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Nonsmooth Quasi-Variational-like Inequalities with Applications to Nonsmooth Vector Quasi-Optimization

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Abstract. We consider nonsmooth vector quasi-variational-like inequalities and nonsmooth vector quasi-optimization problems. We utilize the method of scalarization to define nonsmooth quasi-variational-like inequalities by means of Clarke generalized directional derivative. We then study their relations with the problem of vector quasi-optimization and its scalarized version. Under the assumption of pseudomonotonicity and then densely pseudomonotonicity, we present some existence results for solutions to nonsmooth quasi-variational-like inequalities. To the best of our knowledge, the results we obtained are new in the sense of utilizing the scalarization method.

1. Introduction

Vector variational inequalities, which were first introduced and studied in the early eighties by Giannessi [20], proved to be useful in studying vector optimization problems, see for example [1–4, 6–11, 15, 19–21, 24, 28, 36–38, 41]. The motivation for extending the classical variational inequalities comes from the different applications where variational inequalities and its extensions are employed successfully. Such applications occur in mathematical programming, game theory, partial differential equation theory, economics and control theory.

Vector variational-like inequalities in a reflexive Banach Space was introduced by Ansari [4]. Since then various authors studied vector variational-like and extended them in different directions, see [1–3, 7, 8, 10, 11, 19, 35–37, 41] and the references within. Vector variational-like inequalities proved to be very successful in studying vector optimization problems for nonconvex but invex functions; see [1, 11, 19, 21, 37, 41] and the references therein. When the objective function is not necessarily differentiable, then Ansari and Yao [10] considered the generalized vector variational-like inequalities defined by means of a subdifferential of a nonconvex function and studied the existence of solutions of vector optimization problems by using such vector variational inequalities. A vector optimization problem may have a nonsmooth objective function. In that case, one has to consider a generalized directional derivative or some kind of subdifferentials. The vector variational (-like) inequalities defined by means of Dini directional derivatives are considered in

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[2, 7, 15, 24, 28]. The relation between a vector optimization problem and these kinds of vector variational (-like) inequalities are also studied in these references. The generalized vector variational (-like) inequalities defined by means of Clarke generalized directional derivative or by means of Clarke subdifferentials are considered in [3, 8, 31, 36, 38] and the references therein. Recently, Santos et al [38] considered scalarized variational-like inequality problem defined by means of Clarke generalized directional derivative, and showed that each of its solution is a weak efficient solution of a vector optimization problem. They proved the existence of a solution of scalarized variational-like inequality problem.

In quasi-variational(-like) inequalities , a variational(-like) inequality and a fixed-point problem are required to be simultaneously solved. The theory of quasi-variational inequalities has developed over the last four decades to be among the most useful fields of pure, applied and industrial mathematics. This theory has proved successful as a tool for solving a wide range of problems arising in different fields such as optimization, optimal control, game theory, financial mathematics, structural mechanics, elasticity, economics etc. [16, 27, 32–34]. Despite the challenging nature of the existence theory of quasi-variational inequalities, it has received a growing interest from researchers from different branches of mathematics and engineering. Some of the first existence results of generalized quasi-variational-like inequalities were introduced and studied by Chen and Li [13] and Lee et al. [29]. More recently, Ansari et al. [5] introduced weighted quasi-variational inequalities (WQVIP) over product of sets and system of weighted quasi-variational inequalities and studied the relationships between the solutions of these problems and the solution of weighted constrained Nash equilibrium problem. They derived existence results for solutions of WQVIP and constrained Nash equilibrium problems for vector valued functions using the concept of densely pseudomonotonicity of bifunctions. (see [5, 33] and the reference therein for more on the connection between quasi-variational inequalities and Nash equilibria). For more on the existence theory of solutions to quasi-variational and quasi-variational-like inequalities we refer the reader to [16, 23, 26, 27, 34, 35, 39, 40] and the references therein. For methods of solving general quasi-variational inequalities, we refer to [32] and the references therein.

We consider, in the present paper, the nonsmooth vector quasi-variational-like inequalities and nonsmooth vector quasi-optimization problems. We utilize the method of scalarization, see [3, 38], to define nonsmooth quasi-variational-like inequalities by means of Clarke generalized directional derivative. We then study their relations with the problem of vector quasi-optimization and its scalarized version. Under the assumption of pseudomonotonicity and then densely pseudomonotonicity, we present some existence results for solutions to our nonsmooth quasi-variational-like inequalities. To the best of our knowledge, the results we obtained are new in the sense of utilizing the scalarization method.

The contents of this paper are organized into five sections. In Section 2, we present the necessary notation and background needed to what follows. In this section, we also define precisely the problem of vector quasi-optimization and formulate two sets of variational inequality problems; the first is the vector quasi-variational inequalities and the second is the nonsmooth quasi-variational-like inequalities. The remaining of this section is devoted to state some preliminary results to be used in the rest of the paper. In Section 3, some connections between the nonsmooth quasi-variational-like inequality problems from one hand and the vector quasi-optimization problem and the vector quasi-variational inequality problems from another are explored and established. Section 4 gives the main existence results under pseudomonotonicity and then densely pseudomonotonicity. We give some concluding remarks in Section 5.

