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Centered Operators Via Moore-Penrose Inverse and Aluthge Transformations

M. R. Jabbarzadeh[a], M. Jafari Bakhshkandi[a]

[a]*Faculty of Mathematical Sciences, University of Tabriz, P. O. Box: 5166615648, Tabriz, Iran*

Abstract. In this paper, we obtain some characterizations of centered and binormal operators via Moore-Penrose inverse and Aluthge transform.

1. Introduction and Preliminaries

Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and the range of an operator $T \in B(\mathcal{H})$, respectively. Recall that for $T \in B(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. As a consequence, $U^*U|T| = |T|$. Also, it is a classical fact that the polar decomposition of T^* is $U^*|T|$, and so $UU^*|T^*| = |T^*|$.

In [8] Morrel and Muhly introduced the concept of a centered operator. An operator T on a Hilbert space $\mathcal{H}$ is said to be centered if the doubly infinite sequence $\{T^nT^{*n}, T^{*m}T^m : n, m \geq 0\}$ consists of mutually commuting operators. It is shown in [4] that if $T = U|T|$ is an operator on $\mathcal{H}$ such that for each $n \in \mathbb{N}$, T^n has polar decomposition $U_n|T^n|$, then T is centered if and only if $U_n = U^n$ for each $n \in \mathbb{N}$.

Associated with $T \in B(\mathcal{H})$ there is a useful related operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the Aluthge transform of T as it has been studied by Aluthge in [1]. Binormality of operators was defined by Campbell in [3]. An operator T is said to be binormal or weakly centered [9], if $[|T|, |T^*|] = 0$, where $[A, B] = AB - BA$ for operators A and B. Let $T \in B(\mathcal{H})$ have closed range. Then the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in B(\mathcal{H})$ which satisfies $TT^\dagger T = T$, $T^\dagger TT^\dagger = T^\dagger$, $(TT^\dagger)^* = TT^\dagger = P_{\mathcal{R}(T)}$ and $(T^\dagger T)^* = T^\dagger T = P_{\mathcal{R}(T^\dagger)}$, where the $P_{\mathcal{M}}$ means the orthogonal projection onto a closed subspace $\mathcal{M}$.

In this paper, we study the centered and binormal bounded linear operators on a Hilbert space $\mathcal{H}$ via Moore-Penrose inverse and Aluthge transformation. The work is organized as follows. In section 2, firstly, we give the polar decomposition of $T^\dagger$, and then we show that $T^\dagger$ is centered if and only if T is centered. Secondly, we introduce the notion †-Aluthge transformation $\widetilde{T}^{(\dagger)}$ of T by setting $\widetilde{T}^{(\dagger)} = (\widetilde{T^\dagger})^\dagger$. We show

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Email addresses: mjabbar@tabrizu.ac.ir (M. R. Jabbarzadeh), m_jafari@tabrizu.ac.ir (M. Jafari Bakhshkandi)

that if T is a closed range binormal operator, then the $*$-Aluthge and $\dagger$-Aluthge transformations (see [11]) coincide. Also, we show that the reverse order law holds for $|T|$ and $|T^*|$; i.e. $(|T||T^*|)^\dagger = |T^*|^\dagger|T|^\dagger$, whenever T is a closed range binormal operator. Finally, we give the polar decomposition of powers of $\widetilde{T}$ and then we find some conditions under which $\widetilde{T}$ be centered. Also we show that if T is quasinormal operator, then $\widetilde{T}$ is centered.

2. On Some Characterizations of $T^\dagger$

Let $CR(\mathcal{H})$ be the subset of all bounded linear operators on $\mathcal{H}$ with closed range. In the following proposition we obtain the polar decomposition of $T^\dagger$. The following lemma is significant for amount of consideration for the next results and computations.

