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Optimality for *E-[0,1]* Convex Multi-Objective Programming

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Abstract. In this paper, we interest with deriving the sufficient and necessary conditions for optimal solution of special classes of Programming. These classes involve generalized *E-[0,1]* convex functions. Characterization of efficient solutions for *E-[0,1]* convex multi-objective Programming are obtained. Finally, sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution are derived.

1. Introduction

The study of multi-objective Programming was very active in recent years. The weak minimum (weakly efficient, weak Pareto) solution is an important concept in mathematical models, economics, decision theory, optimal control and game theory (see, for example, [4, 12]). In most works, an assumption of convexity was made for the objective functions. The extension of convexity is an area of active current research in the field of optimization theory. Various relaxations of convexity were possible, and were called generalized convex functions. The definition of generalized convex functions has occupied the attention of a number of mathematicians, for an overview of generalized convex functions we refer to [3, 8]. A significant generalization of convexity is the concept of *E-[0,1]* convexity [10]. *E-[0,1]* convexity depends on the effect of an operator E on the range of the function and the closed unit interval *[0.1].* Inspired and motivated by above reasons, the purpose of this paper is to formulate the problems which involve generalized *E-[0,1]* convex functions. The paper is organized as follows. In section 2, we define generalized *E-[0,1]* convex functions, which are called pseudo *E-[0,1]* convex functions, and obtain sufficient and necessary conditions for optimal solution of *E-[0,1]* convex Programming. In section 3, we consider the Mond-Weir type dual problem and generalized its results under the *E-[0,1]* convexity assumption. In section 4, we formulate the multi-objective programming which involves *E-[0,1]* convex functions. An efficient solution for the considered problem is characterized by weighting, and ε-constraint approaches. At the end of this paper, we obtain sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution for problems involving generalized *E-[0,1]* convex functions. Let us survey, briefly, the definitions and some results of E-[0,1] convexity.

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Definition 1.1. *[10] A real valued function $f : M \subseteq R^n \to R$ is said to be E-[0,1] convex function on M, with respect to $E : R \times [0, 1] \to R$, if M is convex set and, for each $x, y \in M$ and $\lambda_1, \lambda_2 \in [0, 1]$, $\lambda_1 + \lambda_2 = 1$,*

$$f(\lambda_1 x + \lambda_2 y) \leq E(f(x), \lambda_1) + E(f(y), \lambda_2).$$

If $f(\lambda_1 x + \lambda_2 y) \geq E(f(x), \lambda_1) + E(f(y), \lambda_2)$,then f is called *E-[0,1]* concave function on M. If the inequality signs in the previous two inequalities are strict, then f is called strictly *E-[0,1]* convex and strictly *E-[0,1]* concave, respectively.

Every *E-[0,1]* convex function, with respect to $E : R \times [0, 1] \to R$ is convex function if $E(f(x), \lambda) = \lambda f(x)$. Let $E : R \times [0, 1] \to R$ be a mapping such that $E(t, \lambda) = (1 + \lambda) t$, $t \in R$, $\lambda \in [0, 1]$, then the function $h(x) = \sum_{i=1}^{k} a_i f_i(x)$ is *E-[0,1]* convex on M *for* $a_i \geq 0$, $i = 1, 2, ..., k$ if the functions $f_i : R^n \to R$ are all *E-[0,1]* convex on a convex set $M \subseteq R^n$. Let $E : R \times [0, 1] \to R$ be a mapping such that $E(t, \lambda) = \min\{\lambda t, t\}$, $t \in R$, $\lambda \in [0, 1]$ then a numerical function $f : M \subset R^n \to R^+$ defined on convex set $M \subseteq R^n$ is *E-[0,1]* convex if and only if its *epi(f)* is convex. Let B be an open convex subset of R^n and let $E : R \times [0, 1] \to R$ be a mapping such that $E(t, \lambda) = \min\{\lambda, t\}$, $t \in R$, $\lambda \in [0, 1]$, then f is continuous on B if f is *E-[0,1]* convex on B. If $f : R^n \to R$ is a differentiable *E-[0,1]* convex function at $y \in M$ with respect to $E : R \times [0, 1] \to R$ such that $E(t, \lambda) = \min\{\lambda t, t\}$, $t \in R$, $\lambda \in [0, 1]$, then, for each $x \in M$ we have $(x - y)\nabla f(y) \leq f(x) - f(y)$. For more details about *E-[0,1]* convex functions, see[10].

Definition 1.2. *[11] A real valued function $f : M \subseteq R^n \to R$ is said to be quasi E-[0,1] convex function on M, with respect to $E : R \times [0, 1] \to R$, if M is convex set and, for each $x, y \in M$ and $\lambda_1, \lambda_2 \in [0, 1]$, $\lambda_1 + \lambda_2 = 1$,*

$$f(\lambda_1 x + \lambda_2 y) \leq \max\{E(f(x), \lambda_1), E(f(y), \lambda_2)\}.$$

If $f(\lambda_1 x + \lambda_2 y) \geq \min\{E(f(x), \lambda_1), E(f(y), \lambda_2)\}$, then f is called quasi *E-[0,1]* concave function on M. If the inequality signs in the previous two inequalities are strict, then f is called strictly quasi *E-[0,1]* convex and strictly quasi *E-[0,1]* concave respectively.

Every quasi *E-[0,1]* convex function, with respect to $E : R \times [0, 1] \to R$ is convex function if $E(f(x), \lambda) = \lambda f(x)$. Let $E : R \times [0, 1] \to R$ be a mapping such that $E(f(x), \lambda) = f(\lambda x)$ for each $x \in M$, $\lambda \in [0, 1]$, then $f(\sum_{i=1}^{n} \lambda_i x_i) \leq \max_{1 \leq i \leq n} E(f(x_i), \lambda_i)$ for each $x_i \in M$, $\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$, if $f : R^n \to R$ is *E-[0,1]* convex on a convex set $M \subseteq R^n$. Let $E : R \times [0, 1] \to R$ be a mapping such that $E(t, \lambda) = \min\{\lambda, t\}$, $t \in R$, $\lambda \in [0, 1]$, then the level set $L_{\alpha}^{E-[0,1]}$ is convex set for each $\alpha \in R$ if $f : R^n \to R$ is quasi *E-[0,1]* convex on a convex set $M \subseteq R^n$. Let $E : R \times [0, 1] \to R$ be a mapping such that $E(t, \lambda) = \max\{\lambda, t\}$, $t \in R$, $\lambda \in [0, 1]$ and let $\alpha = \min_{x} \min_{\lambda} E(f(x), \lambda)$, then the level set $L_{\alpha}^{E-[0,1]}$ is convex set If and only if f is quasi *E-[0,1]* convex . If $f : R^n \to R$ is a differentiable quasi *E-[0,1]* convex function at $y \in M$ with respect to $E : R \times [0, 1] \to R$ such that $E(t, \lambda) = \min\{\lambda, t\}$, $t \in R$, $\lambda \in [0, 1]$, then, for each $x \in M$ we have $(x - y)\nabla f(y) \leq 0$. For more details about quasi *E-[0,1]* convex functions, see [11].

