Filomat 31:20 (2017), 6473–6481
https://doi.org/10.2298/FIL1720473K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extension of the Kantorovich Inequality for Positive Multilinear Mappings

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Abstract. It is known that the power function $f(t) = t^2$ is not matrix monotone. Recently, it has been shown that t^2 preserves the order in some matrix inequalities. We prove that if $\mathbb{A} = (A_1, \cdots, A_k)$ and $\mathbb{B} = (B_1, \cdots, B_k)$ are k-tuples of positive matrices with $0 < m \le A_i, B_i \le M \ (i = 1, \ldots, k)$ for some positive real numbers $m < M$, then

$$\Phi^2\left(A_1^{-1}, \cdots, A_k^{-1}\right) \le \left(\frac{(1+v^k)^2}{4v^k}\right)^2 \Phi^{-2}(A_1, \cdots, A_k)$$

and

$$\Phi^2\left(\frac{A_1+B_1}{2}, \cdots, \frac{A_k+B_k}{2}\right) \le \left(\frac{(1+v^k)^2}{4v^k}\right)^2 \Phi^2\left(A_1 \sharp B_1, \cdots A_k \sharp B_k\right),$$

where Φ is a unital positive multilinear mapping and $v = \frac{M}{m}$ is the condition number of each A_i.

1. Introduction

Throughout the paper, assume that $\mathcal{M}_n := \mathcal{M}_n(\mathbb{C})$ is the algebra of all $n \times n$ complex matrices and I denotes the identity matrix. A Hermitian matrix A is called positive (denoted by $A \ge 0$) if all of its eigenvalues are nonnegative. If in addition A is invertible, then A is called strictly positive (denoted by $A > 0$). For Hermitian matrices $A, B \in \mathcal{M}_n$, the inequality $A \le B$ means that $B - A \ge 0$. If m is a real scalar, then by $m \le A$ we mean that $mI \le A$.

Let $J \subseteq \mathbb{R}$ be an interval. A continuous real function $f : J \to \mathbb{R}$ is called matrix monotone if $A \le B$ implies that $f(A) \le f(B)$ for all Hermitian matrices A and B whose eigenvalues are in J. A celebrated result of Löwner–Heinz (see for example [9, 10]) asserts that $f(t) = t^r$ is matrix monotone for all $0 \le r \le 1$. In fact the converse is also true, if $f(t) = t^r$ is matrix monotone, then $0 \le r \le 1$. This concludes that the power function $f(t) = t^r$ does not preserve the matrix order in general except for $0 \le r \le 1$. For example, $A \le B$ does not imply $A^2 \le B^2$. To see this, it is enough to set $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

2010 *Mathematics Subject Classification*. Primary 47A63; Secondary 47A64
Keywords. Kantorovich inequality; multilinear mapping; matrix power mean
Received: 12 May 2016; Revised: 14 June 2016; Accepted: 20 July 2016
Communicated by M. S. Moslehian
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However, there have recently been some works in which some operator inequalities are squared. Moreover, it has been recently shown that the power function $f(t) = t^r$ preserves the order in some matrix inequalities even if $r \geq 1$. In this section, we take a look at these works.

A linear mapping $\Phi : \mathcal{M}_n \to \mathcal{M}_p$ is called positive if Φ preserves the positivity, i.e., if $A \geq 0$ in $\mathcal{M}_n$, then $\Phi(A) \geq 0$ in $\mathcal{M}_p$ and Φ is called unital if $\Phi(I) = I$. Also Φ is said to be strictly positive if $\Phi(A) > 0$ whenever $A > 0$.