2. Notation, Background and Formulations

Let X and Y be two real Banach spaces and Y^* be the topological dual of Y. The canonical bilinear form of Y^* and Y is will be denoted by $\langle \cdot, \cdot \rangle$. For a set $K \subseteq X$, we denote by 2^K the collection of all subsets of K. A function $\psi : K \to \mathbb{R} \cup \{\pm\infty\}$ is called lower semicontinuous on K if for each point $x \in K$, we have

$$\psi(x) \leq \liminf_{y \to x (y \in K)} \psi(y).$$

On the other hand, the function ψ is upper semicontinuous on K if $-\psi$ is lower semicontinuous on K. Furthermore, ψ is said to be sublinear on K if for any $x, y \in K$ and any constants α_1, α_2, we have

$$\psi(\alpha_1 x + \alpha_2 y) \leq \alpha_1 \psi(x) + \alpha_2 \psi(y).$$

We also adopt the following definition of affine functions.

Definition 2.1. *Let K be a nonempty convex subset of X. The a function $\psi : K \to \mathbb{R}$ is said to be affine if*

$$\psi(\alpha x + (1-\alpha) y) = \alpha \psi(x) + (1-\alpha)\psi(y), \quad \textit{for all } x, y \in K, \quad \textit{and for all } \alpha \in [0,1].$$

A function $\psi : X \to \mathbb{R}$ is said to be locally Lipschitz(or locally Lipschitzian) near a point $x \in X$ if there is a neighborhood B of x and a constant $l \geq 0$ such that

$$\|f(y) - f(z)\| \leq l|y - z|, \quad \text{for all} \quad y, z \in B,$$

where $\|\cdot\|$ denotes the norm on X.

Definition 2.2. *Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function near a point $x \in X$. The Clarke generalized directional derivative of ψ at x in the direction $d \in X$ [14], denoted by $\psi^\circ(x; d)$, is defined as*

$$\psi^\circ(x; d) = \limsup_{\substack{y \to x \\ \lambda \to 0^+}} \frac{\psi(y + \lambda d) - \psi(y)}{\lambda}.$$

Remark 2.3. *For a locally Lipschitz function $\psi : X \to \mathbb{R}$, it is well known (see [14]) that*

- *the function $\psi^\circ(x; \cdot)$ is finite and sublinear and satisfies*

  $$|\psi^\circ(x; d)| \leq l\|d\|,$$

  with l the Lipschitz constant.
- *the function $\psi^\circ(\cdot; \cdot)$ is upper semicontinuous.*
- *$\psi^\circ(x; d) \geq \psi'(x; d)$, where $\psi'(x; d)$ is the directional derivative of ψ at x in the direction d.*

For the rest of the paper, unless otherwise specified, let $f : K \to Y$ be a vector-valued map, $\eta : K \times K \to X$ be a given vector-valued map, and $A : K \to 2^K$ be a set-valued map with nonempty, and convex values (i.e. $A(x)$ is nonempty, closed, and convex for all $x \in K$) with K a nonempty convex subset of X, and let $C \subset Y$ be a closed, convex and pointed (i.e. $C \cap (-C) = \{0\}$) cone in Y such that $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ stands for the interior of C. The (positive) dual of C is defined as

$$C^* = \{\xi \in Y^* : \langle \xi, x \rangle \geq 0, \text{ for all } x \in C\}.$$

It is well known form convex analysis that if C is a convex cone with $\operatorname{int} C \neq \emptyset$, then $\langle \xi, x \rangle > 0$ for all $\xi \in C^* \setminus \{\mathbf{0}\}$ if and only if $x \in \operatorname{int} C$.

The vector quasi-optimization problem (VQOP) can be stated as follows:

$$\min f(x), \quad \text{such that } x \in A(x). \tag{VQOP}$$

where the minimum can be realized in different senses. We focus on the following concept to solution for (VQOP).

Definition 2.4. *A point $\bar{x} \in K$ is said to be a weakly efficient solution of (VQOP) if and only if $\bar{x} \in A(\bar{x})$ and*

$$f(y) - f(\bar{x}) \notin -\operatorname{int} C, \quad \textit{for all } y \in A(\bar{x}).$$

The Stampacchia vector quasi-variational inequality problem (SVQVIP) is to

$$\text{find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and } \quad h(\bar{x}; y - \bar{x}) \notin -\text{int}\, C, \quad \text{for all } y \in A(\bar{x}). \tag{SVQVIP}$$

whereas the Minty vector quasi-variational inequality problem (MVQVIP) is to

$$\text{find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and } h(y; y - \bar{x}) \notin -\text{int}\, C, \quad \text{for all } y \in A(\bar{x}). \tag{MVQVIP}$$

where h in both problems is a vector-valued bifunction from $K \times X$ to Y.

When $A(x) = K$ for all $x \in K$, then these problems are considered and studied by Lalitha and Mehta [28], Ansari and Lee [7], Al-Homidan et al [2] and by Crespi et al [15].

If $Y = \mathbb{R}$ and $h(x; y - x) = f^{\circ}(x; y - x)$ for all $x, y \in K$, then the above problems are called nonsmooth Stamppachia quasi-variational inequality problem and nonsmooth Minty quasi-variational inequality problem, respectively.

If we replace $y - x$ by $\eta(y, x)$ in the formulations of (SVQVIP) and (MVQVIP), then (SVQVIP) and (MVQVIP) are called Stampacchia vector quasi-variational-like inequality problem (SVQVLIP) and Minty vector quasi-variational-like inequality problem (MVQVLIP), respectively. When $A(x) = K$, for all $x \in K$, these problems are studied by Alshahrani et al [3] and by Al-Homidan et al [2] with further applications to vector optimization and related topics.

