} x'(t) y'(t) \mathrm{d}t - \lambda \int_0^{\pi} x(t) y(t) \mathrm{d}t = \int_0^{\pi} f(t) y(t) \mathrm{d}t, \tag{9}$$

and the energy functional is

$$J(x) = \frac{1}{2} \int_0^{\pi} |x'(t)|^2 \mathrm{d}t - \frac{\lambda}{2} \int_0^{\pi} |x(t)|^2 \mathrm{d}t + \sum_{j=1}^{p} \int_0^{x(t_j)} I_j(s) \mathrm{d}s - \int_0^{\pi} f(t) x(t) \mathrm{d}t, \forall x \in H. \tag{10}$$

Moreover

$$(J'(x),y)=\int_0^{\pi}x'(t)y'(t)\mathrm{d}t-\lambda\int_0^{\pi}x(t)y(t)\mathrm{d}t+\sum_{j=1}^{p}I_j(x(t_j))y(t_j)-\int_0^{\pi}f(t)y(t)\mathrm{d}t,\forall x,y\in H.$$

For convenience, let $\int_0^{\sigma}I_j(s)\mathrm{d}s=G_j(\sigma)$ for $j=1,2,\dots,p$.

Definition 2.1 If there exists $x\in H$ such that, for all $y\in H$, (9) is satisfied, then x is called a weak solution for (1).

Note from the form of J', the solutions of problem (1) are the corresponding critical points of J. From(H), J is of class C^1.

Lemma 2.2(see [9, 10]) Let $X=Y\bigoplus Z$ be a Banach space with Z is closed in X and $\dim Y<\infty$. For $\rho>0$, define $\mathcal{M}=\{u\in Y:\|u\|\le\rho\},\mathcal{M}_0=\{u\in Y:\|u\|=\rho\}$. Let $J\in C^1(X,\mathbb{R})$ be such that $b=\inf_{u\in Z}J(u)>a=\max_{u\in\mathcal{M}_0}J(u)$. If J satisfies the (PS)$_c$ condition with $c=\inf_{\gamma\in\Gamma}\max_{u\in\mathcal{M}}J(\gamma(u))$, where $\Gamma=\{\gamma\in C(\mathcal{M},X):\gamma|_{\mathcal{M}_0}=\mathrm{Id}\}$, then c is a critical value of J.

Lemma 2.3 Now $\|x\|_\infty\le\sqrt{\pi}\|x\|,\forall x\in H$ where $\|x\|_\infty=\max_{t\in[0,\pi]}|x(t)|$.

Proof. For any $x\in H$ and $\tau\in[0,\pi]$, from the Hölder inequality we have

$$|x(\tau)|=\left|\int_0^{\tau}x'(t)\mathrm{d}t\right|\le\int_0^{\pi}|x'(t)|\mathrm{d}t\le\sqrt{\pi}\left(\int_0^{\pi}|x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

Consequently, $\|x\|_\infty\le\sqrt{\pi}\|x\|$. □

To study the existence of solutions for (1) with the parameter $\lambda=\lambda_n=n^2$ for $n=1,2,...$, we recall some basic concepts for operators of type $(S)_+$(see [11-14]).

Definition 2.4 Let H be a reflexive real Banach space and H^* its dual. The operator $T:H\to H^*$ is said to satisfy the $(S)_+$ condition if the assumptions $u_n\rightharpoonup u_0$ weakly in H and $\limsup_{n\to\infty}(T(u_n),u_n-u_0)\le0$ imply $u_n\to u_0$ strongly in H.

Definition 2.5 The operator $T:H\to H^*$ is said to be demicontinuous if T maps strongly convergent sequences in H to weakly convergent sequences in H^*.

Lemma 2.6 Let $T:H\to H^*$ satisfy the $(S)_+$ condition and let $K:H\to H^*$ be a compact operator. Then the sum $T+K:H\to H^*$ satisfies the $(S)_+$ condition.

Lemma 2.7 Let $T:H\to H^*$ be a bounded and demicontinuous operator satisfying the $(S)_+$ condition. Let $\mathcal{D}\subset H$ be an open, bounded and nonempty set with the boundary $\partial\mathcal{D}$ such that $T(u)\ne0$ for $u\in\partial\mathcal{D}$. Then there exists an integer $\deg(T,\mathcal{D},0)$ such that

(1) $\deg(T,\mathcal{D},0)\ne0$ implies that there exists an element $u_0\in\mathcal{D}$ such that $T(u_0)=0$.