A continuous real function $f : J \to \mathbb{R}$ is said to be matrix convex if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all Hermitian matrices A, B with eigenvalues in J and all $\lambda \in [0,1]$. Positive linear mappings have been used to characterize matrix convex and matrix monotone functions. For example, it is well-known that a continuous real function $f : J \to \mathbb{R}$ is matrix convex if and only if the Choi–Davis–Jensen inequality [10] $f(\Phi(A)) \leq \Phi(f(A))$ holds true for every unital positive linear mapping Φ and every Hermitian matrix A whose eigenvalues are in J. Two other special cases of this result are the Kadison inequality and the Choi inequality, see [2, 10]:

Theorem 1.1. *If $\Phi : \mathcal{M}_n \to \mathcal{M}_p$ is a unital positive linear mapping, then*

$$\text{The Choi inequality} \quad \Phi(A)^{-1} \leq \Phi\left(A^{-1}\right) \quad (A > 0). \tag{1}$$

$$\text{The Kadison inequality} \quad \Phi(A)^2 \leq \Phi\left(A^2\right).$$

In what follows, assume that m and M are positive real numbers such that $0 < m < M$ and $A, B \in \mathcal{M}_n$ are matrices with $0 < m \leq A, B \leq M$ except where otherwise clearly indicated. Moreover, assume that $\xi = \frac{(M+m)^2}{4Mm}$.

A counterpart to the choi inequality (1) has been presented by Marshal and Olkin [15] as follows:

$$\Phi\left(A^{-1}\right) \leq \xi\, \Phi(A)^{-1}. \tag{2}$$

A similar result for the Kadison inequality (see [16]) holds true:

$$\Phi\left(A^2\right) \leq \xi\, \Phi(A)^2. \tag{3}$$

The constant ξ is known as the Kantorovich constant. In addition, the inequalities of type (2) and (3), which present reverse of some inequalities, are known as Kantorovich type inequalities. For a recent survey concerning Kantorovich type inequalities the reader is referred to [17].

Regarding the possible squared version of (2), Lin [13] noticed that the inequality

$$\Phi(A) + Mm\Phi\left(A^{-1}\right) \leq M + m \tag{4}$$

holds for every unital positive linear mapping Φ. The inequality (4) turns out to be a tool for squaring matrix inequalities. Using (4) Lin [13] showed that (2) can be squared:

Theorem 1.2. *[13, Theorem 2.8] If $\Phi : \mathcal{M}_n \to \mathcal{M}_p$ is a unital positive linear mapping, then*

$$\Phi\left(A^{-1}\right)^2 \leq \xi^2\, \Phi(A)^{-2}. \tag{5}$$

As pointed out by Fu and He [5], the inequality (5) and the matrix monotonicity of $f(t) = t^s$ $(0 \leq s \leq 1)$ imply that

$$\Phi\left(A^{-1}\right)^r \leq \xi^r\, \Phi(A)^{-r} \tag{6}$$

for every $0 \leq r \leq 2$. In the case where $r \geq 2$, it was shown in [5] that:

Theorem 1.3. *[5, Theorem 3] For every $r \geq 2$*

$$\Phi\left(A^{-1}\right)^r \leq \left(\frac{(M+m)^2}{4^{\frac{2}{r}}Mm}\right)^r \Phi(A)^{-r}. \tag{7}$$

The matrix arithmetic–geometric mean inequality (the A-G mean inequality) (see for example [2, 10]) $A\sharp B \leq \frac{A+B}{2}$ implies that

$$\Phi(A\sharp B) \leq \Phi\left(\frac{A+B}{2}\right)$$

for every unital positive linear mapping Φ.

A converse of this inequality reads as follows (see [8])

$$\Phi\left(\frac{A+B}{2}\right) \leq \sqrt{\xi}\,\Phi(A\sharp B) \leq \xi\,\Phi\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right). \tag{8}$$

Lin [14] has tried to obtain an square version of (8) and proved that

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \xi^2\,\Phi^2(A\sharp B) \tag{9}$$
$$\Phi^2\left(\frac{A+B}{2}\right) \leq \xi^2\,(\Phi(A)\sharp\Phi(B))^2.$$

In Section 2, we give an extension of (9) using positive multilinear mappings. As noticed in [5], utilizing the Löwner-Heinz inequality, (9) can be extended as

$$\Phi^r\left(\frac{A+B}{2}\right) \leq \xi^r\,\Phi^r(A\sharp B) \tag{10}$$
$$\Phi^r\left(\frac{A+B}{2}\right) \leq \xi^r\,(\Phi(A)\sharp\Phi(B))^r$$

for every $0 \leq r \leq 2$. In the case where $r \geq 2$, Fu and He [5] showed that