Lemma 2.1. *Let $T \in CR(\mathcal{H})$. Then the following assertions hold.*

(a) If $T \geq 0$, then $T^\dagger \geq 0$, $\mathcal{N}(T) = \mathcal{N}(T^{\frac{1}{2}})$, $T^{\frac{1}{2}} \in CR(\mathcal{H})$ and $(T^\dagger)^{\frac{1}{2}} = (T^{\frac{1}{2}})^\dagger$.
(b) $|T^\dagger| = |T^*|^\dagger$.
(c) $\mathcal{R}(|T|) = \mathcal{R}(|T|^{\frac{1}{2}})$ and $\mathcal{R}(|T^\dagger|) = \mathcal{R}(|T^*|) = \mathcal{R}(|T^*|^{\frac{1}{2}})$.
(d) If $T = T^*$, then $TP_{\mathcal{R}(T)} = T$.
(e) $|T^\dagger|^{\frac{1}{2}} = (|T^*|^{\frac{1}{2}})^\dagger$, $|T^\dagger|^{\frac{1}{2}}P_{\mathcal{R}(|T^*|)} = |T^\dagger|^{\frac{1}{2}}$ and $|T^\dagger|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = P_{\mathcal{R}(|T^\dagger|)}$.

Proof. (a) Let $f \in \mathcal{H}$. Then $\langle T^\dagger f, f\rangle = \langle T^\dagger T T^\dagger f, f\rangle = \langle TT^\dagger f, T^\dagger f\rangle \geq 0$, and so $\langle T^\dagger f, f\rangle \geq 0$. Also from $\langle Tf, f\rangle = \langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} f\rangle = \|T^{\frac{1}{2}} f\|$ we deduce that $T^{\frac{1}{2}} f = 0$ if and only if $Tf = 0$. Now, from this and the inequality $\|Tf\| = \|T^{\frac{1}{2}}(T^{\frac{1}{2}} f)\| \leq \|T^{\frac{1}{2}}\|\|T^{\frac{1}{2}} f\|$, we conclude that the range of $T^{\frac{1}{2}}$ is also closed. Finally, Since for each $n \in \mathbb{N}$, $(T^n)^\dagger = (T^\dagger)^n$, we have $(T^{\frac{1}{2}})^\dagger(T^{\frac{1}{2}})^\dagger = ((T^{\frac{1}{2}})^\dagger)^2 = ((T^{\frac{1}{2}})^2)^\dagger = T^\dagger = ((T^\dagger)^{\frac{1}{2}})^2$.

(b) It is sufficient to show that $(TT^*)^\dagger = (T^*)^\dagger T^\dagger$. Since $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$, so $TT^*(T^*)^\dagger T^\dagger TT^* = TP_{\mathcal{R}(T^*)}P_{\mathcal{R}(T^*)}T^* = TT^*$ and

$$(T^*)^\dagger T^\dagger TT^*(T^*)^\dagger T^\dagger = (T^*)^\dagger P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T^*)}T^\dagger = (T^*)^\dagger T^\dagger.$$

Hence, $|T^\dagger| = ((T^*)^\dagger T^\dagger)^{\frac{1}{2}} = ((TT^*)^\dagger)^{\frac{1}{2}} = ((TT^*)^{\frac{1}{2}})^\dagger = |T^*|^\dagger$.

(c) By part (a), $\mathcal{N}(|T^*|) = \mathcal{N}(|T^*|^{\frac{1}{2}})$. Hence it follows that $\overline{\mathcal{R}(|T^*|)} = \overline{\mathcal{R}(|T^*|^{\frac{1}{2}})}$. By hypotheses $\mathcal{R}(TT^*)$, $\mathcal{R}(|T^*|)$ and so $R(|T^*|^{\frac{1}{2}})$ are closed. Thus $R(|T^*|) = R(|T^*|^{\frac{1}{2}})$. The equality $\mathcal{R}(|T^\dagger|) = \mathcal{R}(|T^*|)$ follows from (b).