2. *E-[0,1]* Convex Programming

In this section, we define generalized *E-[0,1]* convex functions, which are called pseudo strongly *E*-convex functions, and obtain sufficient and necessary conditions for optimal solution for problems involving generalized *E-[0,1]* convex functions.

Definition 2.1. *A real valued function $f : M \subseteq R^n \to R$ is said to be pseudo E-[0,1] convex function on a convex set $M \subseteq R^n$ if there exists a strictly positive function $b : R^n \times R^n \to R$ such that*

$$E(f(x), \lambda_1) < E(f(y), \lambda_2) \Rightarrow f(\lambda_1 x + \lambda_2 y) \leq E(f(y), \lambda_2) - \lambda_1 \lambda_2 b(x, y),$$

for all $x, y \in M$ and $\lambda_1, \lambda_2 \in [0, 1]$, $\lambda_1 + \lambda_2 = 1$.

Remark 2.2. *Every pseudo E-[0,1] convex function with respect to* $E : R \times [0,1] \to R$ *is convex function if* $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$.

Proposition 2.3. *Let* $E : R \times [0,1] \to R$ *be a map such that* $E(t,\lambda) = \max\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$. *A convex function* $f : R^n \to R$ *on a convex set* $M \subseteq R^n$*, is pseudo E-[0,1] convex function on M.*

Proof. Let $E(f(x),\lambda_1) < E(f(y),\lambda_2)$. Since f is a convex function on a convex set $M \subseteq R^n$, then for all $x, y \in M$ *and* $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$, we have

$$f(\lambda_1 x + \lambda_2 y) \le \lambda_1 f(x) + \lambda_2 f(y) \le \lambda_1 E(f(x),\lambda_1) + \lambda_2 E(f(y),\lambda_2).$$

That is

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y) &\le E(f(y),\lambda_2) + \lambda_1[E(f(x),\lambda_1) - E(f(y),\lambda_2)] \\ &\le E(f(y),\lambda_2) + \lambda_1\lambda_2[E(f(x),\lambda_1) - E(f(y),\lambda_2)] \\ &= E(f(y),\lambda_2) - \lambda_1\lambda_2[E(f(y),\lambda_2) - E(f(x),\lambda_1)] \\ &= E(f(y),\lambda_2) - \lambda_1\lambda_2 b(x,y), \end{aligned}$$

since $b(x,y) = E(f(y),\lambda_2) - E(f(x),\lambda_1) > 0$. The required result.

Theorem 2.4. *Let* $E : R \times [0,1] \to R$ *be a map such that* $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$ *and* $M \subseteq R^n$ *be a convex set. If* $f : R^n \to R$ *is a differentiable pseudo E-[0,1] convex function at* $y \in M$*, then* $(x-y)\nabla f(y) < 0$*, for each* $x \in M$*.*

Proof. Since $f : R^n \to R$ be a differentiable pseudo *E-[0,1]* convex function at $y \in M$, then

$$E(f(x),\lambda_1) < E(f(y),\lambda_2)$$

$$\Rightarrow f(\lambda_1 x + \lambda_2 y) \le E(f(y),\lambda_2) - \lambda_1\lambda_2 b(x,y) \le f(y) - \lambda_1\lambda_2 b(x,y),$$

for each $x \in M$ *and* $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$. That is

$$E(f(x),\lambda_1) < E(f(y),\lambda_2)$$

$$\Rightarrow f(y + \lambda_1(x-y)) \le f(y) - \lambda_1\lambda_2 b(x,y)$$

$$\Rightarrow f(y) + \lambda_1(x-y)\nabla f(y) + O(\lambda_1^2) \le f(y) - \lambda_1\lambda_2 b(x,y)$$

Dividing the above inequality by $\lambda_1 > 0$ and letting $\lambda_1 \to 0$, we get

$$(x-y)\nabla f(y) \le -b(x,y) < 0,$$

for each $x \in M$.

Remark 2.5. *Let* $E : R \times [0,1] \to R$ *be a map such that* $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R, \lambda \in [0,1]$ *and* $M \subseteq R^n$ *be a convex set. If* $f : R^n \to R$ *is a differentiable pseudo E-[0,1] convex function at* $y \in M$*, then* $(x-y)\nabla f(y) \ge 0 \Rightarrow E(f(x),\lambda_1) \ge E(f(y),\lambda_2)$*, for each* $x \in M$ *and* $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$.

Lemma 2.6. *Let* $E : R \times [0,1] \to R$ *be a mapping such that* $E(t,\lambda) = \lambda \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$*. If* $g_i : R^n \to R$ *is an E-[0,1] convex function on* $R^n, i = 1,2,...,m$*, then the set* $M = \{x \in R^n : g_i(x) \le 0, i = 1,2,...,m\}$ *is convex set.*

Proof. Since $g_i(x), i = 1,2,...,m$ are *E-[0,1]* convex functions with respect to $E(t,\lambda) = \lambda \min\{t,\lambda\}$, then, for each $x, y \in M$ *and* $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} g_i(\lambda_1 x + \lambda_2 y) &\le E(g_i(x),\lambda_1) + E(g_i(y),\lambda_2) \\ &= \lambda_1 \min\{g_i(x),\lambda_1\} + \lambda_2 \min\{g_i(y),\lambda_2\} \\ &\le \lambda_1 g_i(x) + \lambda_2 g_i(y) \le 0, \qquad i = 1,2,...,m, \end{aligned}$$

hence $\lambda_1 x + \lambda_2 y \in M$ for all $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$. This means that M is convex set.