If $Y = \mathbb{R}$ and $h(x; \eta(y, x)) = f^{\circ}(x; \eta(y, x))$ for all $x, y \in K$, then the problems (SVQVIP) and (MVQVIP) are called nonsmooth Stamppachia quasi-variational-like inequality problem (SQVLIP) and nonsmooth Minty quasi-variational-like inequality problem (MQVLIP), respectively.

By utilizing the scalarization method, we consider the following nonsmooth Stampacchia quasi-variational-like inequality problem (NSQVLIP):

$$\begin{aligned} &\text{Find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and for all } y \in A(\bar{x}), \\ &\text{there exists } \omega^* \in C^* \setminus \{\mathbf{0}\} \text{ such that } (\omega^* \circ f)^{\circ}(\bar{x}; \eta(y, \bar{x})) \geq 0. \end{aligned} \tag{NSQVLIP}$$

The following (NSGQVLIP) is more general than (NSQVLIP):

$$\begin{aligned} &\text{Find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and for all } \omega^* \in C^* \setminus \{\mathbf{0}\} \\ &(\omega^* \circ f)^{\circ}(\bar{x}; \eta(y, \bar{x})) \geq 0, \quad \text{for all } y \in A(\bar{x}). \end{aligned} \tag{NSGQVLIP}$$

We also consider the following nonsmooth Minty quasi-variational-like inequality problem (NMQVLIP):

$$\begin{aligned} &\text{Find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and for all } \omega^* \in C^* \setminus \{\mathbf{0}\} \\ &(\omega^* \circ f)^{\circ}(y; \eta(y, \bar{x})) \geq 0, \quad \text{for all } y \in A(\bar{x}). \end{aligned} \tag{NMQVLIP}$$

Our main objective is to study the existence of solutions of (NSQVLIP) under densely pseudomonotonicity or pseudomonotonicity assumption. We also aim to give some relations between the solutions of (NSQVLIP) and a weakly efficient solution of (VQOP).

Definition 2.5. *Let K be a convex subset of a Hausdorff topological vector space X. The set-valued map $\Psi : K \to 2^X$ is said to be a KKM map on K if for each $\{x_1, \cdots, x_n\} \subseteq K$, we have*

$$co\{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} \Psi(x_i),$$

where $co\{\cdot\}$ stands for the convex hull.

We recall the following results which are the main tools to establish our existence results for (NSQVLIP).

Lemma 2.6. [18] *Let K be a convex subset of a Hausdorff topological vector space X and $\Psi : K \to 2^X$ be a KKM map with closed values. If there is a subset K_0 contained in a compact convex subset of K such that $\bigcap_{x\in K_0} \Psi(x)$ is compact, then $\bigcap_{x\in K} \Psi(x) \neq \emptyset$.*

We shall also use the following well-known Fan-KKM lemma.

Lemma 2.7. [17] *Let K be a convex subset of a Hausdorff topological vector space X, K_0 be a nonempty subset of K and $\Phi : K_0 \to 2^K$ be a set-valued map such that the following conditions hold:*

(i) $\Phi(x)$ *is nonempty and closed for each* $x \in K_0$*;*

(ii) $\Phi(x)$ *is compact for some* $x \in K_0$*;*

(iii) $\Phi(x)$ *is a KKM map.*

Then,

$$\bigcap_{x\in K_0} \Phi(x) \neq \emptyset.$$

3. Scalarized Quasi-Optimization Problem

Let K be a nonempty subset of X and $\eta : K \times K \to X$ be a given vector-valued map. We recall the following concepts.

Definition 3.1. *The set K is said to be invex with respect to η (or simply, invex) if the vector $x + \lambda\eta(y, x)$ belongs to K for all $x, y \in K$ and all $\lambda \in [0, 1]$.*

Definition 3.2. *Let $\psi : K \to \mathbb{R}$ be a locally Lipschitz function near $x \in K$. Then ψ is said to be*

(a) invex at x with respect to η if

$$\psi(y) - \psi(x) \geq \psi^\circ(x; \eta(y, x)), \quad \textit{for all} \;\; y \in K;$$

(b) pseudoinvex at x with respect to η if

$$\psi^\circ(x; \eta(y, x)) \geq 0 \quad \textit{implies} \quad \psi(y) \geq \psi(x), \quad \textit{for all} \;\; y \in K;$$

(c) strictly pseudoinvex at x with respect to η if

$$\psi^\circ(x; \eta(y, x)) \geq 0 \quad \textit{implies} \quad \psi(y) > \psi(x), \quad \textit{for all} \;\; y \in K.$$

Furthermore, the function ψ is said to be invex (respectively, pseudoinvex and strictly pseudoinvex) on K with respect to η if it is invex (respectively, pseudoinvex and strictly pseudoinvex) at every point $x \in K$ with respect to η.

The function ψ is said to be incave (respectively, pseudoincave and strictly pseudoincave) on K with respect to η if $-\psi$ is invex (respectively, pseudoinvex and strictly pseudoinvex) on K with respect to the same η.

Definition 3.3. *A vector-valued function $f : K \to Y$ is said to be C^*-invex (respectively, C^*-pseudoinvex and strictly C^*-pseudoinvex) on K with respect to η if for each $\omega^* \in C^*$, $\omega^* \circ f : K \to \mathbb{R}$ is invex (respectively, pseudoinvex and strictly pseudoinvex) on K with respect to the same η. We say that f is C^*-incave (respectively, C^*-pseudoincave and strictly C^*-pseudoincave) on K with respect to η if for each $\omega^* \in C^*$, $\omega^* \circ f : K \to \mathbb{R}$ is incave (respectively, pseudoincave and strictly pseudoincave) on K with respect to the same η.*

We consider the following scalarized quasi-optimization problem:

$$\min\,(\omega^* \circ f)(x), \quad \text{subject to } x \in A(x), \tag{SQOP$_\omega$}$$

for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$.