(d) Since $\mathcal{N}(T) = \mathcal{N}(T^*) = \mathcal{R}(T)^\perp$, hence $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$. So for each $f \in \mathcal{H}$ there exists a unique $g \in \mathcal{R}(T)$ and a unique $h \in \mathcal{R}(T)^\perp$ such that $f = g + h$. It follows that $TP_{R(T)}(f) = T(g) = T(g + h) = T(f)$.

(e) It follows from the previous parts. □

Proposition 2.2. *Let $U|T|$ be the polar decomposition of an operator $T \in CR(\mathcal{H})$. Then $T^\dagger = U^*|T^*|^\dagger = U^*|T^\dagger|$ is the polar decomposition for $T^\dagger$ and hence the Aluthge transformation of $T^\dagger$ is $|T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$.*

Proof. Put $S = |T^*|^\dagger U$. Since $\mathcal{R}(U) = \mathcal{R}(T) = \mathcal{R}(|T^*|)$ and $\mathcal{R}(|T^*|)^\perp = \mathcal{N}(|T^*|)$, so $\mathcal{R}(S) = \mathcal{R}(U) = \mathcal{R}(|T^*|)$. Moreover, we have

$$\begin{aligned} T^*ST^* &= U^*|T^*|(|T^*|^\dagger U)U^*|T^*| \\ &= U^*|T^*||T^*|^\dagger|T^*| \\ &= U^*|T^*| = T^*, \end{aligned}$$

$$\begin{aligned} ST^*S &= |T^*|^\dagger(UU^*|T^*|)|T^*|^\dagger U \\ &= |T^*|^\dagger|T^*||T^*|^\dagger U \\ &= |T^*|^\dagger U = S, \end{aligned}$$

$$\begin{aligned} T^*S &= U^*|T^*||T^*|^\dagger U \\ &= U^*P_{\mathcal{R}(|T^*)}U \\ &= U^*U \\ &= P_{\mathcal{R}(T^*)}, \end{aligned}$$

and

$$\begin{aligned} ST^* &= |T^*|^\dagger UU^*|T^*| \\ &= |T^*|^\dagger P_{R(|T^*|)}|T^*| \\ &= |T^*|^\dagger|T^*| \\ &= P_{\mathcal{R}(|T^*|^\dagger)} = P_{\mathcal{R}(|T^*|)} = P_{\mathcal{R}(S)}. \end{aligned}$$

These equalities show that $(T^\dagger)^* = (T^*)^\dagger = S$, and hence $T^\dagger = S^* = U^*|T^*|^\dagger = U^*|T^\dagger|$ is the polar decomposition for $T^\dagger$. Thus, the Aluthge transformation of $T^\dagger$ is $|T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$. □

Corollary 2.3. *Let $T \in CR(\mathcal{H})$. Then $T^\dagger$ is centered if and only if T is centered.*

Proof. By Proposition 2.2, $U_n|T^n|$ is the polar decomposition of T^n if and only if $U_n^*|(T^*)^n|^\dagger$ is the polar decomposition of $(T^\dagger)^n$, for each $n \in \mathbb{N}$. Now, the desired conclusion follows from the Morrel-Muhly Theorem [8]. □

Proposition 2.4. *Let $T = U|T| \in B(\mathcal{H})$ and $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|$ be the polar decompositions. Suppose that $\widetilde{T} \in CR(\mathcal{H})$, then $\widetilde{T}^\dagger = U^*V^*|\widetilde{T}^\dagger| = U^*(|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|)^\dagger$ is the polar decomposition.*