Lemma 2.7. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \min\{t,\lambda\}, t \in R, \lambda \in [0,1]$. If $g_i : R^n \to R$ is a quasi E-[0,1] convex function on R^n, $i = 1,2,...,m$, then the set $M = \{x \in R^n : g_i(x) \leq 0, i = 1,2,...,m\}$ is convex set.*

Proof. Since $g_i(x), i = 1,2,...,m$ are quasi *E-[0,1]* convex functions with respect to $E(t,\lambda) = \min\{t,\lambda\}$, then, for each $x, y \in M$ and $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} g_i(\lambda_1 x + \lambda_2 y) &\leq \max[E(g_i(x),\lambda_1), E(g_i(y),\lambda_2)] \\ &\leq \max[g_i(x), g_i(y)] \\ &\leq 0, \qquad i = 1,2,...,m, \end{aligned}$$

hence $\lambda_1 x + \lambda_2 y \in M$ for all $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$. This means that M is convex set.

Now, we discuss the necessary and sufficient conditions for a feasible solution to be an optimal solution for *E-[0,1]* convex Programming. Consider the following *E-[0,1]* convex programming

$$(\bar{P}) \qquad \begin{aligned} &Min\ f(x), \\ &subiect\ to \\ &x \in M = \{x \in R^n : g_i(x) \leq 0, i = 1,2,...,m\}. \end{aligned}$$

Where $f : R^n \to R$ and $g_i : R^n \to R$, $i = 1,2,...,m$ are *E-[0,1]* convex functions with respect to $E : R\times[0,1] \to R$.

Theorem 2.8. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \lambda \min\{t,\lambda\}, t \in R, \lambda \in [0,1]$. Suppose that there exists a feasible solution x^* for $(\bar{P})$ and f, g are differentiable E-[0,1] convex functions with respect to the same E at x^*. If there is $u \in R^m$ and $u \geq 0$ such that (x^*, u) satisfies the following conditions:*

$$\begin{aligned} &\nabla f(x^*) + \nabla u^T g(x^*) = 0, \\ &u^T g(x^*) = 0, \qquad g(x^*) \leq 0, \end{aligned} \tag{1}$$

then x^* is an optimal solution for problem $(\bar{P})$.

Proof. For each $x \in M$, we have

$$\begin{aligned} f(x) - f(x^*) &\geq (x - x^*)\nabla f(x^*) = -(x - x^*)\nabla u^T g(x^*) \\ &\geq -u^T(g(x) - g(x^*)) = -u^T g(x) \geq 0, \end{aligned}$$

where the above inequalities hold because f, g are *E-[0,1]* convex at x^* with respect to the same E (see Theorem (4.1) in [10]). Thus, x^* is the minimizer of $f(x)$ under the constraint $g(x) \leq 0$ which implies that x^* is an optimal solution for problem $(\bar{P})$.

Remark 2.9. *[5] In Theorem (2.8) above, since $u \geq 0$, $g(x^*) \leq 0$, and $u^T\nabla g(x^*) = 0$, we have that*

$$u_i g_i(x^*) = 0,\ i = 1,2,\ldots,m. \tag{2}$$

If $I(x^*) = \{i : g_i(x^*) = 0\}$ and $J = \{i : g_i(x^*) < 0\}$, then $I \bigcup J = \{1,2,...,m\}$ and (2) gives that $u_i = 0$ for $i \in J$. It is obvious then, from the proof of Theorem (2.8), that E-[0,1] convexity of g_I at x^* is all that is needed instead of E-[0,1] convexity of g at x^* as was assumed in the theorem above.

Theorem 2.10. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \min\{t,\lambda\}, t \in R, \lambda \in [0,1]$. Suppose that there exists a feasible solution x^* for $(\bar{P})$, and scalars, $u_i \geq 0$, $i \in I(x^*)$, such that (1) of Theorem (2.8) holds. If f is pseudo E-[0,1] convex and g_I are quasi E-[0,1] convex at $x^* \in M$. Then, $E(f(x^*),\lambda_2)$, $\lambda_2 \in [0,1]$ is an optimal solution in objective space of problem $(\bar{P})$.*

Proof. Since $E(g_I(x),\lambda_1) \leq E(g_I(x^*),\lambda_2) = 0, u_i \geq 0, \lambda_1,\lambda_2 \in [0,1], \lambda_1 + \lambda_2 = 1$, and g_I are quasi *E-[0,1]* convex at x^*, we have

$$(x - x^*) \sum_{i \in I(x^*)} u_i[\nabla g_i(x^*)]^T \leq 0, \forall x \in M, \tag{3}$$

by using the above inequality in (1) and pseudo *E-[0,1]* convexity of f at x^*, we obtain

$$(x-x^*)[\nabla f(x^*)]^T \geq 0 \quad \Rightarrow E(f(x),\lambda_1) \geq E(f(x^*),\lambda_2) \quad \Rightarrow f(x) \geq E(f(x^*),\lambda_2).$$

Hence, $E(f(x^*),\lambda_2)$ is an optimal solution in objective space of problem $(\bar{P})$.

The next two theorems use the idea proposed by Mahajan and Vartak [6].

Theorem 2.11. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R, \lambda \in [0,1]$. Suppose that there exists a feasible solution x^* for $(\bar{P})$ and scalars $u_i \geq 0$, $i \in I(x^*)$ such that* (1) *of Theorem (2.8) holds. If f is pseudo E-[0,1] convex, and $u_I^T g_I$ is quasi E-[0,1] convex at $x^* \in M$, then $E(f(x^*),\lambda_2)$, $\lambda_2 \in [0,1]$ is an optimal solution in objective space of problem $(\bar{P})$.*

Proof. The proof of this theorem is similar to the proof of Theorem (2.10) except that the argument to get the inequality (3) is as follows:

Since $E(g_I(x),\lambda_1) \leq E(g_I(x^*),\lambda_2)$, $u_I \geq 0$, $\lambda_1,\lambda_2 \in [0,1]$, $\lambda_1+\lambda_2 = 1$, we obtain

$$u_I^T E(g_I(x),\lambda_1) \leq 0 = u_I^T E(g_I(x^*),\lambda_2)$$

for all $x \in M$. Quasi *E-[0,1]* convexity of $u_I^T g_I$ at x^*, yields

$$(x-x^*)\nabla(u_I^T g_I(x^*)) \leq 0, \ \forall \ x \in M.$$

We can proceed as in the above theorem to prove that $E(f(x^*),\lambda_2)$ is an optimal solution in objective space of problem $(\bar{P})$.