(2) If $\mathcal{D}$ is symmetric with respect to the origin and T satisfies $T(u)=-T(-u)$ for any $u\in\partial\mathcal{D}$, then $\deg(T,\mathcal{D},0)$ is an odd number.

(3) Let T_λ be a family of bounded and demicontinuous mappings which satisfy the $(S)_+$ condition and which depend continuously on a real parameter $\lambda\in[0,1]$, and let $T_\lambda(u)\ne0$ for any $u\in\partial\mathcal{D}$ and $\lambda\in[0,1]$. Then $\deg(T_\lambda,\mathcal{D},0)$ is constant with respect to $\lambda\in[0,1]$.

3. The Existence of Weak Solutions for (1)

For the parameter $\lambda=\lambda_1=1$, we have the following theorem.

Theorem 3.1 Let (H) hold. Then (1) has at least one weak solution if and only if

$$\sum_{j=1}^{p}I_j(+\infty)\sin t_j<\int_0^{\pi}f(t)\sin t\mathrm{d}t<\sum_{j=1}^{p}I_j(-\infty)\sin t_j. \tag{11}$$

Proof. We first prove that J is weakly coercive on Z. From (H) we have $I_j(s)$ is bounded for all $s \in \mathbb{R}$, $j = 1, 2, \ldots, p$. Therefore, there exist $M_j > 0 (j = 1, 2, \ldots, p)$ such that

$$|I_j(s)| \leq M_j, \ j = 1, 2, \ldots, p. \tag{12}$$

Now for $z \in Z$, $f \in L^2(0, \pi)$ and (8) enable us to obtain

$$\begin{aligned} J(z) &= \frac{1}{2}\int_0^\pi |z'(t)|^2 \mathrm{d}t - \frac{1}{2}\int_0^\pi |z(t)|^2 \mathrm{d}t + \sum_{j=1}^p \int_0^{z(t_j)} I_j(s)\mathrm{d}s - \int_0^\pi f(t)z(t)\mathrm{d}t \\ &\geq \frac{3}{8}\|z\|^2 - \sqrt{\pi}\|z\| \sum_{j=1}^p M_j - \frac{1}{2}\|f\|_{L^2}\|z\|, \end{aligned}$$

and thus $J(z) \to \infty$ as $\|z\| \to \infty, z \in Z$. The weak sequential lower semi-continuity of $\|\cdot\|$ implies J is weakly sequentially lower semi-continuous on Z, so there exists $z_0 \in Z$ such that

$$-\infty < J(z_0) = \min_{z \in Z} J(z). \tag{13}$$

For $y \in Y$ and we let $y = \rho \sin t$. Then

$$\begin{aligned} J(\rho \sin t) &= \frac{\rho^2}{2}\int_0^\pi \cos^2 t\mathrm{d}t - \frac{\rho^2}{2}\int_0^\pi \sin^2 t\mathrm{d}t + \sum_{j=1}^p \int_0^{\rho \sin t_j} I_j(s)\mathrm{d}s - \rho\int_0^\pi f(t)\sin t\mathrm{d}t \\ &= \sum_{j=1}^p G_j(\rho \sin t_j) - \rho \int_0^\pi f(t) \sin t\mathrm{d}t. \end{aligned}$$

From L'Hospital's Rule, we have

$$\lim_{\rho \to \pm\infty} \frac{G_j(\rho \sin t_j)}{\rho} = \lim_{\rho \to \pm\infty} I_j(\rho \sin t_j)\sin t_j = I_j(\pm\infty)\sin t_j.$$

Consequently, from the Lebesgue dominated convergence theorem and (11) we have

$$\lim_{\rho \to \pm\infty} J(\rho \sin t) = \lim_{\rho \to \pm\infty} \rho \left[\sum_{j=1}^p \frac{G_j(\rho \sin t_j)}{\rho} - \int_0^\pi f(t)\sin t\mathrm{d}t\right] \to -\infty.$$

Taking ρ_0 large enough we then have $J(\pm\rho_0 \sin t) < J(z_0)$, where z_0 is defined in (13). As a result, the assumptions of Lemma 2.2 are satisfied with $\mathcal{M} = \{\rho \sin t : \rho \in [-\rho_0, \rho_0]\}$, $\mathcal{M}_0 = \{-\rho_0 \sin t, \rho_0 \sin t\}$.