Proof. It is sufficient to show that $\widetilde{T} = VU|\widetilde{T}|$ is the polar decomposition. This has been proved in [7, Theorem 2.1]. Here we give a new proof. Since $U|T|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}U$, we obtain $(\widetilde{T})^* = |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}} = U^*|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}} = U^*V^*||T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|$. But $||T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|^2 = |T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}} = (|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}) = |(\widetilde{T})^*|^2$. Hence $(\widetilde{T})^* = U^*V^*|(\widetilde{T})^*|$. Also it is easy to check that $(U^*V^*)(VU)(U^*V^*) = U^*V^*$ and $\mathcal{N}((\widetilde{T})^*) = \mathcal{N}(U^*V^*)$. Hence $\widetilde{T} = VU|\widetilde{T}|$ is the polar decomposition. By using Proposition 2.2, we have that $\widetilde{T}^\dagger = U^*V^*|\widetilde{T}^\dagger| = U^*(|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|)^\dagger$ is the polar decomposition. □

Theorem 2.5. *If $T \in CR(\mathcal{H})$ is binormal, then $\widetilde{T}^\dagger = (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}$.*

Proof. First, we note that by a modification of [5, Theorem 2] we obtain

$$P_{\mathcal{N}(|T|)^\perp}P_{\mathcal{N}(|T^*|)^\perp} = P_{\mathcal{N}(|T^*|)^\perp}P_{\mathcal{N}(|T|)^\perp}.$$

Since for each $0 \le A \in B(\mathcal{H})$, $\overline{\mathcal{R}(A)} = \mathcal{N}(A)^\perp$, this implies that

$$P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)} = P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}.$$

Put $S = (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}$. Then we have

$$\begin{aligned} S\widetilde{T}S &= (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}} \\ &= (|T|^\dagger)^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}UP_{\mathcal{R}(|T|)}U^*(|T|^\dagger)^{\frac{1}{2}} \\ &= (|T|^\dagger)^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}UU^*(|T|^\dagger)^{\frac{1}{2}} \\ &= (|T|^\dagger)^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}(|T|^\dagger)^{\frac{1}{2}} \\ &= (|T|^\dagger)^{\frac{1}{2}}U^*P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}(|T|^\dagger)^{\frac{1}{2}} \\ &= (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}} = S, \end{aligned}$$

$$\begin{aligned}\widetilde{T}S\widetilde{T} &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}(|T|^{\dagger})^{\frac{1}{2}}U^*(|T|^{\dagger})^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\\ &= |T|^{\frac{1}{2}}UP_{\mathcal{R}(|T|)}U^*P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}}\\ &= |T|^{\frac{1}{2}}UU^*P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}}\\ &= |T|^{\frac{1}{2}}P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}}\\ &= |T|^{\frac{1}{2}}P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}U|T|^{\frac{1}{2}}\\ &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \widetilde{T},\end{aligned}$$

and

$$\begin{aligned}S\widetilde{T} &= (|T|^{\dagger})^{\frac{1}{2}}U^*(|T|^{\dagger})^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\\ &= (|T|^{\dagger})^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}}\\ &= (|T|^{\dagger})^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}|T^*|^{\frac{1}{2}}U\\ &= (|T|^{\dagger})^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}P_{\mathcal{R}(|T|)}U\\ &= (|T|^{\dagger})^{\frac{1}{2}}|T|^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}U\\ &= P_{\mathcal{R}(|T|)}U^*P_{\mathcal{R}(|T|)}U = U^*P_{\mathcal{R}(|T|)}U.\end{aligned}$$

By similar computation we have $\widetilde{T}S = P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}$. Hence $\widetilde{T}S$ and $S\widetilde{T}$ are self-adjoint operators. From the uniqueness of Moore-Penrose inverse we have $\widetilde{T}^{\dagger} = S$. □

In [11], Yamazaki introduce the notion of the $*$-Aluthge transform $\widetilde{T}^{(*)}$ of T by setting $\widetilde{T}^{(*)} = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$. With the motivation of this definition we define †-Aluthge transformation of T by setting $\widetilde{T}^{(\dagger)} := (\widetilde{T^{\dagger}})^{\dagger}$. In the following theorem we show that if $T \in \mathrm{CR}(\mathcal{H})$ is binormal, then the $*$-Aluthge and †-Aluthge transformations coincide.