Theorem 2.12. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R, \lambda \in [0,1]$. Suppose that there exists a feasible point x^* for $(\bar{P})$ and the numerical function $f + u_I^T g_I$ is pseudo E-[0,1] convex at x^*. If there is scalars $u \in R^m$ such that (x^*,u) satisfies the conditions* (1) *of Theorem (2.8), then $E(f(x^*),\lambda_2)$, $\lambda_2 \in [0,1]$ is an optimal solution in objective space of problem $(\bar{P})$.*

Proof. The proof of this theorem is similar to the proof of Theorem (2.11) except that the arguments are as follows: (1) can be written as

$$\nabla f(x^*) + \nabla(u_I^T g_I(x^*)) = 0.$$

This can be rewritten in the form

$$(x-x^*)\nabla((f+u_I^T g_I)(x^*)) \leq 0, \forall x \in M,$$

which gives that

$$E((f+u_I^T g_I)(x^*),\lambda_2) \leq E((f+u_I^T g_I)(x),\lambda_1), \forall x \in M,$$

because, $f + u_I^T g_I$ is pseudo *E-[0,1]* convex at x^*, i.e.,

$$E((f+u_I^T g_I)(x^*),\lambda_2) \leq f(x) + (u_I^T g_I)(x), \ \forall x \in M.$$

It follows, by using the definition of I, that

$$E(f(x^*),\lambda_2) \leq f(x), \forall x \in M.$$

Hence, $E(f(x^*),\lambda_2)$ is an optimal solution in objective space of problem $(\bar{P})$.

Theorem 2.13. *(necessary optimality criteria) Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \lambda \min\{t,\lambda\}$, $t \in R, \lambda \in [0,1]$. Assume that x^* is an optimal solution for problem $(\bar{P})$ and there exist a feasible point x for $(\bar{P})$ such that $g_i(x) < 0$, $i = 1,2,...,m$. If g_i, $i \in I(x^*)$ is E-[0,1] convex at $x^* \in M$, then there exists scalars $u_i \geq 0, i \in I(x^*)$ such that (x^*,u_i) satisfies*

$$\nabla f(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0. \tag{4}$$

Proof. If we can show that

$$(x - x^*)\nabla g_I(x^*) \leq 0 \Rightarrow (x - x^*)\nabla f(x^*) \geq 0. \tag{5}$$

The result will follow as in [1] by applying Farkas' Lemma. Assume (5) does not hold, i.e, there exists $x \in R^n$ such that

$$(x - x^*)\nabla g_I(x^*) \leq 0 \Rightarrow (x - x^*)\nabla f(x^*) < 0. \tag{6}$$

Since by the assumed Slater-type condition,

$$g_i(\tilde{x}) - g_i(x^*) < 0, i \in I(x^*),$$

and from *E-[0,1]* convexity of g_i *at* x^*,we get

$$(\tilde{x} - x^*)^T\nabla g_i(x^*) < 0, i \in I(x^*). \tag{7}$$

Therefore from (6) and (7)

$$[(x - x^*) + \rho(\tilde{x} - x^*)]^T\nabla g_i(x^*) < 0, \ i \in I(x^*), \ \forall\rho > 0.$$

Hence for some positive β small enough

$$g_i(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) < g_i(x^*) = 0, i \in I(x^*).$$

Similarly, for $i \notin I(x^*)$, $g_i(x^*) < 0$ and for $\beta > 0$ small enough

$$g_i(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) \leq 0, i \notin I(x^*).$$

Thus, for β sufficiently small and all $\rho > 0$, $x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]$is feasible for problem $(\bar{P})$. For sufficiently small $\rho > 0$ (6) gives

$$f(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) < f(x^*), \tag{8}$$

which contradicts the optimality of x^* for $(\bar{P})$. Hence, the system (6) has no solution. The result then follows from an application of the Farkas Lemma, namely

$$\nabla f(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0, \ u_i \geq 0, i \in I(x^*).$$

3. Duality in *E-[0,1]* Convexity

We consider the Mond-Weir type dual and generalized its results under the *E-[0,1]* convexity assumptions. Consider the following Mond-Weir type dual of problem $(\bar{P})$.

$$(\bar{D}) \quad \begin{array}{l} max \ f(y), \\ subiect \ to \\ \nabla_x f(y) + u^T\nabla_x g(y) = 0, \\ u^T\nabla g(y) \geq 0, \qquad u \geq 0, \end{array}$$

wheref, g are differentiable functions defined on R^n. We now prove the following duality theorems relating problem $(\bar{P})$ and $(\bar{D})$.

Theorem 3.1. *(Weak Duality) Let* $E : R \times [0, 1] \rightarrow R$ *be a map such that* $E(t, \lambda) = \lambda \min\{t, \lambda\}$, $t \in R$, $\lambda \in [0, 1]$ *and let that there exists a feasible solution x for* $(\bar{P})$ *and (y,u) a feasible solution for* $(\bar{D})$. *If* f, g *are E-[0,1] convex functions at y, then y is an optimal solution for problem* $(\bar{P})$.

Proof. Since$f$ is *E-[0,1]* convex at y then

$$f(x) - f(y) \geq (x-y)^T \nabla_x f(y),$$

and by using (1) and *E-[0,1]* convexity of $g_i \quad \forall i$ at y, we have

$$\begin{aligned} f(x) - f(y) &\geq (x-y)^T \nabla_x f(y) = -(x-y)^T u^T \nabla g_x(y) \\ &\geq -u^T(g(x) - g(y)) = -u^T g(x) \geq 0, \end{aligned}$$

Thus, $f(x) \geq f(y)$, for all $x \in M$, which implies that y is the minimizer of $f(x)$ under the constraint $g(x) \leq 0$. Hence, y is an optimal solution for $(\bar{P})$.