In what follows, we explore different connections between the solutions to the problems (VQOP), (SQOP)$_\omega$, (NSQVLIP), (NSGQVLIP) and (NMQVLIP).

Lemma 3.4. *Every solution of (SQOP)$_\omega$ is a weakly efficient solution of (VQOP) for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$.*

Proof. Let $\bar{x} \in K$ be a solution of (SQOP)$_\omega$ but not a weakly efficient solution of (VQOP). Then, $\bar{x} \in A(\bar{x})$ and for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$, we have

$$\langle \omega^*, f(y)\rangle \geq \langle \omega^*, f(\bar{x})\rangle, \quad \text{for all } y \in A(\bar{x}). \tag{1}$$

On the other hand, there exists $y \in A(\bar{x})$ such that

$$f(y) - f(\bar{x}) \in -\text{int}\, C.$$

Therefore, for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$, we have

$$\langle \omega^*, f(y) - f(\bar{x})\rangle < 0,$$

that is,

$$\langle \omega^*, f(y)\rangle < \langle \omega^*, f(\bar{x})\rangle,$$

a contradiction of (1). Hence, $\bar{x}$ is a weakly efficient solution of (VQOP).
□

Proposition 3.5. *Let $f : K \to Y$ be locally Lipschitz near $\bar{x} \in K$ and C^*-pseudoinvex with respect to η. If $\bar{x}$ is a solution of (NSGQVLIP), then it is a solution of (SQOP)$_\omega$, and hence, a weakly efficient solution of (VQOP).*

Proof. Assume that $\bar{x}$ is a solution of (NSGQVLIP). Then for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$, we have

$$(\omega^* \circ f)^\circ (\bar{x}; \eta(y, \bar{x})) \geq 0, \quad \text{for all } y \in A(\bar{x})$$

By C^*-pseudoinvexity of f, we have

$$(\omega^* \circ f)(y) \geq (w^* \circ f)(\bar{x}), \quad \text{for all } y \in A(\bar{x}) \text{ and for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

Hence, $\bar{x}$ is a solution of (SQOP)$_\omega$. By Lemma 3.4, $\bar{x}$ is a solution of (VQOP). □

Proposition 3.6. *Let $A : K \to 2^K$ be a set-valued map such that for all $x \in K$, $A(x)$ is a nonempty invex set with respect to η. If $f : K \to Y$ is locally Lispchitz near $\bar{x} \in K$ and if $\bar{x}$ is a solution of (SQOP)$_\omega$, then it is a solution of (NSGQVLIP).*

Proof. Suppose that $\bar{x} \in K$ is a solution of (SQOP)$_\omega$, then $\bar{x} \in A(\bar{x})$ and we have for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$

$$(\omega^* \circ f)(y) - (\omega^* \circ f)(\bar{x}) \geq 0, \quad \text{for all } y \in A(\bar{x}).$$

Since $A(\bar{x})$ is invex, $y_\lambda := \bar{x} + \lambda\eta(y, \bar{x}) \in A(\bar{x})$ for all $y \in A(\bar{x})$ and all $\lambda \in [0, 1]$. Therefore,

$$\frac{(\omega^* \circ f)(\bar{x} + \lambda\eta(y, \bar{x})) - (\omega^* \circ f)(\bar{x})}{\lambda} \geq 0, \quad \text{for all } \lambda \in (0, 1].$$

Now, for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$ and all $y \in A(\bar{x})$,

$$\begin{aligned}(\omega^* \circ f)^\circ (\bar{x}; \eta(y, \bar{x})) &\geq (\omega^* \circ f)' (\bar{x}; \eta(y, \bar{x})) \\ &= \lim_{\lambda \to 0^+} \frac{(\omega^* \circ f)(\bar{x} + \lambda\eta(y, \bar{x})) - (\omega^* \circ f)(\bar{x})}{\lambda} \geq 0.\end{aligned}$$

Hence, $\bar{x}$ is a solution of (NSGQVLIP). □

Proposition 3.7. *Let $f : K \to \mathbb{R}^n$ be locally Lipschitz near $\bar{x} \in K$ and C^*-strictly pseudoincave on K with respect to η. If $\bar{x}$ is a weakly efficient solution of (VQOP), then it is a solution of (NSQVLIP).*

Proof. Assume that $\bar{x}$ is a weakly efficient solution of (VQOP) but not a solution of (NSQVLIP). Then $\bar{x} \in A(\bar{x})$ and there exists $y \in A(\bar{x})$ such that for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$

$$(\omega^* \circ f)^\circ (\bar{x}; \eta(y, \bar{x})) < 0.$$

Since f is C^*-strictly pseudoincave on K, we have for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$

$$0 > (\omega^* \circ f)(y) - (\omega^* \circ f)(\bar{x}) = \langle \omega^*, f(y) - f(\bar{x}) \rangle;$$

which means that,

$$f(y) - f(\bar{x}) \in -\text{int } C,$$

contradicting our assumption that $\bar{x}$ is a weakly efficient solution of (VQOP). Hence, $\bar{x}$ is a solution of (NSQVLIP). □

To establish the relationship between (NSGQVLIP) and (NMQVLIP), we recall the following definition.