Theorem 2.6. *If $T \in CR(\mathcal{H})$ is binormal, then $\widetilde{T}^{(\dagger)} = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}} = \widetilde{T}^{(*)}$.*

Proof. Since T is binormal, then we have

$$\begin{aligned}\widetilde{T}^{(*)}\widetilde{T^{\dagger}}\widetilde{T}^{(*)} &= (|T^{\dagger}|^{\frac{1}{2}})^{\dagger}U(|T^{\dagger}|^{\frac{1}{2}})^{\dagger}|T^{\dagger}|^{\frac{1}{2}}U^*|T^{\dagger}|^{\frac{1}{2}}(|T^{\dagger}|^{\frac{1}{2}})^{\dagger}U(|T^{\dagger}|^{\frac{1}{2}})^{\dagger}\\ &= |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}|T^{\dagger}|^{\frac{1}{2}}U^*|T^{\dagger}|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T^*|)}U^*P_{\mathcal{R}(|T^*|)}U|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T^*|)}U^*U|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T|)}|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}} = \widetilde{T}^{(*)},\end{aligned}$$

and

$$\begin{aligned}
\widetilde{T^\dagger}\widetilde{T}^{(*)}\widetilde{T^\dagger} &= |T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}(|T^\dagger|^{\frac{1}{2}})^\dagger U(|T^\dagger|^{\frac{1}{2}})^\dagger|T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}\\
&= |T^\dagger|^{\frac{1}{2}}U^*P_{\mathcal{R}(|T^*|)}UP_{\mathcal{R}(|T^*|)}U^*|T^\dagger|^{\frac{1}{2}}\\
&= |T^\dagger|^{\frac{1}{2}}U^*UP_{\mathcal{R}(|T^*|)}U^*|T^\dagger|^{\frac{1}{2}}\\
&= |T^\dagger|^{\frac{1}{2}}P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}U^*|T^\dagger|^{\frac{1}{2}}\\
&= |T^\dagger|^{\frac{1}{2}}P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}U^*|T^\dagger|^{\frac{1}{2}}\\
&= |T^\dagger|^{\frac{1}{2}}P_{\mathcal{R}(|T^*|)}U^*|T^\dagger|^{\frac{1}{2}}\\
&= |T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}} = \widetilde{T^\dagger}.
\end{aligned}$$

By similar computations we have $\widetilde{T^\dagger}\widetilde{T}^{(*)} = P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}$ and $\widetilde{T}^{(*)}\widetilde{T^\dagger} = UP_{\mathcal{R}(|T^*|)}U^*$. Hence $\widetilde{T^\dagger}\widetilde{T}^{(*)}$ and $\widetilde{T}^{(*)}\widetilde{T^\dagger}$ are self-adjoint operators. Thus, $\widetilde{T}^{(\dagger)} = (\widetilde{T^\dagger})^\dagger = \widetilde{T}^{(*)}$. □

The so-called reverse order law, which is one of the most important properties of the Moore-Penrose inverse that has been deeply studied, states that under which condition the equation $(T_1T_2)^\dagger = T_2^\dagger T_1^\dagger$ holds (see [6]). In the following theorem we show that the reverse order law holds for $|T|$ and $|T^*|$ whenever $T \in \mathrm{CR}(\mathcal{H})$ is binormal.