Theorem 1.4. *[5, Theorem 4] If $r \geq 2$, then*

$$\Phi^r\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{r}}Mm}\right)^r \Phi^r(A\sharp B) \tag{11}$$
$$\Phi^r\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{r}}Mm}\right)^r (\Phi(A)\sharp\Phi(B))^r.$$

It is well-known that the arithmetic mean is the biggest and the harmonic mean is the smallest among symmetric means (see [11]). Fu and Hoa in [6] extended the inequalities (10) and (11) to arbitrary means between harmonic and arithmetic means. If σ, τ be two arbitrary means between harmonic and arithmetic means, then for every positive unital linear mapping Φ and $0 \leq r \leq 2$ they proved that

$$\Phi^r(A\sigma B) \leq \xi^r\,\Phi^r(A\tau B) \tag{12}$$
$$\Phi^r(A\sigma B) \leq \xi^r\,(\Phi(A)\tau\Phi(B))^r.$$

Also for $r \geq 2$ they showed that

$$\Phi^r(A\sigma B) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{r}}Mm}\right)^r \Phi^r(A\tau B) \tag{13}$$
$$\Phi^r(A\sigma B) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{r}}Mm}\right)^r (\Phi(A)\tau\Phi(B))^r.$$

Similar results can be found in [18].

Let $G(A_1, \cdots, A_k)$ denote the Ando–Li–Mathias geometric mean of strictly positive $A_i \in \mathcal{M}_n$ $(i = 1, \dots, k)$ [1]. It is known that it satisfies in the arithmetic-geometric-Harmonic mean inequality:

$$\left(\frac{A_1^{-1} + \cdots + A_k^{-1}}{k}\right)^{-1} \leq G(A_1, \cdots, A_k) \leq \frac{A_1 + \cdots + A_k}{k}. \tag{14}$$

The converse of (14) is a Kantorovich type inequality (see [7]) which reads as

$$\frac{A_1 + \cdots + A_k}{k} \leq \xi\, G(A_1, \cdots, A_k) \quad \text{and} \quad G(A_1, \cdots, A_k) \leq \xi \left(\frac{A_1^{-1} + \cdots + A_k^{-1}}{k}\right)^{-1} \tag{15}$$

where $A_i \in \mathcal{M}_n$ with $0 < m \leq A_i \leq M$ $(i = 1, \dots k)$.

Lin [14, Theorem 3.2] proved that (15) can be squared:

$$\left(\frac{A_1 + \cdots + A_k}{k}\right)^2 \leq \xi^2\, G(A_1, \cdots, A_k)^2. \tag{16}$$

In almost all of the above results, the following two key lemmas have been utilized:

Lemma 1.5. *[3] Let $A, B \in \mathcal{M}_n$. If $A, B \geq 0$, then*

$$\|AB\| \leq \frac{1}{4}\|A + B\|^2$$

for every unitarily invariant norm $\|\cdot\|$ on $\mathcal{M}_n$.

Lemma 1.6. *[2, Theorem 1.6.9] Let $A, B \in \mathcal{M}_n$. If $A, B \geq 0$ and $1 \leq r < \infty$, then*

$$\|A^r + B^r\| \leq \|(A + B)^r\| \tag{17}$$

for every unitarily invariant norm $\|\cdot\|$ on $\mathcal{M}_n$.

2. Positive Multilinear Mapping Inequalities

A mapping $\Phi : \mathcal{M}_n^k := \mathcal{M}_n \times \cdots \times \mathcal{M}_n \to \mathcal{M}_p$ is said to be multilinear if it is linear in each of its variable. A multilinear mapping $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is called positive if $A_i \geq 0$ for $i = 1, \cdots, k$ implies that $\Phi(A_1, \cdots, A_k) \geq 0$ and Φ is called unital if $\Phi(I, \dots, I) = I$. [4].