Definition 3.8. *[3] Let $x \in K$ and $f : K \to Y$ be locally Lipschitz near x. Then f is said to be C^*-pseudomonotone with respect to η at x if for all $y \in K$,*

$$(\omega^* \circ f)^\circ (x; \eta(y, x)) \geq 0 \text{ for some } \omega^* \in C^* \setminus \{\mathbf{0}\}$$

implies that

$$(\omega^* \circ f)^\circ (y; \eta(y, x)) \geq 0 \text{ for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

We say that f is C^*-pseudomonotone with respect to η on K if it is C^*-pseudomonotone with respect to η at every point $x \in K$.

Example 3.9. *Let $X = Y = \mathbb{R}^2$, $K = [-1, 0] \times [-1, 0]$, $C = C^* = \{(x_1, x_2) : x_1 \leq 0 \text{ and } x_2 \geq 0\}$ and define $f : K \times K \to Y$ by*

$$f(x_1, x_2) = \left(x_1^2, 2x_2\right).$$

It can be easily shown that f is C^*-pseudomonotone on K with respect to $\eta : K \times K \to Y$ defined as

$$\eta\left[(x_1, x_2), (y_1, y_2)\right] = (|y_1 - x_1|, |y_2 - x_2|).$$

In fact, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in K$ and for all $\omega^* = (\omega_1^*, \omega_2^*) \in C^* \setminus \{\mathbf{0}\}$, we have

$$(\omega^* \circ f)^\circ ((x_1, x_2), \eta\left[(x_1, x_2), (y_1, y_2)\right]) = 2\omega_1^* |y_1 - x_1| x_1 + 2\omega_2^* |y_2 - x_2| \geq 0.$$

However, it is not pseudomonotone, in the classical sense [22], on K. In particular, it is not pseudomonotone at $x = (0, 0)$. Furthermore, it is not C^*-pseudomonotone on K with respect to $\eta(y, x) = y - x$. Indeed, let $x = (0, 0)$, $y = (0, -1)$, then

$$(\omega^* \circ f)^\circ (x, \eta(x, y)) = 2\omega_2^* y_2 = 0, \quad \text{for } \omega^* = (-1, 0), \quad \text{for all } y \in K.$$

However,

$$(\omega^* \circ f)^\circ (y, \eta(x, y)) = 2\omega_2^* y_2 = -2\omega_2^* < 0, \quad \text{for all } \omega^* \in \{(\omega_1^*, \omega_2^*) \in C^* \setminus \{\mathbf{0}\} : \omega_2^* > 0\}.$$

and $y = (0, -1)$.

Rest of the paper, unless otherwise specified, $f : K \to Y$ be a locally Lipschitz function.

Definition 3.10. *[30] A subset K_0 of K is said to be segment-dense in K if for all $x \in K$, there can be found $x_0 \in K_0$ such that x is a cluster point of the set $[x, x_0] \cap K_0$, where $[x, x_0]$ denotes the line segment joining x and x_0 including end points.*

The densely C^*-pseudomonotonicity, which generalizes the notion of densely pseudomonotonicity considered by Luc [30], is introduced in [3].

Definition 3.11. *[3] A vector-valued function $f : K \to Y$ is said to be densely C^*-pseudomonotone with respect to η on K if there exists a segment-dense subset $K_0 \subseteq K$ such that f is C^*-pseudomonotone with respect to η on K_0.*

Proposition 3.12. *Let K be a nonempty convex subset of X and $f : K \to Y$ be a C^*-pseudomonotone with respect to η on K. Suppose that η is affine in the first argument and $\eta(x, x) = 0$ for all $x \in K$ and all $\lambda \in [0, 1]$. Then $\bar{x} \in K$ is a solution of (NSGQVLIP) if and only if it is a solution of (NMQVLIP).*

Proof. The C^*-pseudomonotonicity of f implies that every solution of (NSGQVLIP) is a solution of (NMQVLIP).
Conversely, let $\bar{x}$ be a solution of (NMQVLIP). Then for each $y \in A(\bar{x})$ and all $\omega^* \in C^* \setminus \{\mathbf{0}\}$, we have

$$(\omega^* \circ f)^\circ (y; \eta(y, \bar{x})) \geq 0.$$

Since $A(\bar{x})$ is convex, $y_\lambda := \lambda y + (1 - \lambda)\bar{x} \in K$ for all $y \in A(\bar{x})$ and all $\lambda \in (0, 1)$. Therefore,

$$(\omega^* \circ f)^\circ (y_\lambda; \eta(y_\lambda, \bar{x})) \geq 0.$$

By the affinity of $\eta(y, x)$ in y and the sublinearity of $(\omega^* \circ f)^\circ (x, d)$ in d, we have

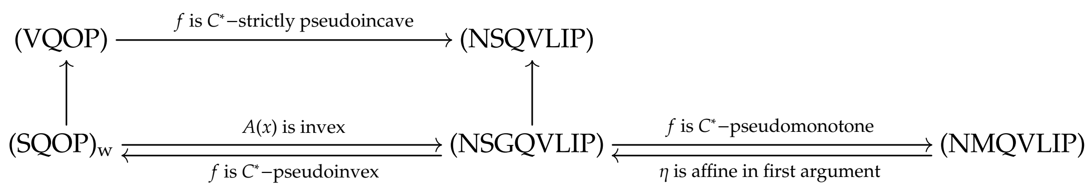
$$\lambda (\omega^* \circ f)^\circ (y_\lambda; \eta(y, \bar{x})) \geq 0.$$