Lemma 2.7. *Let $T \in CR(\mathcal{H})$ be a binormal operator. Then $(|T||T^*|)^\dagger = |T^*|^\dagger|T|^\dagger$.*

Proof. Since T is a binormal operator, so by a modification of [5, Theorem 2], $P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)} = P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}$. Put $S = |T^*|^\dagger|T|^\dagger$. Then we have,

$$\begin{aligned}
|T||T^*|S|T||T^*| &= |T|P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}|T^*|\\
&= |T|P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}|T^*|\\
&= |T||T^*|,
\end{aligned}$$

$$\begin{aligned}
S|T||T^*|S &= |T^*|^\dagger P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}|T|^\dagger\\
&= |T^*|^\dagger P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}|T|^\dagger\\
&= S,
\end{aligned}$$

$$\begin{aligned}
S|T||T^*| &= |T^*|^\dagger P_{\mathcal{R}(|T|)}|T^*|\\
&= |T^*|^\dagger|T^*|P_{\mathcal{R}(|T|)}\\
&= P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)},
\end{aligned}$$

and

$$\begin{aligned}
|T||T^*|S &= |T|P_{\mathcal{R}(|T^*|)}|T|^\dagger\\
&= P_{\mathcal{R}(|T^*|)}|T||T|^\dagger\\
&= P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}.
\end{aligned}$$

Consequently, $(|T||T^*|)^\dagger = S$. □

Theorem 2.8. *Let $T \in CR(\mathcal{H})$. Then T is binormal if and only if $T^\dagger$ is so.*

Proof. Suppose TT^* commutes with T^*T. Then by Lemma 2.7, we have $|T^*|^\dagger|T|^\dagger = |T|^\dagger|T^*|^\dagger$. Since $|T|^\dagger = |(T^*)^\dagger| = |(T^\dagger)^*|$, it follows that $|T^\dagger||(T^\dagger)^*| = |(T^\dagger)^*||T^\dagger|$. Conversely, since $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$ and $(T^\dagger)^\dagger = T$, then $T^\dagger \in \mathrm{CR}(\mathcal{H})$ and so the converse also holds.
□

In [7], the authors obtained the polar decomposition of $\widetilde{T} = VU|\widetilde{T}|$, when $T = U|T|$ and $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|$. In the following theorem we obtain the polar decomposition of the powers of $\widetilde{T}$.

Theorem 2.9. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$. Then $(\widetilde{T})^n = |T|^{\frac{1}{2}}T^{n-1}U|T|^{\frac{1}{2}}$. Moreover, if $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}} = W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|$ is the polar decomposition, then $(\widetilde{T})^n = W_{n-1}U|(\widetilde{T})^n|$ is the polar decomposition of $(\widetilde{T})^n$.*

Proof. By direct computations we obtain that

$$\begin{aligned} U|(\widetilde{T})^n| &= U|(\widetilde{T})^n|U^*U \\ &= U(|T|^{\frac{1}{2}}U^*(T^{n-1})^*|T|T^{n-1}U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^*U \\ &= (U|T|^{\frac{1}{2}}U^*(T^{n-1})^*|T|T^{n-1}U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}}U \\ &= (|T^*|^{\frac{1}{2}}(T^{n-1})^*|T|T^{n-1}|T^*|^{\frac{1}{2}})^{\frac{1}{2}}U \\ &= (||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|^2)^{\frac{1}{2}} \\ &= ||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|U. \end{aligned}$$

Also, by the polar decomposition $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}} = W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|$ we get that

$$\begin{aligned} W_{n-1}U|(\widetilde{T})^n| &= W_{n-1}U|(\widetilde{T})^n|UU^* \\ &= W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|U \\ &= |T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}U \\ &= |T|^{\frac{1}{2}}T^{n-1}U|T|^{\frac{1}{2}}U \\ &= (\widetilde{T})^n. \end{aligned}$$

Now, we show that the kernel condition $\mathcal{N}((\widetilde{T})^n) = \mathcal{N}(W_{n-1}U)$ holds. Let $f \in \mathcal{N}(W_{n-1}U)$. Since by hypothesis $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}} = W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|$ is the polar decomposition and $\mathcal{N}(W_{n-1}) = \mathcal{N}(|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}})$, we obtain that $(\widetilde{T})^n(f) = |T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}U(f) = 0$. Hence $\mathcal{N}(W_{n-1}U) \subseteq \mathcal{N}((\widetilde{T})^n)$. Let $f \in \mathcal{N}((\widetilde{T})^n)$. Then $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}U(f) = 0$, and so $W_{n-1}U(f) = 0$. Consequently, $\mathcal{N}((\widetilde{T})^n) \subseteq \mathcal{N}(W_{n-1}U)$. □