It remains to prove that J satisfies the $(PS)_c$ condition. Let $\{x_n\}$ be a $(PS)_c$ sequence, i.e., there exists $c > 0$ such that

$$|J(x_n)| \leq c, \forall n \in \mathbb{N}, \tag{14}$$

and there exists a strictly decreasing sequence $\{\epsilon_n\}$, $\lim_{n\to\infty} \epsilon_n = 0$, such that

$$|(J'(x_n), y)| \leq \epsilon_n \|y\|, \forall n \in \mathbb{N}, y \in H. \tag{15}$$

Suppose for contradiction that $\|x_n\| \to \infty$. Put $v_n = \frac{x_n}{\|x_n\|}$. Then $\{v_n\}$ is bounded in H and so there exists a subsequence (without loss of generality suppose its the whole sequence) which converges to a function v_0 weakly in H and strongly in $L^2(0, \pi)$ and $C[0, \pi]$.

Dividing (10) with $x = x_n$ by $\|x_n\|^2$, so we have

$$\limsup_{n\to\infty} \left[\frac{1}{2} - \frac{1}{2}\int_0^\pi |v_n(t)|^2 \mathrm{d}t + \frac{1}{\|x_n\|^2}\sum_{j=1}^p \int_0^{x_n(t_j)} I_j(s)\mathrm{d}s - \frac{1}{\|x_n\|^2}\int_0^\pi f(t)x_n(t)\mathrm{d}t\right] \leq 0. \tag{16}$$

Now $f \in L^2(0,\pi)$, Lemma 2.3 and (12) enable us to obtain

$$\left|\frac{1}{\|x_n\|^2}\int_0^\pi f(t)x_n(t)\mathrm{d}t\right| \le \frac{\|f\|_{L^2}\|x_n\|_{L^2}}{\|x_n\|^2} \to 0,\ \left|\frac{1}{\|x_n\|^2}\sum_{j=1}^{p}\int_0^{x_n(t_j)} I_j(s)\mathrm{d}s\right| \le \frac{\|x_n\|_\infty \sum_{j=1}^{p} M_j}{\|x_n\|^2} \to 0.$$

Passing to the limit in (16), we have $\int_0^\pi |v_0(t)|^2\mathrm{d}t \ge 1$. Using the weak lower semicontinuity of the norm, note $\lambda_1 = 1$, we have

$$1 \le \lambda_1 \int_0^\pi |v_0(t)|^2\mathrm{d}t \le \int_0^\pi |v_0'(t)|^2\mathrm{d}t \le \liminf_{n\to\infty}\int_0^\pi |v_n'(t)|^2\mathrm{d}t = 1.$$

Thus $\|v_0\| = 1$, and $\int_0^\pi |v_0'(t)|^2\mathrm{d}t = \lambda_1\int_0^\pi |v_0(t)|^2\mathrm{d}t$. This implies that $v_0 = \kappa \sin t$ with $\kappa \ne 0$. Choosing $y = v_n - v_0$ in (15), we obtain

$$\Big|\int_0^\pi v_n'(t)(v_n'(t) - v_0'(t))\mathrm{d}t - \int_0^\pi v_n(t)(v_n(t) - v_0(t))\mathrm{d}t$$
$$+\frac{1}{\|x_n\|}\sum_{j=1}^{p} I_j(x_n(t_j))(v_n(t_j) - v_0(t_j)) - \frac{1}{\|x_n\|}\int_0^\pi f(t)(v_n(t) - v_0(t))\mathrm{d}t\Big| \le \epsilon_n\frac{\|v_n - v_0\|}{\|x_n\|}.$$

Since $v_n \to v_0$ in $L^2(0,\pi)$ and $C[0,\pi]$, by the hypotheses on f and I_j, we have

$$\frac{1}{\|x_n\|}\sum_{j=1}^{p} I_j(x_n(t_j))(v_n(t_j) - v_0(t_j)) \to 0,\ \frac{1}{\|x_n\|}\int_0^\pi f(t)(v_n(t) - v_0(t))\mathrm{d}t \to 0,$$
$$\int_0^\pi v_n(t)(v_n(t) - v_0(t))\mathrm{d}t \to 0,\ \epsilon_n\frac{\|v_n - v_0\|}{\|x_n\|} \to 0.$$