Since $(x, d) \mapsto (\omega^* \circ f)^\circ (x, d)$ is upper semicontinuous, we have for all $y \in A(\bar{x})$ and all $\omega^* \in C^* \setminus \{\mathbf{0}\}$

$$(\omega^* \circ f)^\circ (\bar{x}; \eta(y, \bar{x})) \geq \limsup_{\lambda \to 0} (\omega^* \circ f)^\circ (y_\lambda; \eta(y, \bar{x})) \geq 0,$$

that is, $\bar{x}$ is a solution of (NSGQVLIP). □

We conclude this section by summarizing the above relations in the following diagram



4. Existence Results

Throughout this section, unless otherwise specified, X and Y are real two Banach spaces, K is a nonempty subset of X , $K_0 \subseteq K$ is a segment-dense set in K, C is a closed, convex and pointed cone in Y with int $C \neq \emptyset$, and C^* is the dual cone of C. Let $\eta : K \times K \to X$ be a map.
We start by giving the following lemma which is essential to our existence theorem.

Lemma 4.1. *Let $f : K \to Y$ be densely C^*-pseudomonotone with respect to η on K. If η is upper semicontinuous in the first argument, then (NMQVLIP) is equivalent to the following problem:*

Find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and (NMQVLIP)$_0$
$(\omega^* \circ f)^\circ (y; \eta(y, \bar{x})) \geq 0,$ *for all $y \in A(x) \cap K_0$ and all $\omega^* \in C^* \setminus \{\mathbf{0}\}$.*

Proof. Obviously, (NMQVLIP) implies $(\text{NMQVLIP})_0$.

Conversely, let $\bar{x} \in K$ be a solution of $(\text{NMQVLIP})_0$. Then $\bar{x} \in A(\bar{x})$ and for all $y \in A(x) \cap K_0$ and all $\omega^* \in C^* \setminus \{\mathbf{0}\}$, we have

$$(\omega^* \circ f)^\circ (y; \eta(y, \bar{x})) \geq 0. \tag{2}$$

Since K_0 is segment-dense and $A(x) \cap K_0 \subseteq A(x) \cap K = A(x)$, we deduce that $A(x) \cap K_0$ is segment-dense in $A(x)$ for all $x \in K$. Then for all $y \in A(\bar{x})$, we can find $y_0 \in A(\bar{x}) \cap \in K_0$ and a net $\{y_t\}_{t \in I} \in [y, y_0] \cap A(\bar{x}) \cap K_0$, where I is some index set, converging to y. Then from (2), we get

$$(\omega^* \circ f)^\circ (y_t; \eta(y_t, \bar{x})) \geq 0.$$

Since y_t converges to y and $(\omega^* \circ f)^\circ(\cdot, \cdot)$ is upper semicontinuous in both arguments and η is upper semicontinuous in the first argument, we have for all $y \in A(\bar{x})$

$$(\omega^* \circ f)^\circ (y; \eta(y, \bar{x})) \geq \limsup_{y_t \to y} (\omega^* \circ f)^\circ (y_t; \eta(y_t, \bar{x})) \geq 0, \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

Hence $\bar{x}$ is a solution of (NMVLIP). □

Under C^*−pseudomonotonicity of f on K, we have the following existence theorem for solutions of (NMQVLIP). This theorem generalizes Theorem 3 in [3].

Theorem 4.2. *Let X be a Banach space and K be a nonempty convex subset of X and $f : K \to Y$ be C^*-pseudomonotone on K with respect to η. Assume that*

(i) the map η is upper semicontinuous and affine in both arguments such that $\eta(x, x) = 0$ for all $x \in K$,

(ii) for each $x \in K$, the set $A(x)$ is a nonempty convex and closed set,

(iii) for each $x \in K$, the set $A^{-1}(x)$ is open in K,

(iv) there exists a nonempty subset D of K and a subset D contained in a compact convex subset of K such that for all $y \in K \setminus D$, there exists $x_y \in D \cap A(y)$, such that for some $\omega^* \in C^* \setminus \{\mathbf{0}\}$

$$(\omega^* \circ f)^\circ \left(y, \eta(y, x_y)\right) < 0.$$

Then, there exists a solution $\bar{x} \in K$ of (NSQVLIP), and hence, it is a weakly efficient solution of (VQOP).

Proof. For $x, y \in K$, we define the following set-valued maps, $\Gamma, \Psi, \Theta : K \to 2^K$,

$$\begin{aligned}
\Gamma(y) &:= \{x \in K : \text{for some } \omega^* \in C^* \setminus \{\mathbf{0}\} \text{ we have } (\omega^* \circ f)^\circ(y, \eta(y, x)) < 0\},\\
\Psi(y) &:= \begin{cases} A(y) \cap \Gamma(y), & \text{if } y \in A(y),\\ A(y), & \text{if } y \in K \setminus A(y) \end{cases},\\
\Theta(x) &:= \{y \in K : y \notin \Psi^{-1}(x)\}.
\end{aligned}$$