Corollary 2.10. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$. Then $\widetilde{T}$ is centered if and only if for each positive integer n, $W_{n-1}U = (W_0U)^n$. In particular, if T is a centered operator, then $\widetilde{T}$ is centered if and only if $W_{n-1}U = U^n$.*

Proof. By the definition $\widetilde{T}$ is centered if and only if for each positive integer n, $W_{n-1}U = (W_0U)^n$. If T is centered then T is binormal, and so $W_0 = I$. Thus we conclude that $\widetilde{T}$ is centered if and only if $W_{n-1}U = U^n$. □

Lemma 2.11. *Let $T = U|T| \in B(\mathcal{H})$ be a quasinormal operator. Then the following assertions hold.*

(i) $|T|^{\frac{1}{2}}T = T|T|^{\frac{1}{2}}$.
(ii) $|T^2| = |T|^2$.
(iii) $|T|^nU|T| = U|T|^{n+1}$, *for each* $n \in \mathbb{N}$.
(iv) $|T^*|^{\frac{1}{2}}|T|^{\frac{3}{2}} = |T|^{\frac{3}{2}}|T^*|^{\frac{1}{2}}$.
(v) T is centered so the partial isometry part in T^n is U^n.
(vi) $(T^n)^*T^n = (T^*T)^n$ *and hence* $|T^n| = |T|^n$, *for each* $n \in \mathbb{N}$.

Proof. (i) From $|T|^2T = T^*TT = TT^*T = T|T|^2$, we deduce that $|T|T = T|T|$ and hence $|T|^{\frac{1}{2}}T = T|T|^{\frac{1}{2}}$.

(ii) Since $T^*T^*TT = T^*TT^*T$, so $(T^*)^2T^2 = (T^*T)^2$ and hence $|T^2|^2 = (|T|^2)^2$. Consequently, $|T^2| = |T|^2$.

(iii) For $n = 2$ we have $|T|^2U|T| = |T||T|U|T| = |T|U|T|^2 = U|T|^3$. Let for positive integer n, $|T|^nU|T| = U|T|^{n+1}$. Then

$$\begin{aligned}|T|^{n+1}U|T| &= |T||T|^nU|T|\\ &= |T|U|T|^n\\ &= U|T|^{n+1}.\end{aligned}$$

(iv) Since$T^*|T| = |T|T^*$, so $TT^*|T| = T|T|T^* = |T|TT^*$. This means $|T^*|^2|T| = |T||T^*|^2$ and hence $|T^*||T| = |T||T^*|$. Therefore, $|T^*||T|^3 = |T||T^*||T|^2 = |T|^3|T^*|$ and so $|T^*|^{\frac{1}{2}}|T|^{\frac{3}{2}} = |T|^{\frac{3}{2}}|T^*|^{\frac{1}{2}}$. □

Theorem 2.12. *Let $T = U|T| \in B(\mathcal{H})$ be a quasinormal operator. Suppose that for each $n \in \mathbb{N}$, $|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} = W_n||T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}}|$ is the polar decomposition. Then $W_n = U^n$.*

Proof. By using Lemma 2.11, we have

$$\begin{aligned}|T^*|^{\frac{1}{2}}(T^n)^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} &= |T^*|^{\frac{1}{2}}(T^n)^*|T|T^n|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}(T^n)^*T^n|T||T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}(T^*T)^n|T||T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}|T|^{2n+1}|T^*|^{\frac{1}{2}}\\ &= |T^*|^{\frac{1}{2}}|T|^{\frac{2n+1}{2}}|T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}\\ &= (|T^*|^{\frac{1}{2}}|T|^{\frac{2n+1}{2}})^2.\end{aligned}$$