Recently, an extension of the Choi inequality (1) has been presented in [4] for positive multilinear mappings:

Lemma 2.1. *If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$\Phi(A_1, \cdots, A_k)^{-1} \leq \Phi\left(A_1^{-1}, \cdots, A_k^{-1}\right)$$

for all strictly positive matrices $A_i \in \mathcal{M}_n$ $(i = 1, \dots k)$.

Moreover, a multilinear version of (2), which is a Kantorovich type inequality for positive multilinear mappings, has also been presented in [4] as

Lemma 2.2. *[4, Corollary 5.3] If $A_i \in \mathcal{M}_n$ $(i = 1, \dots, k)$ are positive matrices with $0 < m \leq A_i \leq M$ for some positive real numbers $m < M$ and $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$\Phi(A_1^{-1}, \dots, A_k^{-1}) \leq \frac{(1 + v)^2}{4v}\Phi(A_1, \dots, A_k)^{-1}, \tag{18}$$

where $v = \frac{M}{m}$ is the condition number of each A_i.

Unfortunately, there is an error in the above lemma. The Kantorovich constant $\frac{(1+v)^2}{4v}$ does not work in (18) in general (see Remark 2.7). We give a correct form of (18) in the next lemma. The proof is quite similar to that of [4, Corollary 5.3] and we omit the proof.

Lemma 2.3. *Let $A_i \in \mathcal{M}_n$ $(i = 1, \cdots, k)$ with $0 < m \le A_i \le M$ for some positive real numbers $m < M$. If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$\Phi(A_1^{-1}, \cdots, A_k^{-1}) \le \frac{(1+v^k)^2}{4v^k} \Phi(A_1, \cdots, A_k)^{-1}, \tag{19}$$

where $v = \frac{M}{m}$ is the condition number of each A_i.

The following key lemma which is a direct conclusion of [16, Theorem 2.1] has an important role in obtaining our main results.

Lemma 2.4. *Let f be a positive strictly convex twice differentiable function on $[m, M]$ with $0 < m < M$ and let $C_i \in \mathcal{M}_n$ such that $\sum_{i=1}^k C_i^* C_i = I$. If $A_i \in \mathcal{M}_n$ with $0 < m \le A_i \le M$ $(i = 1, \cdots, k)$, then*

$$\sum_{i=1}^k C_i^* f(A_i) C_i \le a_f \sum_{i=1}^k C_i^* A_i C_i + b_f I \le \alpha f\left(\sum_{i=1}^k C_i^* A_i C_i\right), \tag{20}$$

where $a_f = \frac{f(M)-f(m)}{M-m}$, $b_f = \frac{Mf(m)-mf(M)}{M-m}$ and $\alpha = \max_{m \le t \le M}\left\{\frac{a_f t + b_f}{f(t)}\right\}$.

Lemma 2.5. *Let $A_i \in \mathcal{M}_n$ with $0 < m \le A_i \le M$ for some positive real numbers $m < M$ $(i = 1, \cdots, k)$. If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$\Phi(A_1^r, \cdots, A_k^r) \le a_r \Phi(A_1, \cdots, A_k) + b_r I \tag{21}$$

for all $r \ge 1$ and $r \le 0$ in which

$$a_r = \frac{M^{kr} - m^{kr}}{M^k - m^k}, \qquad b_r = \frac{M^k m^{kr} - m^k M^{kr}}{M^k - m^k}.$$

Proof. Assume that $A_i = \sum_{j=1}^n \lambda_{ij} P_{ij}$ $(i = 1, \cdots, k)$ is the spectral decomposition of each $A_i \in \mathcal{M}_n$ for which $\sum_{j=1}^n P_{ij} = I$. Put $C(j_1, \cdots, j_k) := \left(\Phi(P_{1j_1}, \cdots, P_{kj_k})\right)^{\frac{1}{2}}$ so that $\sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n C(j_1, \cdots, j_k)^* C(j_1, \cdots, j_k) = I$. It is well known that $f(t) = t^r$ is a positive strictly convex differentiable function on $(0, \infty)$. Then