Note that $y \notin \Gamma(y)$; because $(\omega^* \circ f)^\circ (y, \eta(y, y)) = (\omega^* \circ f)^\circ (y, 0) = 0$. This means that $x \in \Theta(x)$ for all $x \in K$. We show now that Θ is a KKM map on K; so let $\{x_i\}_{1 \leq i \leq n} \in K$ and consider a convex combination $\tilde{y} = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{1 \leq i \leq n} \alpha_i = 1$ and assume the contrary that $\tilde{y} \notin \bigcup_{1 \leq i \leq n} \Theta(x_i)$; that is $\tilde{y} \notin \Theta(x_i)$ for $1 \leq i \leq n$. By definition of Θ, we have $\tilde{y} \in \Psi^{-1}(x_i)$ for $1 \leq i \leq n$. In other words, $x_i \in \Psi(\tilde{y})$ for $1 \leq i \leq n$. If $\tilde{y} \in A(\tilde{y})$, then $x_i \in \Gamma(\tilde{y})$ for $1 \leq i \leq n$ and therefore for some $\omega^* \in C^* \setminus \{\mathbf{0}\}$ one has

$$(\omega^* \circ f)^\circ (\tilde{y}, \eta(\tilde{y}, x_i)) < 0, \qquad \text{for all } 1 \leq i \leq n.$$

By the sublinearity of $(\omega^* \circ f)^\circ (\tilde{y}, \cdot)$ and affinity of $\eta(\tilde{y}, \cdot)$, we have

$$\begin{aligned}
(\omega^* \circ f)^\circ (\tilde{y}, \eta(\tilde{y}, \tilde{y})) &= (\omega^* \circ f)^\circ (\tilde{y}, \eta(\tilde{y}, \textstyle\sum_{i=1}^n \alpha_i x_i))\\
&= (\omega^* \circ f)^\circ (\tilde{y}, \textstyle\sum_{i=1}^n \alpha_i \eta(\tilde{y}, x_i))\\
&\leq \textstyle\sum_{i=1}^n \alpha_i (\omega^* \circ f)^\circ (\tilde{y}, \eta(\tilde{y}, x_i)) < 0;
\end{aligned}$$

implying that $\tilde{y} \in \Gamma(\tilde{y})$ which is a contradiction. On the other hand, if $\tilde{y} \in K \setminus A(\tilde{y})$; that is $\tilde{y} \notin A(\tilde{y})$, then $x_i \in A(\tilde{y})$ for $1 \leq i \leq n$. Hence, by convexity of $A(\tilde{y})$, $\tilde{y} \in A(\tilde{y})$; which is also a contradiction. This shows that Θ is a KKM map.

We also claim that Θ is closed. Indeed, for all $x \in K$, we have

$$\begin{aligned}
\Psi^{-1}(x) &= \{y \in A(x) : x \in A(y) \cap \Gamma(y)\} \cup \{y \in (K \setminus A(x)) : x \in A(y)\} \\
&= \left\{y \in A(x) : y \in A^{-1}(x) \cap \Gamma^{-1}(x)\right\} \cup \left\{y \in (K \setminus A(x)) : y \in A^{-1}(x)\right\} \\
&= \left[A(x) \cap A^{-1}(x) \cap \Gamma^{-1}(x)\right] \cup \left[(K \setminus A(x)) \cap A^{-1}(x)\right] \\
&= \left[A(x) \cap A^{-1}(x) \cap \Gamma^{-1}(x)\right] \cup \left[(K \setminus A(x)) \cap A^{-1}(x)\right] \\
&= A^{-1}(x) \cap \left[\Gamma^{-1}(x) \cup (K \setminus A(x))\right]
\end{aligned}$$

So,

$$\begin{aligned}
\Theta(x) &= K \setminus \left\{A^{-1}(x) \cap \left[\Gamma^{-1}(x) \cup (K \setminus A(x))\right]\right\} \\
&= \left[K \setminus A^{-1}(x)\right] \cup \left\{K \setminus \left[\Gamma^{-1}(x) \cup (K \setminus A(x))\right]\right\} \\
&= \left[K \setminus A^{-1}(x)\right] \cup \left[A(x) \cap (K \setminus \Gamma^{-1}(x))\right].
\end{aligned}$$

Since $A^{-1}(x)$ is open in K, the set $K \setminus A^{-1}(x)$ is closed. Furthermore, by assumption, $A(x)$ is closed. We also have

$$\begin{aligned}
K \setminus \Gamma^{-1}(x) &= \left\{y \in K : y \notin \Gamma^{-1}(x)\right\} \\
&= \{y \in K : x \notin \Gamma(y)\} \\
&= \{y \in K : \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\} \text{ such that } (\omega^* \circ f)^\circ(y, \eta(y, x)) \geq 0\},
\end{aligned}$$

Note that for all $x \in K$, $x \in K \setminus \Gamma^{-1}(x)$ and hence $K \setminus \Gamma^{-1}(x) \neq \emptyset$. To see that the set $K \setminus \Gamma^{-1}(x)$ is closed for each $x \in K$, suppose that X is equipped with the weak topology and let $\{y_n\} \subseteq K \setminus \Gamma^{-1}(x)$ be a sequence such that $y_n \to y$. Then

$$(\omega^* \circ f)^\circ (y_n; \eta(y_n, x)) \geq 0, \qquad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

Let $\omega^* \in C^* \setminus \{\mathbf{0}\}$ be fixed, then by the upper semicontinuity of $(\omega^* \circ f)^\circ (\cdot, \cdot)$ and the upper semicontinuity of η in the first argument, we have

$$0 \leq \limsup_{n\to\infty} (\omega^* \circ f)^\circ (y_n; \eta(y_n, x)) \leq (\omega^* \circ f)^\circ (y; \eta(y, x)).$$