Theorem 3.2. *(Strong Duality) Let $E : R \times [0,1] \to R$ be a map such that $E(t,\lambda) = \lambda \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$ and let that x^* be an optimal solutionfor $(\bar{P})$ and let g satisfy the Kuhn-Tucker constraint qualification at x^*. Then, there exists $u^* \in R^m$ such that (x^*, u^*) be a feasible solution for $(\bar{D})$ and the $(\bar{P})$-objective at x^* equal to the $(\bar{D})$-objective at (x^*, u^*). If f, g are E-[0,1] convex functions at x^* with respect to $E : R \times [0,1] \to R$ be a map such that $E(t,\lambda) = \lambda \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$, then (x^*, u^*) is an optimal solution for problem $(\bar{D})$.*

Proof. Since g satisfy the Kuhn-Tucker constraint qualification at x^*, then there exists $u^* \in R^m$, such that the following Kuhn-Tucker conditions are satisfied:

$$\nabla_x f(x^*) + u^{*t} \nabla_x g(x^*) = 0, \tag{9}$$

$$u^{*t} g(x^*) = 0, \tag{10}$$

$$g(x^*) \leq 0, \tag{11}$$

$$u^* \geq 0. \tag{12}$$

Equations (9),(10), (11) yield that (x^*, u^*) is a feasible solution for $(\bar{D})$. Also (10) yield that the $(\bar{P})$-objective at x^*equal to the $(\bar{D})$-objective at (x^*, u^*). Now, if (x^*, u^*) is not optimal solution for problem $(\bar{D})$, then there exists a feasible solution for $(\bar{D})$ $(\bar{x}, \bar{u})$ such that $f(\bar{x}) > f(x^*)$. This contradicts Theorem (7). Hence (x^*, u^*) is an optimal solution for problem $(\bar{D})$.

4. *E-[0,1]* Convex Multi-Objective Programming

In this section, we formulate a multi-objective programming which it involves *E-[0,1]* convex functions. An efficient solution for the considered problem is characterized by weighting, and ε-constraint approaches. At the end of this section, we obtain sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution for this kind of problems. An *E-[0,1]* convex multi-objective programming is formulated as follows:

$$(P) \qquad \begin{array}{l} Min\ f_j(x), \\ subiect\ to \\ x \in M = \{x \in R^n : g_i(x) \leq 0, i = 1,2,...,m\}, \end{array}$$

where $f_j : R^n \to R$, $j = 1,2,...,k$, and $g_i : R^n \to R$, $i = 1,2,...,m$ are *E-[0,1]* convex functions with respect to $E : R \times [0,1] \to R$.

Definition 4.1. *[2] A feasible solution x^* for (P) is said to be an efficient solution for (P) if and only if there is no other feasible x for (P) such that, for some $i \in \{1,2,...,k\}$,*

$$f_i(x) < f_i(x^*), f_j(x) \leq f_j(x^*), \quad for\ all\ j \neq i.$$

Definition 4.2. *[2] An efficient solution $x^* \in M$ for (P) is a properly efficient solution for (P) if there exists a scalar $\mu > 0$ such that for each i, $i = 1,2,...,k$ and each $x \in M$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one $j \neq i$ with $f_j(x) > f_j(x^*)$ and $[f_i(x) - f_i(x^*)]/[f_j(x^*) - f_j(x)] \leq \mu$.*

Lemma 4.3. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \lambda \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$. If $f : R^n \to R^k$ is an E-[0,1] convex function on a convex set $M \subseteq R^n$, then the set $A = \bigcup_{x\in M} A(x)$ is convex set such that*

$$A(x) = \{z : z \in R^k, z > f(x) - f(x^*)\},\ x \in M.$$

Proof. Let $z^1, z^2 \in A,$ then for all $x^1, x^2 \in M$ and $\lambda_1, \lambda_2 \in [0, 1], \lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned}
\lambda_1 z^1 + \lambda_2 z^2 &> \lambda_1[f(x^1) - f(x^*)] + \lambda_2[f(x^2) - f(x^*)] \\
&= \lambda_1 f(x^1) + \lambda_2 f(x^2) - f(x^*) \\
&\geq \lambda_1 \min(f(x^1), \lambda_1) + \lambda_2 \min(f(x^2)\ , \lambda_2) - f(x^*) \\
&= E(f(x^1), \lambda_1) + E(f(x^2)\ , \lambda_2) - f(x^*) \\
&\geq f(\lambda_1 x^1 + \lambda_2 x^2) - f(x^*),
\end{aligned}$$

since f is *E-[0,1]* convex function on a convex set M. Then, $\lambda_1 z^1 + \lambda_2 z^2 \in A$, and hence A is convex set.

4.1. Characterizing Efficient Solutions by Weighting Approach

To characterizing an efficient solution for problem (P) by weighting approach [2] let us scalar problem (P) to become in the form.

$$(\mathrm{P}_w) \qquad Min \sum_{j=1}^{k} w_j f_j(x), \quad subiect\ to \quad x \in M,$$

where $w_j \geq 0,\ j = 1,2,...,k,\ \sum_{j=1}^{k} w_j = 1$ and $f_j,\ j = 1,2,...,k$ are *E-[0,1]* convex functions with respect to $E : R \times [0,1] \to R$ such that $E(t,\lambda) = \lambda \min\{t,\lambda\},\ t \in R,\ \lambda \in [0,1]$ on convex set M.

Theorem 4.4. *If $\bar{x} \in M$ is an efficient solution for problem* (P)*, then there exist $w_j \geq 0,\ \ j = 1,2,...,k,\ \ \sum_{j=1}^{k} w_j = 1$ such that $\bar{x}$ is an optimal solution for problem* (P_w).

Proof. Let $\bar{x} \in M$ be an efficient solution for problem (P), then the system $f_j(x) - f_j(\bar{x}) < 0,\ j = 1,2,...,k$ has no solution $x \in M$. Upon Lemma (4.3) and applying the generalized Gordan theorem [7], there exist $p_j \geq 0,\ \ j = 1,2,...,k$ such that $p_j[f_j(x) - f_j(\bar{x})] \geq 0,\ j = 1,2,...,k,$ and $\frac{p_j}{\sum_{j=1}^{k} p_j} f_j(x) \geq \frac{p_j}{\sum_{j=1}^{k} p_j} f_j(\bar{x})$.

Denote $w_j = \frac{p_j}{\sum_{j=1}^{k} p_j}$, then $w_j \geq 0,\ \ j = 1,2,...,k,\ \ \sum_{j=1}^{k} w_j = 1$, and $\sum_{j=1}^{k} w_j f_j(\bar{x}) \leq \sum_{j=1}^{k} w_j f_j(x)$. Hence $\bar{x}$ is an optimal solution for problem(P_w).

Theorem 4.5. *If $\bar{x} \in M$ is an optimal solution for $(\mathrm{P}_{\bar{w}})$ corresponding to $\bar{w}_j$, then $\bar{x}$ is an efficient solution for problem* (P) *if either one of the following two conditions holds:*
(i) $\bar{w}_j > 0,\ \forall\, j = 1,2,...,k$; or (ii) $\bar{x}$ is the unique solution of $(\mathrm{P}_{\bar{w}})$.