Hence, we get

$$\int_0^\pi v_n'(t)(v_n'(t) - v_0'(t))\mathrm{d}t \to 0.$$

Similarly, we can prove that

$$\int_0^\pi v_0'(t)(v_n'(t) - v_0'(t))\mathrm{d}t \to 0.$$

As a result,

$$0 = \lim_{n\to\infty}\int_0^\pi |v_n'(t) - v_0'(t)|^2\mathrm{d}t = \lim_{n\to\infty}\|v_n - v_0\|^2 \ge 0,$$

which implies $\|v_n\| \to \|v_0\|$. The uniform convexity of H yields that v_n converges strongly to $v_0 = \kappa\sin t$ in H.

Now we rewrite (14) and (15) with $y = x_n$ and obtain

$$-2c \le \int_0^\pi |x_n'(t)|^2\mathrm{d}t - \int_0^\pi |x_n(t)|^2\mathrm{d}t + 2\sum_{j=1}^{p}\int_0^{x_n(t_j)} I_j(s)\mathrm{d}s - 2\int_0^\pi f(t)x_n(t)\mathrm{d}t \le 2c,$$

and

$$-\epsilon_n\|x_n\| \le -\int_0^\pi |x_n'(t)|^2\mathrm{d}t + \int_0^\pi |x_n(t)|^2\mathrm{d}t - \sum_{j=1}^{p} I_j(x_n(t_j))x_n(t_j) + \int_0^\pi f(t)x_n(t)\mathrm{d}t \le \epsilon_n\|x_n\|.$$

Summing and dividing by $\|x_n\|$, we have

$$\left| \frac{2}{\|x_n\|} \sum_{j=1}^{p} \int_0^{x_n(t_j)} I_j(s)\mathrm{d}s - \sum_{j=1}^{p} I_j(x_n(t_j))v_n(t_j) - \int_0^{\pi} f(t)v_n(t)\mathrm{d}t \right| \le \frac{2c}{\|x_n\|} + \epsilon_n. \tag{17}$$

Note that $x_n(t_j) = v_n(t_j)\|x_n\|$ and $v_n \to \kappa \sin t$ with $\kappa \ne 0$. Hence, we have

$$\frac{2}{\|x_n\|} \sum_{j=1}^{p} \int_0^{x_n(t_j)} I_j(s)\mathrm{d}s = 2\sum_{j=1}^{p} \frac{\int_0^{x_n(t_j)} I_j(s)\mathrm{d}s}{x_n(t_j)} v_n(t_j) \to 2\sum_{j=1}^{p} I_j(\pm\infty)\kappa \sin t_j.$$

Passing to the limit in (17), we have

$$\sum_{j=1}^{p} I_j(\pm\infty)\kappa \sin t_j = \int_0^{\pi} f(t)\kappa \sin t\mathrm{d}t, \text{ i.e., } \sum_{j=1}^{p} I_j(\pm\infty) \sin t_j = \int_0^{\pi} f(t) \sin t\mathrm{d}t,$$

which contradicts (11), so $\{x_n\}$ is bounded in H. Consequently there exits a subsequence (without loss of generality suppose its the whole sequence) which converges to a function x weakly in H and strongly in $L^2(0,\pi)$ and $C[0,\pi]$. From the form of J' we have

$$\begin{aligned}(J'(x_n) - J'(x), x_n - x) = &\int_0^{\pi} |x_n'(t) - x'(t)|^2\mathrm{d}t \\ &- \int_0^{\pi} |x_n(t) - x(t)|^2\mathrm{d}t + \sum_{j=1}^{p}(I_j(x_n(t_j)) - I_j(x(t_j)))(x_n(t_j) - x(t_j)).\end{aligned}$$

Therefore, $\|x_n\| \to \|x\|$ from the fact that $(J'(x_n) - J'(x), x_n - x) \to 0, \|x_n - x\|_{L^2} \to 0, \sum_{j=1}^{p}(I_j(x_n(t_j)) - I_j(x(t_j)))(x_n(t_j) - x(t_j)) \to 0$. As a result, x_n converges strongly to x in H, so J satisfies the $(\mathrm{PS})_c$ condition.