$$\begin{aligned}
\Phi(A_1^r, A_2^r, \cdots, A_k^r) &= \Phi\left(\sum_{j=1}^n \lambda_{1j}^r P_{1j}, \cdots, \sum_{j=1}^n \lambda_{kj}^r P_{kj}\right) \\
&= \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n \lambda_{1j_1}^r \lambda_{2j_2}^r \cdots \lambda_{kj_k}^r \Phi(P_{1j_1}, \cdots, P_{kj_k}) \quad \text{(by multilinearity of } \Phi) \\
&= \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n C(j_1, \cdots, j_k) \left(\lambda_{1j_1} \lambda_{2j_2} \cdots \lambda_{kj_k}\right)^r C(j_1, \cdots, j_k) \\
&\le a_r \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n C(j_1, \cdots, j_k) \lambda_{1j_1} \lambda_{2j_2} \cdots \lambda_{kj_k} C(j_1, \cdots, j_k) + b_r I \quad \text{(by Lemma 2.4)} \\
&= a_r \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n \lambda_{1j_1} \lambda_{2j_2} \cdots \lambda_{kj_k} \Phi(P_{1j_1}, \cdots, P_{kj_k}) + b_r I \\
&= a_r \Phi(A_1, \cdots, A_k) + b_r I.
\end{aligned}$$

□

Now we give our first main result which is an square version of (19).

Theorem 2.6. *Let $A_i \in \mathcal{M}_n$ with $0 < m \leq A_i \leq M$ for some positive real numbers $m < M$ $(i = 1, \dots, k)$. If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$\Phi^2\left(A_1^{-1}, \cdots, A_k^{-1}\right) \leq \left(\frac{(1+v^k)^2}{4v^k}\right)^2 \Phi^{-2}(A_1, \cdots, A_k) \tag{22}$$

in which $v = \frac{M}{m}$ is the condition number of each A_i.

Proof. Assume that the convex function f is defined on $(0, \infty)$ by $f(t) = t^{-1}$. Applying Lemma 2.5 for $r = -1$ we get

$$a_r = \frac{-1}{M^k m^k}, \qquad b_r = \frac{M^k + m^k}{M^k m^k}$$

and

$$\Phi\left(A_1^{-1}, \cdots, A_k^{-1}\right) \leq \frac{-1}{M^k m^k}\Phi(A_1, \cdots, A_k) + \frac{M^k + m^k}{M^k m^k} I.$$

It follows that

$$\Phi(A_1, \cdots, A_k) + M^k m^k \Phi\left(A_1^{-1}, \cdots, A_k^{-1}\right) \leq M^k + m^k. \tag{23}$$

On the other hand Lemma 1.5 yields that

$$\begin{aligned} &M^k m^k \left\|\Phi(A_1, \cdots, A_k)\Phi\left(A_1^{-1}, \cdots, A_k^{-1}\right)\right\| \\ &\quad \leq \frac{1}{4}\left\|\Phi(A_1, \cdots, A_k) + M^k m^k \Phi\left(A_1^{-1}, \cdots, A_k^{-1}\right)\right\|^2. \end{aligned} \tag{24}$$

Combining (23) and (24) we obtain

$$\left\|\Phi(A_1, \cdots, A_k)\Phi\left(A_1^{-1}, \cdots, A_k^{-1}\right)\right\| \leq \frac{(M^k + m^k)^2}{4M^k m^k} = \frac{(1+v^k)^2}{4v^k}.$$

Therefore

$$\Phi^2\left(A_1^{-1}, \cdots, A_k^{-1}\right) \leq \left(\frac{(1+v^k)^2}{4v^k}\right)^2 \Phi^{-2}(A_1, \cdots, A_k).$$

□

Remark 2.7. *It should be remarked that the number k in the constant $\frac{(1+v^k)^2}{4v^k}$ is the best possible in* (19) *and so in* (22)*. To see this, consider the bilinear mapping $\Phi : \mathcal{M}_2^2 \to \mathcal{M}_p$ defined by $\Phi(A, B) = \langle x, \mathrm{diag}(A)\mathrm{diag}(B)x\rangle I_p$, where $x = [1/\sqrt{2}, 1/\sqrt{2}]^t \in \mathbb{C}^2$. If $A = B = \mathrm{diag}(1, 2)$ so that $v = 2$, then*

$$\Phi\left(A^{-1}, B^{-1}\right) = 0.625 I_p = \frac{(1+v^2)^2}{4v^2}\Phi(A, B)^{-1}.$$