Proof. To proof see V. Chankong, Y. Y. Haimes [2].

4.2. Characterizing Efficient Solutions by ε-Constraint Approach

An ε-constraint approach is one of the common approaches for characterizing efficient solutions of multi-objective Programming [2]. In the following we shall characterizing an efficient solution for multi-objective *E-[0,1]* convex programming (P) in term of an optimal solution of the following scalar problem.

$$\mathrm{P}_q(\varepsilon, E) \qquad \begin{array}{l} Min\ f_q(x), \\ subiect\ to \quad x \in M, \\ f_j(x) \leq E(\varepsilon_j, \lambda_j), \quad j = 1,2,...,k,\ j \neq q. \end{array}$$

Where $f_j,\ j = 1,2,...,k$ are *E-[0,1]* convex functions with respect to $E : R \times [0,1] \to R$ such that $E(t,\lambda) = \min\{\,t\,,\lambda\,\},\ t \in R,\ \lambda \in [0,1]$ on convex set M.

Theorem 4.6. *If $\bar{x} \in M$ is an efficient solution for problem* (P)*, then $\bar{x}$ is an optimal solution for problem* $\mathrm{P}_q(\bar{\varepsilon}, \bar{E})$ *and $\bar{\varepsilon}_j = f_j(\bar{x})$.*

Proof. Let $\bar{x}$ be not optimal solution for $\mathrm{P}_q(\bar{\varepsilon}, \bar{E})$ where $\bar{\varepsilon}_j = f_j(\bar{x}),\ j = 1, 2, ..., k$. So there exists $x \in M$ such that

$$f_q(x) < f_q(\bar{x}),$$

$$f_j(x) \leq \bar{E}(\bar{\varepsilon}_j, \bar{\lambda}_j) \leq \bar{\varepsilon}_j = f_j(\bar{x}),\ j = 1, 2, ..., k,\ j \neq q,$$

since $\bar{E}(\bar{\varepsilon}_j, \bar{\lambda}_j) = \min(\bar{\varepsilon}_j, \bar{\lambda}_j)$ and convexity of M. This implies that the system $f_j(x) - f_j(\bar{x}) < 0,\ j = 1, 2, ..., k$ has a solution $x \in M$. Thus, $\bar{x}$ is inefficient solution for problem (P) which is a contradiction. Hence is $\bar{x}$ an optimal solution for problem $\mathrm{P}_q(\bar{\varepsilon}, \bar{E})$.

Theorem 4.7. *Let $\bar{x} \in M$ be an optimal solution, for all q of* $\mathrm{P}_q(\bar{\varepsilon}, \bar{E})$*, where $\bar{\varepsilon}_j = f_j(\bar{x}),\ j = 1, 2, ..., k$. Then $\bar{x}$ is an efficient solution for problem* (P).

Proof. Since $\bar{x} \in M$ is an optimal solution for $\mathrm{P}_q(\bar{\varepsilon}, \bar{E})$, where $\bar{\varepsilon}_j = f_j(\bar{x}),\ j = 1, 2, ..., k$, then, for each $x \in M$, we get

$$f_q(\bar{x}) < f_q(x),$$

$$f_j(x) \leq \bar{E}(\bar{\varepsilon}_j, \bar{\lambda}_j) \leq \bar{\varepsilon}_j = f_j(\bar{x}),\ j = 1, 2, ..., k,\ j \neq q,$$

where $\bar{E}(\bar{\varepsilon}_j, \bar{\lambda}_j) = \min(\bar{\varepsilon}_j, \bar{\lambda}_j)$. This implies the system $f_j(x) - f_j(\bar{x}) < 0,\ j = 1, 2, ..., k$ has no solution $x \in M$, i.e. $\bar{x}$ is an efficient solution for problem (P).

4.3. Sufficient and Necessary Conditions for Efficiency

In this section, we discuss the sufficient and necessary conditions for a feasible solution x^* to be efficient or properly efficient for problem (P) in the form of the following theorems.

Theorem 4.8. *Let $E : R \times [0, 1] \to R$ be a mapping such that $E(t, \lambda) = \lambda \min\{t, \lambda\}$, $t \in R$, $\lambda \in [0, 1]$. Suppose that there exist a feasible solution x^* for (P) and scalars $\gamma_i > 0,\ i = 1, 2, ..., k,\ u_i \geq 0,\ i \in I(x^*)$ such that*

$$\sum_{i=1}^{k} \gamma_i \nabla f_i(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0. \tag{13}$$

If $f_i,\ i = 1, 2, ..., k$, and g_i ,$i \in I(x^*)$ are differentiable E-[0,1] convex functions at $x^* \in M$, then x^* is a properly efficient solution for problem (P).

Proof. Since $f_i,\ i = 1, 2, ..., k,$ *and* $g_i,\ i \in I(x^*)$ are differentiable *E-[0,1]* convex functions at $x^* \in M$, so for any $x \in M$, we have

$$\begin{aligned}
\sum_{i=1}^{k} \gamma_i f_i(x) - \sum_{i=1}^{k} \gamma_i f_i(x^*) &\geq (x - x^*) \sum_{i=1}^{k} \gamma_i [\nabla f_i(x^*)]^T \\
&= -(x - x^*) \sum_{i \in I(x^*)} u_i [\nabla g_i(x^*)]^T \\
&\geq \sum_{i \in I(x^*)}^{k} u_i g_i(x^*) - \sum_{i \in I(x^*)}^{k} u_i g_i(x) \\
&= -\sum_{i \in I(x^*)} u_i g_i(x) \geq 0.
\end{aligned}$$

Thus, $\sum_{i=1}^{k} \gamma_i f_i(x) \geq \sum_{i=1}^{k} \gamma_i f_i(x^*)$, for all $x \in M$, which implies that x^* is the minimizer of $\sum_{i=1}^{k} \gamma_i f_i(x)$ under the constraint $g(x) \leq 0$. Hence, from Theorem (4.11) of [2], x^* is a properly efficient solution for problem (P).