From Lemma 2.2, J has a positive critical value c, i.e., there exists $x \in H$ such that $J(x) = c > 0$ and $J'(x) = 0$. Note that $J(0) = 0$, so x is a nontrivial weak solution for (1).

Finally, we prove that (11) is also a necessary condition for the solvability of (1). Assume that $x \in H$ is a weak solution for (1), i.e., $\int_0^{\pi} x'(t)y'(t)\mathrm{d}t - \int_0^{\pi} x(t)y(t)\mathrm{d}t + \sum_{j=1}^{p} I_j(x(t_j))y(t_j) = \int_0^{\pi} f(t)y(t)\mathrm{d}t, \forall y \in H$. Let $y = \sin t$. Then $\sum_{j=1}^{p} I_j(x(t_j)) \sin t_j = \int_0^{\pi} f(t) \sin t\mathrm{d}t$. From (H) we obtain that (11) is satisfied. This completes the proof. □

Let us define operators $J, S, G : H \to H^*$ and an element $f^* \in H$ by

$$(Jx, y) = \int_0^{\pi} x'(t)y'(t)\mathrm{d}t,$$

$$(Sx, y) = \int_0^{\pi} x(t)y(t)\mathrm{d}t, (Gx, y) = \sum_{j=1}^{p} I_j(x(t_j))y(t_j),\ (f^*, y) = \int_0^{\pi} f(t)y(t)\mathrm{d}t.$$

From our inner product (5) and the compactness of $H \hookrightarrow L^2(0,\pi)$ and $H \hookrightarrow C[0,\pi]$, we have J is an identical operator and S, G, f^* are compact operators. Hence, we easily prove that J and $J - n^2S + G - f^*$ satisfy the $(S)_+$ condition. For the parameter $\lambda = \lambda_n = n^2$ for $n = 1, 2, ...,$ we have the following theorem.

Theorem 3.2 Let (H) hold. Then (1) has at least one weak solution if and only if

$$\begin{aligned}&\sum_{j=1}^{p} I_j(+\infty)(\sin nt_j)^+ - \sum_{j=1}^{p} I_j(-\infty)(\sin nt_j)^- \\ &< \int_0^{\pi} f(t) \sin nt\mathrm{d}t < \sum_{j=1}^{p} I_j(-\infty)(\sin nt_j)^+ - \sum_{j=1}^{p} I_j(+\infty)(\sin nt_j)^-,\end{aligned} \tag{18}$$

where $(\sin nt_j)^+$ and $(\sin nt_j)^-$, respectively denote the positive and negative parts of $\sin nt$ for $n = 1, 2, ...$

Proof. Note that, according to the definitions of J, S, G, f^*, we only prove that there exits $x \in H$ such that $Jx = n^2Sx - Gx + f^*$. Fix $\delta \in (0, 2n+1)$ and define $\mathscr{H} : [0,1]\times H \to H^*$ by $\mathscr{H}(\tau, x) = Jx - n^2Sx - (1-\tau)\delta Sx + \tau Gx - \tau f^*$, for all $x \in H$ and $\tau \in [0,1]$. We now prove that there exists a large enough $R > 0$ such that this homotopy is admissible with respect to the ball $Q(0,R) \subset H$. If the claim is false, for any $k \in \mathbb{N}$, there exist $\tau_k \in [0,1]$ and $x_k \in H$, $\|x_k\| \geq k$ such that $\mathscr{H}(\tau_k, x_k) = 0$, i.e., $Jx_k - n^2Sx_k - (1-\tau_k)\delta Sx_k + \tau_k Gx_k - \tau f^* = 0$, and thus

$$\begin{aligned}&\int_0^\pi x_k'(t)y'(t)\mathrm{d}t - n^2\int_0^\pi x_k(t)y(t)\mathrm{d}t\\ &\quad -(1-\tau_k)\delta\int_0^\pi x_k(t)y(t)\mathrm{d}t + \tau_k\sum_{j=1}^p I_j(x_k(t_j))y(t_j) - \tau_k\int_0^\pi f(t)y(t)\mathrm{d}t = 0,\end{aligned} \tag{19}$$

for all $y \in H$. Let $v_k = \frac{x_k}{\|x_k\|}$. Then $\|v_k\| = 1$ and

$$\begin{aligned}&\int_0^\pi v_k'(t)y'(t)\mathrm{d}t - n^2\int_0^\pi v_k(t)y(t)\mathrm{d}t\\ &\quad -(1-\tau_k)\delta\int_0^\pi v_k(t)y(t)\mathrm{d}t + \frac{\tau_k}{\|x_k\|}\sum_{j=1}^p I_j(x_k(t_j))y(t_j) - \frac{\tau_k}{\|x_k\|}\int_0^\pi f(t)y(t)\mathrm{d}t = 0.\end{aligned}$$