Let $A_i \in \mathcal{M}_n$ with $0 < m \leq A_i \leq M$ for some real numbers $m < M$ $(i = 1, \dots, k)$. If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then matrix monotonicity of $f(t) = t^s$ $(0 \leq s \leq 1)$ and (22) imply that

$$\Phi^r\left(A_1^{-1}, \cdots, A_k^{-1}\right) \leq \left(\frac{(1+v^k)^2}{4v^k}\right)^r \Phi^{-r}(A_1, \cdots, A_k)$$

for every $0 \leq r \leq 2$. By a similar technique used in the proof of Theorem 2.6 and Applying Lemma 1.6 one can obtain the following result as a multilinear version of (7).

Theorem 2.8. *Let $A_i \in \mathcal{M}_n$ with $0 < m \le A_i \le M$ for some positive real numbers $m < M$ $(i = 1, \ldots, k)$. If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping and $r > 2$, then*

$$\Phi^r\left(A_1^{-1}, \cdots, A_k^{-1}\right) \le \left(\frac{(1+v^k)^2}{4^{\frac{2}{r}} v^k}\right)^r \Phi^{-r}(A_1, \cdots, A_k).$$

In [12] Lim and Pálfia established the notion of the matrix power means for k positive definite matrices ($k \ge 3$). First we recall some basic properties of matrix power means.

Assume that $\mathbb{A} = (A_1, \ldots, A_k)$ is a k-tuple of strictly positive matrices in $\mathcal{M}_n$ and $\omega = (\omega_1, \ldots, \omega_k)$ is a k-tuple of positive scalars with $\sum_{i=1}^k \omega_i = 1$. The matrix power mean of $A_1, \ldots, A_k$ [12], denoted by $P_t(\omega; \mathbb{A})$, is the unique positive invertible solution of the non-linear matrix equation

$$X = \sum_{i=1}^k \omega_i (X \sharp_t A_i),$$

where $t \in (0, 1]$ and $X \sharp_t A = X^{\frac{1}{2}}\left(X^{-\frac{1}{2}} A X^{-\frac{1}{2}}\right)^t X^{\frac{1}{2}}$ is the t-weighted geometric mean of strictly positive matrices X and A. If $t \in [-1, 0)$, then put $P_t(\omega; \mathbb{A}) := P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \ldots, A_k^{-1})$. The matrix power mean $P_t(\omega, \mathbb{A})$ interpolates between the weighted harmonic and arithmetic means. In particular, it satisfies the inequality

$$\left(\sum_{i=1}^k \omega_i A_i^{-1}\right)^{-1} \le P_t(\omega, \mathbb{A}) \le \sum_{i=1}^k \omega_i A_i \quad (t \in [-1, 1] \setminus \{0\}). \tag{25}$$

The Karcher mean of $A_1, \ldots, A_k$, denoted by $G(\omega; \mathbb{A})$, is the unique positive invertible solution of the Karcher equation

$$\sum_{i=1}^k \omega_i \log(X^{-1/2} A_i X^{-1/2}) = 0.$$

It is known that the Karcher mean coincides with the limit of matrix power means as $t \to 0$. For more information on the matrix power mean the reader is referred to [12]. We are going to present an extension of (9) for positive multilinear mappings.