Theorem 4.9. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \lambda \min\{t,\lambda\}$, $t \in R, \lambda \in [0,1]$. Suppose that there exist a feasible solution x^* for (P) and scalars $\gamma_i \geq 0, i = 1,2,...,k$, $\sum_{i=1}^{k} \gamma_i = 1$, $u_i \geq 0, i \in I(x^*)$, such that the triplet (x^*, γ_i, u_i) satisfies* (13) *of Theorem (4.8). If $\sum_{i=1}^{k} \gamma_i f_i$ is strictly E-[0,1] convex, and g_I is E-[0,1] convex at $x^* \in M$, then x^* is an efficient solution for problem (P).*

Proof. Suppose that x^* is not an efficient solution for (P),then, there exists a feasible $x \in M$, and index r such that

$$f_r(x) < f_r(x^*),$$
$$f_i(x) \leq f_i(x^*), \quad for\ all\ i \neq r.$$

Since $\sum_{i=1}^{k} \gamma_i f_i$ is strictly *E-[0,1]* convex at x^*, then the previous two inequalities lead to

$$0 \geq \sum_{i=1}^{k} \gamma_i f_i(x) - \sum_{i=1}^{k} \gamma_i f_i(x^*) \quad \Rightarrow \quad 0 > (x - x^*) \sum_{i=1}^{k} \gamma_i [\nabla f_i(x^*)]^T. \tag{14}$$

Also, *E-[0,1]* convexity of g_i, $i \in I(x^*)$ at x^* implies

$$(x - x^*)\nabla g_i(x^*) \leq g_i(x) - g_i(x^*) \quad \Rightarrow \quad (x - x^*)\nabla g_i(x^*) \leq 0, \ i \in I(x^*),$$

and, for $u_i \geq 0, i \in I(x^*)$, we get

$$(x - x^*) \sum_{i \in I(x^*)} u_i [\nabla g_i(x^*)]^T \leq 0. \tag{15}$$

Adding (14) and (15), contradicts (13). Hence, x^* is an efficient solution for problem (P).

Remark 4.10. *Similarly as in Theorem (4.8), it can be easily seen that x^* becomes properly efficient solution for (P), in the above theorem, if $\gamma_i > 0$, for all $i = 1,2,...,k$.*

Theorem 4.11. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$. Suppose that there exists a feasible solution x^* for (P) and scalars $\gamma_i > 0, i = 1,2,...,k$, $u_i \geq 0, i \in I(x^*)$, such that* (13) *of Theorem (4.8) holds. If $\sum_{i=1}^{k} \gamma_i f_i$ is pseudo E-[0,1] convex, and g_I are quasi E-[0,1] convex at $x^* \in M$, then $E(f(x^*), \lambda_2)$, $\lambda_2 \in [0,1]$ is a properly nondominated solution in objective space of problem (P).*

Proof. Since $E(g_I(x), \lambda_1) \leq E(g_I(x^*), \lambda_2) = 0$, $\lambda_1, \lambda_2 \in [0, 1]$, $\lambda_1 + \lambda_2 = 1$, and from quasi *E-[0,1]* convex of g_I at x^*, $u_I \geq 0$, we get

$$(x - x^*) \sum_{i \in I(x^*)} u_i [\nabla g_i(x^*)]^T \leq 0, \forall x \in M,$$

by using the above inequality in (13), and from pseudo *E-[0,1]* convexity of $\sum_{i=1}^{k} \gamma_i f_i$ at x^*, we get

$$(x - x^*) \sum_{i=1}^{k} \gamma_i [\nabla f_i(x^*)]^T \geq 0 \quad \Rightarrow \quad \sum_{i=1}^{k} \gamma_i E(f_i(x), \lambda_1) \geq \sum_{i=1}^{k} \gamma_i E(f_i(x^*), \lambda_2).$$

$$\Rightarrow \quad \sum_{i=1}^{k} \gamma_i f_i(x) \geq \sum_{i=1}^{k} \gamma_i E(f_i(x^*), \lambda_2)$$

Hence, $E(f(x^*), \lambda_2)$ is a properly nondominated solution in objective space of problem (P).

Theorem 4.12. *Let $E : R \times [0,1] \to R$ be a mapping such that $E(t,\lambda) = \min\{t,\lambda\}$, $t \in R$, $\lambda \in [0,1]$. Suppose that there exists a feasible solution x^* for (P) and scalars $\gamma_i \geq 0, i = 1,2,...,k$, $\sum_{i=1}^{k} \gamma_i = 1$, $u_i \geq 0, i \in I(x^*)$, such that* (13) *of Theorem (4.8) holds. If $\sum_{i=1}^{k} \gamma_i f_i$ is strictly pseudo E-[0,1] convex and g_I is quasi E-[0,1] convex at $x^* \in M$, then $E(f(x^*), \lambda_2)$, $\lambda_2 \in [0,1]$ is a nondominated solution in objective space of problem (P).*

Proof. Suppose that $E(f(x^*), \lambda_2)$ is dominated solution for (P),then, there exist a feasible x for (P), and index r such that

$$f_r(x) < E(f_r(x^*), \lambda_2), \quad f_i(x) \le E(f_i(x^*), \lambda_2), \ \ for\ all\ i \ne r.$$

Since $E(t, \lambda_1) = \min\{t, \lambda_1\}$, $t \in R$, $\lambda_1 \in [0, 1]$, we have

$$E(f_r(x), \lambda_1) < E(f_r(x^*), \lambda_2), \ \ E(f_i(x), \lambda_1) \le E(f_i(x^*), \lambda_2), \ \ \forall\ i \ne r.$$

Strictly pseudo *E-[0,1]* convexity of $\sum_{i=1}^{k} \gamma_i f_i$ at x^* implies that

$$\sum_{i=1}^{k} \gamma_i E(f_i(x), \lambda_1) \le \sum_{i=1}^{k} \gamma_i E(f_i(x^*), \lambda_2) \quad \Rightarrow \quad (x - x^*) \sum_{i=1}^{k} \gamma_i [\nabla f_i(x^*)]^T < 0.$$

Also, quasi *E-[0,1]* convexity of g_I at x^* implies that

$$E(g_I(x), \lambda_1) \le E(g_I(x^*), \lambda_2) = 0 \Rightarrow \ \ (x - x^*)\nabla g_I(x^*) \le 0.$$

The proof now follows along similar to in Theorem (4.9).

Remark 4.13. *Similarly as in Theorem (4.11), it can be easily seen that* $E(f(x^*), \lambda_2)$, $\lambda_2 \in [0, 1]$ *becomes properly nondominated solution for (P), in the above Theorem, if* $\gamma_i > 0$, *for all* $i = 1, 2, ..., k$.