This implies that $||T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}}| = |T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}$. Also we have $|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} = T^n|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = U^n|T|^n|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = U^n|T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}$. Now, if $|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} = W_n||T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}}| = W_n|T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}$ is the polar decomposition, then we obtain $W_n = U^n$. □

Corollary 2.13. *If $T \in B(\mathcal{H})$ is quasinormal, then $\widetilde{T}$ is centered.*

Example 2.14. *Let (X, Σ, μ) be a σ-finite measure space and let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. Put $h = d\mu \circ \varphi^{-1}/d\mu$. Let $C_\varphi : f \mapsto f \circ \varphi$ be a bounded composition operator on $L^2(\Sigma)$ with polar decomposition $C_\varphi = U|C_\varphi|$. It is easy to see that $U = M_{1/\sqrt{h\circ\varphi}}C_\varphi$ and $|C_\varphi| = M_{\sqrt{h}}$. It follows that for each $f \in L^2(\Sigma)$,*

$$\widetilde{C_\varphi}(f) = \sqrt[4]{\frac{h}{h \circ \varphi}} f \circ \varphi.$$

Note that $h \circ \varphi > 0$ almost everywhere. Now, if $C_\varphi \in CR(L^2(\Sigma))$ then it is easy to check that

$$C_\varphi^\dagger(f) = \chi_{\sigma(h)}E(f) \circ \varphi^{-1}, \quad f \in L^2(\Sigma),$$

where E is the conditional expectation operator with respect to $\varphi^{-1}(\Sigma)$. We can write $C_\varphi^\dagger = M_{\frac{\chi_{\sigma(h)}}{h}}C_\varphi^*$, where $C_\varphi^*(f) = hE(f) \circ \varphi^{-1}$. For more details on composition and conditional expectation operators see [2] and references *therein. Let* $|C_\varphi|^{\frac{1}{2}}|C^*_\varphi|^{\frac{1}{2}} = V||C_\varphi|^{\frac{1}{2}}|C^*_\varphi|^{\frac{1}{2}}|$ *be the polar decomposition of* $|C_\varphi|^{\frac{1}{2}}|C^*_\varphi|^{\frac{1}{2}}$*. Then straightforward calculations show that*

$$V(f) = \frac{\sqrt[4]{h}}{\sqrt{E(\sqrt{h\circ\varphi})}} f;$$

$$||C_\varphi|^{\frac{1}{2}}|C^*_\varphi|^{\frac{1}{2}}|(f) = \sqrt[4]{h\circ\varphi}\sqrt{E(\sqrt{h})}E(f).$$

Hence by Proposition 2.4, $\widetilde{C_\varphi} = VU||C_\varphi|^{\frac{1}{2}}|C^*_\varphi|^{\frac{1}{2}}|$ *is the polar decomposition of* $\widetilde{C_\varphi}$*. In [4], Embry-Wardrop and Lambert proved that the composition operator* $C_\varphi \in B(L^2(\Sigma))$ *is centered if and only if* h *is* Σ_∞*-measurable, where* $\Sigma_\infty = \cap_{n=1}^{\infty}\varphi^{-n}(\Sigma)$*. Now, by this fact and Corollary 2.3,* $C^\dagger_\varphi$ *is centered if and only if* h *is* Σ_∞*-measurable. Recall that* C_φ *is quasinormal if and only if* $h = h\circ\varphi$ *(see [10]). So for each* $n \in \mathbb{N}$, $h = h\circ\varphi^n$ *and so* $h \in \Sigma_\infty$*. In this case we have* $\widetilde{C_\varphi} = C_\varphi$*, and hence* $\widetilde{C_\varphi}$ *is centered.*

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