Theorem 2.9. *Let $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ be a unital positive multilinear mapping and $\mathbb{A}^{(i)} = (A_1^{(i)}, \ldots, A_q^{(i)})$ $(i = 1, \ldots, k)$, where $0 < m \le A_j^{(i)} \le M$ for every $i = 1, \ldots, k$ and every $j = 1, \cdots, q$ and some positive real numbers $m < M$. Let $\omega^{(i)} = \left(\omega_1^{(i)}, \cdots, \omega_q^{(i)}\right)$ be a weight vector of positive scalars with $\sum_{j=1}^q \omega_j^{(i)} = 1$ for every $i = 1, \ldots, k$. If $t \in (0, 1]$, then*

$$\begin{aligned}
&\Phi^2\left(\sum_{j=1}^q \omega_j^{(1)} A_j^{(1)}, \cdots, \sum_{j=1}^q \omega_j^{(k)} A_j^{(k)}\right) \\
&\qquad \le \left(\frac{(1+v^k)^2}{4v^k}\right)^2 \Phi^2\left(P_t\left(\omega^{(1)}; \mathbb{A}^{(1)}\right), \ldots, P_t\left(\omega^{(k)}; \mathbb{A}^{(k)}\right)\right),
\end{aligned} \tag{26}$$

where $v = \frac{M}{m}$ is the condition number of each $A_j^{(i)}$.

Proof. Utilizing Lemma 1.5 we have

$$\begin{aligned}
&M^k m^k \left\| \Phi\left(\sum_{j=1}^q \omega_j^{(1)} A_j^{(1)}, \cdots, \sum_{j=1}^q \omega_j^{(k)} A_j^{(k)}\right) \Phi^{-1}\left(P_t\left(\omega^{(1)}; \mathbb{A}^{(1)}\right), \ldots, P_t\left(\omega^{(k)}; \mathbb{A}^{(k)}\right)\right) \right\| \\
&\le \frac{1}{4} \left\| \Phi\left(\sum_{j=1}^q \omega_j^{(1)} A_j^{(1)}, \cdots, \sum_{j=1}^q \omega_j^{(k)} A_j^{(k)}\right) + M^k m^k \Phi^{-1}\left(P_t\left(\omega^{(1)}; \mathbb{A}^{(1)}\right), \ldots, P_t\left(\omega^{(k)}; \mathbb{A}^{(k)}\right)\right) \right\|^2 .
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Phi\left(\sum_{j=1}^{q}\omega_j^{(1)}A_j^{(1)},\cdots,\sum_{j=1}^{q}\omega_j^{(k)}A_j^{(k)}\right)+M^km^k\Phi^{-1}\left(P_t\left(\omega^{(1)};\mathbb{A}^{(1)}\right),\ldots,P_t\left(\omega^{(k)};\mathbb{A}^{(k)}\right)\right)\\
&\le\Phi\left(\sum_{j=1}^{q}\omega_j^{(1)}A_j^{(1)},\cdots,\sum_{j=1}^{q}\omega_j^{(k)}A_j^{(k)}\right)+M^km^k\Phi\left(P_t\left(\omega^{(1)};\mathbb{A}^{(1)}\right)^{-1},\ldots,P_t\left(\omega^{(k)};\mathbb{A}^{(k)}\right)^{-1}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 2.1)}\\
&\le\Phi\left(\sum_{j=1}^{q}\omega_j^{(1)}A_j^{(1)},\cdots,\sum_{j=1}^{q}\omega_j^{(k)}A_j^{(k)}\right)+M^km^k\Phi\left(\sum_{j=1}^{q}\omega_j^{(1)}\left(A_j^{(1)}\right)^{-1},\cdots,\sum_{j=1}^{q}\omega_j^{(k)}\left(A_j^{(k)}\right)^{-1}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (25))}\\
&=\sum_{j_1=1}^{q}\cdots\sum_{j_k=1}^{q}\omega_{j_1}^{(1)}\ldots\omega_{j_k}^{(k)}\left(\Phi\left(A_{j_1}^{(1)},\cdots,A_{j_k}^{(k)}\right)+M^km^k\Phi\left(\left(A_{j_1}^{(1)}\right)^{-1},\cdots,\left(A_{j_k}^{(k)}\right)^{-1}\right)\right)\\
&\le\sum_{j_1=1}^{q}\cdots\sum_{j_k=1}^{q}\omega_{j_1}^{(1)}\ldots\omega_{j_k}^{(k)}(M^k+m^k)\qquad\qquad\text{(by (23))}\\
&=M^k+m^k.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\|\Phi\left(\sum_{j=1}^{q}\omega_j^{(1)}A_j^{(1)},\cdots,\sum_{j=1}^{q}\omega_j^{(k)}A_j^{(k)}\right)\Phi^{-1}\left(P_t\left(\omega^{(1)};\mathbb{A}^{(1)}\right),\ldots,P_t\left(\omega^{(k)};\mathbb{A}^{(k)}\right)\right)\right\|\\
&\le\frac{(M^k+m^k)^2}{4M^km^k},
\end{aligned}$$