Theorem 4.14. *Let* $E : R \times [0, 1] \to R$ *be a mapping such that* $E(t, \lambda) = \min\{t, \lambda\}$, $t \in R$, $\lambda \in [0, 1]$. *Suppose that there exists a feasible solution* x^* *for (P) and scalars* $\gamma_i > 0, i = 1, 2, ..., k$, $u_i \ge 0, i \in I(x^*)$ *such that* (13) *of Theorem (4.8) holds. If* $\sum_{i=1}^{k} \gamma_i f_i$ *is pseudo E-[0,1] convex and* $u_I g_I$ *is quasi E-[0,1] convex at* $x^* \in M$, *then* $E(f(x^*), \lambda_2)$, $\lambda_2 \in [0, 1]$ *is a properly nondominated solution in objective space of problem (P).*

Proof. The proof is similar to the proof of Theorem (4.11).

Theorem 4.15. *Let* $E : R \times [0, 1] \to R$ *be a mapping such that* $E(t, \lambda) = \min\{t, \lambda\}$, $t \in R$, $\lambda \in [0, 1]$. *Suppose that there exists a feasible solution* x^* *for (P) and scalars* $\gamma_i \ge 0, i = 1, 2, ..., k, \sum_{i=1}^{k} \gamma_i = 1$, $u_i \ge 0, i \in I(x^*)$, *such that* (13) *of Theorem (4.8) holds. If* $I(x^*) \ne \phi$, $\sum_{i=1}^{k} \gamma_i f_i$ *is quasi E-[0,1] convex and* $u_I g_I$ *is strictly pseudo E-[0,1] convex at* $x^* \in M$, *then* $E(f(x^*), \lambda_2)$, $\lambda_2 \in [0, 1]$ *is a nondominated solution in objective space of problem (P).*

Proof. The proof is similar to the proof of Theorem (4.12).

Remark 4.16. *Similarly as in Theorem (4.11), it can be easily seen that* $E(f(x^*), \lambda_2)$, $\lambda_2 \in [0, 1]$ *becomes properly nondominated solution for (P), in the above Theorem, if* $\gamma_i > 0$, *for all* $i = 1, 2, ..., k$.

Theorem 4.17. *(Necessary Efficiency Criteria) Let* $E : R \times [0, 1] \to R$ *be a mapping such that* $E(t, \lambda) = \lambda \min\{t, \lambda\}$, $t \in R$, $\lambda \in [0, 1]$ *and* x^* *be a properly efficient solution for problem (P). Assume that there exists a feasible point* x *for (P) such that* $g_i(x) < 0$, $i = 1, 2, ..., m$, *and each* g_i, $i \in I(x^*)$ *is E-[0,1] convex at* $x^* \in M$. *Then, there exists scalars* $\gamma_i > 0, i = 1, 2, ..., k$ *and* $u_i \ge 0, i \in I(x^*)$, *such that the triplet* (x^*, γ_i, u_i) *satisfies*

$$\sum_{i=1}^{k} \gamma_i \nabla f_i(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0. \tag{16}$$

Proof. Let the following system

$$\begin{array}{ll} (x - x^*)^T \nabla f_q(x^*) < 0, & \\ (x - x^*)^T \nabla f_i(x^*) \le 0, & for\ all\ i \ne q \ \ . \\ (x - x^*)^T \nabla g_i(x^*) \le 0, & i \in I(x^*), \end{array} \tag{17}$$

has a solution for every $q = 1, 2, ..., k$. Since by the assumed Slater-type condition,

$$g_i(\tilde{x}) - g_i(x^*) < 0, i \in I(x^*),$$

and from *E-[0,1]* convexity of g_i *at* x^*, we get

$$(\tilde{x} - x^*)^T \nabla g_i(x^*) < 0, i \in I(x^*). \tag{18}$$

Therefore from (17) and (18)

$$[(x - x^*) + \rho(\tilde{x} - x^*)]^T \nabla g_i(x^*) < 0, \;\; \forall i \in I(x^*), \;\; \rho > 0.$$

Hence for some positive β small enough

$$g_i(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) < g_i(x^*) = 0, i \in I(x^*).$$

Similarly, for $i \notin I(x^*), g_i(x^*) < 0$ and for $\beta > 0$ small enough

$$g_i(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) \leq 0, i \notin I(x^*).$$

Thus, for β sufficiently small and all $\rho > 0, x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]$ is feasible for problem (P). For sufficiently small $\rho > 0$ (17) gives

$$f_q(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) < f_q(x^*). \tag{19}$$

Now for all $j \neq q$ such that

$$f_j(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) > f_j(x^*), \tag{20}$$

consider the ratio

$$\frac{N(\beta, \rho)}{D(\beta, \rho)} = \frac{[f_q(x^*) - f_q(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)])]/\beta}{[f_j(x^* + \beta[(x - x^*) + \rho(\tilde{x} - x^*)]) - f_j(x^*)]/\beta}. \tag{21}$$

From(17), $N(\beta, \rho) \to -(x - x^*)^T \nabla f_q(x^*) > 0$. Similarly, $D(\beta, \rho) \to (x - x^*)^T \nabla f_j(x^*) \leq 0$; but, by (20) $D(\beta, \rho) > 0$, so $D(\beta, \rho) \to 0$. Thus, the ratio in (21) becomes unbounded, contradicting the proper efficiency of x^* for (P). Hence, for each $q = 1, 2, ..., k$, the system (17) has no solution. The result then follows from an application of the Farkas Lemma, namely

$$\sum_{i=1}^{k} \gamma_i \nabla f_i(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0, u \geq 0.$$

Theorem 4.18. *Assume that x^* is an efficient solution for problem (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then, there exist scalars $\gamma_i \geq 0, i = 1, 2, ..., k$, $\sum_{i=1}^{k} \gamma_i = 1$, $u_j \geq 0, j = 1, 2, ..., m$, such that*

$$\sum_{i=1}^{k} \gamma_i \nabla f_i(x^*) + \sum_{j=1}^{m} u_j \nabla g_j(x^*) = 0, \quad \sum_{j=1}^{m} u_j g_j(x^*) = 0.$$

Proof. Since every efficient solution is a weak minimum, then by applying Theorem (2.2) of Weir and Mond [9] for x^*, we get that there exists $\gamma \in R^k, \;\; u \in R^m$ such that

$$\gamma^T \nabla f(x^*) + u^T \nabla g(x^*) = 0, \qquad u^T g(x^*) = 0,$$

$$u \geq 0, \qquad \gamma \geq 0, \qquad \gamma^T e = 0,$$

where $e = (1, 1, ..., 1) \in R^k$.

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