From (H) we have $\frac{\tau_k}{\|x_k\|}\sum_{j=1}^p I_j(x_k(t_j))y(t_j) \to 0$, and $\frac{\tau_k}{\|x_k\|}\int_0^\pi f(t)y(t)\mathrm{d}t \to 0$, as $\|x_k\| \to \infty$. From the complete continuity of S, we obtain there is a $v \in H$ such that $v_k \to v$ in H, $\tau_k \to \tau \in [0,1]$ and

$$\int_0^\pi v'(t)y'(t)\mathrm{d}t - n^2\int_0^\pi v(t)y(t)\mathrm{d}t - (1-\tau)\delta\int_0^\pi v(t)y(t)\mathrm{d}t = 0.$$

We consider $\tau = 1$ (since $n^2 + (1-\tau)\delta$ isn't an eigenvalue of (7) if $\tau \neq 1$). Consequently, we have

$$\int_0^\pi v'(t)y'(t)\mathrm{d}t - n^2\int_0^\pi v(t)y(t)\mathrm{d}t = 0,$$

for all $y \in H$. As a result, we get $-v''(t) = n^2v(t), v \in H$ and $\|v\| = 1$. From (7) we have $v(t) = \pm\sqrt{\frac{2}{n^2\pi}}\sin nt$. Let us suppose that $v(t) = \sqrt{\frac{2}{n^2\pi}}\sin nt$ (we proceed analogously for the case $v(t) = -\sqrt{\frac{2}{n^2\pi}}\sin nt$). Taking $y(t) = \sin nt$ in (19) and noting $0 \leq \tau_k \leq 1, \tau_k \to 1$, we have $\sum_{j=1}^p I_j(x_k(t_j))\sin nt_j - \int_0^\pi f(t)\sin nt\mathrm{d}t \geq 0$, i.e., $\liminf_{k\to\infty}\sum_{j=1}^p I_j(x_k(t_j))\sin nt_j \leq \int_0^\pi f(t)\sin nt\mathrm{d}t$. For k sufficiently large, Fatou's lemma yields that $\sum_{j=1}^p I_j(-\infty)(\sin nt_j)^+ - \sum_{j=1}^p I_j(+\infty)(\sin nt_j)^- \leq \int_0^\pi f(t)\sin nt\mathrm{d}t$, a contradiction with (18). This proves that the homotopy $\mathscr{H}$ is admissible with respect to the ball $Q(0,R)$ if R is large enough. Hence, Lemma 2.7 (3) yields that

$$\deg(J - n^2S + G - f^*, Q(0,R), 0) = \deg(J - (n^2+\delta)S, Q(0,R), 0), \tag{20}$$

Note that $\deg(J - (n^2+\delta)S, Q(0,R), 0)$ is an odd number by Lemma 2.7 (2). Hence $\deg(J - n^2S + G - f^*, Q(0,R), 0) \neq 0$, and Lemma 2.7 (1) guarantees the existence of at least one weak solution of (1).

Finally, we prove that (18) is also a necessary condition for the solvability of (1). Assume that $x \in H$ is a weak solution for (1), i.e., $\int_0^\pi x'(t)y'(t)\mathrm{d}t - n^2\int_0^\pi x(t)y(t)\mathrm{d}t + \sum_{j=1}^p I_j(x(t_j))y(t_j) = \int_0^\pi f(t)y(t)\mathrm{d}t, \forall y \in H$. Let $y = \sin nt$. Then $\sum_{j=1}^p I_j(x(t_j))\sin nt_j = \int_0^\pi f(t)\sin nt\mathrm{d}t$. From (H) we can easily obtain (18) holds true. This completes the proof. □

Remark 3.3 If $n = 1$, (18) is the same as (11).

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