which is equivalent to (26). □

Remark 2.10. *Let $\Phi:\mathcal{M}_n\to\mathcal{M}_p$ be a unital positive linear mapping and let $\mathbb{A}=(A_1,\cdots,A_q)$ is a q-tuple of matrices in $\mathcal{M}_n$ with $0<m\le A_i\le M$ for some positive real numbers $m<M$. If $\omega=(\omega_1,\cdots,\omega_q)$ is a weight vector such that $\omega_i\ge 0$ $(i=1,\ldots,q)$ with $\sum_{j=1}^{q}\omega_i=1$, then it follows from Theorem 2.9 that*

$$\Phi^2\left(\sum_{j=1}^{q}\omega_iA_i\right)\le\left(\frac{(1+v)^2}{4v}\right)^2\Phi^2\left(P_t\left(\omega;\mathbb{A}\right)\right),\tag{27}$$

where $v=\frac{M}{m}$. Tending t to zero, we get

$$\Phi^2\left(\sum_{j=1}^{q}\omega_iA_i\right)\le\left(\frac{(1+v)^2}{4v}\right)^2\Phi^2\left(G\left(\omega;\mathbb{A}\right)\right),\tag{28}$$

where $G(\omega;\mathbb{A})$ is the Karcher mean of $A_1,\cdots,A_q$. Inequality (28) *is an extension of* (9)*. Moreover, it follows from* (27) *that the Kantorovich inequality*

$$\Phi\left(\sum_{j=1}^{q}\omega_iA_i\right)\le\frac{(M+m)^2}{4Mm}\Phi\left(P_t\left(\omega;\mathbb{A}\right)\right)$$

holds true.

Remark 2.11. *Define a linear mapping $\Theta:\mathcal{M}_n\oplus\cdots\oplus\mathcal{M}_n\to\mathcal{M}_n\oplus\cdots\oplus\mathcal{M}_n$ by*

$$\Theta\left(\begin{pmatrix}A_1&0&\cdots&0\\0&A_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&A_q\end{pmatrix}\right)=\left(\sum_{j=1}^{q}\omega_iA_i\right)\otimes I_q,$$

where I_q is the identity matrix in $\mathcal{M}_q$. Then Θ is a unital positive linear mapping. Applying (5) *to Θ concludes that*

$$\Theta^2\left(\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{pmatrix}\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^2 \Theta^{-2}\left(\begin{pmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q^{-1} \end{pmatrix}\right)$$

That is

$$\left(\sum_{j=1}^{q} \omega_i A_i\right)^2 \le \left(\frac{(M+m)^2}{4Mm}\right)^2 \left(\left(\sum_{j=1}^{q} \omega_i A_i^{-1}\right)^{-1}\right)^2.$$

A special case of Theorem 2.9 gives an extension of (9) for multilinear mappings:

Corollary 2.12. *Suppose that $A_i, B_i \in \mathcal{M}_n$ with $0 < m \le A_i, B_i \le M$ for some positive real numbers $m < M$ $(i = 1, \cdots, k)$. If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$\Phi^2\left(\frac{A_1+B_1}{2}, \cdots, \frac{A_k+B_k}{2}\right) \le \left(\frac{(1+v^k)^2}{4v^k}\right)^2 \Phi^2\left(A_1 \sharp B_1, \cdots A_k \sharp B_k\right),$$

where $v = \frac{M}{m}$ is the condition number of each A_i and B_i.

Acknowledgement. The authors would like to thank the referee for valuable comments. The authors also thank Professor J.-C. Bourin to point out some errors in the previous version of the paper and for the remark 2.7.

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