Thus $y \in K \setminus \Gamma^{-1}(x)$, and hence, $K \setminus \Gamma^{-1}(x)$ is closed. Thus $\Theta(x)$ is closed for all $x \in K$. By assumption, for all $y \in K \setminus D$, there exists $x_y \in D \cap A(y)$ such that $x_y \in \Psi(y)$. Therefore,

$$K \setminus D \subseteq \bigcup_{y\in D} \Psi^{-1}(y) \subseteq K.$$

Thus,

$$K \setminus \bigcup_{y\in D} \Psi^{-1}(y) = \bigcap_{y\in D} K \setminus \Psi^{-1}(y) \subseteq D,$$

and hence $\bigcap_{y\in D} \Theta(y)$ is compact. By Lemma 2.6, there exists $\bar{x}$ such that

$$\bar{x} \in \bigcap_{y\in K} \Theta(y) = K \setminus \bigcup_{y\in K} \Psi^{-1}(y).$$

That is $\bar{x} \notin \Psi^{-1}(y)$ for all $y \in K$, i.e., $\Psi(\bar{x}) = \emptyset$. If $\bar{x} \in K \setminus A(\bar{x})$, then $\Psi(\bar{x}) = A(\bar{x})$ contradicting the assumption that $A(x) \neq \emptyset$ for all $x \in K$. However, when $\bar{x} \in A(\bar{x})$, then $\emptyset = \Psi(\bar{x}) = A(\bar{x}) \cap \Gamma(\bar{x})$. Hence for all $y \in A(\bar{x})$, $x \notin \Gamma(\bar{x})$. In other words, for all $y \in A(\bar{x})$,

$$(\omega^* \circ f)^\circ (y; \eta(y, \bar{x})) \geq 0, \qquad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

Thus $\bar{x}$ is a solution to (NMQVLIP). According to Proposition 3.12, $\bar{x}$ is a solution of (NSQVLIP). Furthermore, by Proposition 3.5, $\bar{x}$ is also a weakly efficient solution of (VQOP). □

Remark 4.3. *In the previous theorem, if we assume further that the map A is upper semicontinuous, see [12] for definition, then it is closed and hence the set $E = \{x \in K : x \in A(x)\}$ is closed.*

The following theorem assures the existence of solutions of (NMQVLIP) under densely C^*−pseudomonotonicity.

Theorem 4.4. *Let X be a Banach space and K be a nonempty, convex, and closed subset of X and $f : K \to Y$ be densely C^*-pseudomonotone on K with respect to η. Assume the following*

(i) *The map η is upper semicontinuous in both arguments and affine in second argument such that $\eta(x, x) = 0$ for all $x \in K$,*

(ii) *for each $x \in K$, $A(x)$ is a nonempty convex and closed set,*

(iii) *for each $x \in K_0$, $A^{-1}(x)$ is an open set,*

(iv) *the set $E = \{x \in K : x \in A(x)\}$ is closed,*

(v) *there exists a nonempty compact subset $D \subseteq K$ and $\hat{x}$ such that for all $y \in K \setminus D$, $\hat{x} \in A(y)$ and*

$$(\omega^* \circ f)^\circ (\hat{x}, \eta(y, \hat{x})) < 0, \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

Then, there exists a solution $\bar{x} \in K$ of (NMQVLIP).

Proof. We define the following set-valued maps on K, $\Gamma_1, \Gamma_2 : K \to 2^K$

$$\begin{aligned} \Gamma_1(y) &= \{x \in K : (\omega^* \circ f)^\circ (x, \eta(y, x)) < 0 \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}\}, \\ \Gamma_2(y) &= \{x \in K : (\omega^* \circ f)^\circ (y, \eta(y, x)) < 0 \quad \text{for some } \omega^* \in C^* \setminus \{\mathbf{0}\}\}, \end{aligned}$$

for all $y \in K$. For all $x, y \in K$, we also define the set-valued maps ,$\Psi_i, \Theta_i : K \to 2^K$, $i = 1, 2$, by

$$\begin{aligned} \Psi_i(y) &= \begin{cases} A(y) \cap \Gamma_i(y), & \text{if } \; y \in E, \\ A(y), & \text{if } \; y \in K \setminus E, \end{cases} \qquad \text{for } i = 1, 2, \\ \Theta_i(x) &= K \setminus \Psi_i^{-1}(x), \quad \text{for } i = 1, 2. \end{aligned}$$

Note that $y \notin \Gamma_i(y)$ for $i = 1, 2$; because for all $y \in K$, $\eta(y, y) = 0$ and therefore

$$(\omega^* \circ f)^\circ (y, \eta(y, y)) = 0, \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

In a similar manner to Theorem 4.2, one can show that Θ_1 is a KKM map.

We show now that $\Theta_1(\hat{x}) \subseteq D$ with $\hat{x}$ and D are the same as in the assumptions of the theorem. We note first that for all $x \in K$, $x \in \Theta_1(x)$; since otherwise $x \in \Psi_1(x)$ and therefore either $x \in \Gamma_1(x)$ if $x \in E$ which is a contradiction or $x \in A(x)$ if $x \in K \setminus E$ which is an obvious contradiction. Also note that for all $x \in K$

$$\Psi_1^{-1}(x) = \left[(K \setminus E) \cup \Gamma_1^{-1}(x)\right] \cap A^{-1}(x), \tag{3}$$

and

$$\Theta_1(x) = \left[E \cap (K \setminus \Gamma_1^{-1}(x))\right] \cup \left[K \setminus A^{-1}(x)\right]. \tag{4}$$

So let $y \in \Theta_1(\hat{x}) \setminus D$, then by (4), $y \in E \cap (K \setminus \Gamma_1^{-1}(\hat{x}))$ or $y \in K \setminus A^{-1}(\hat{x})$. If $y \in E \cap (K \setminus \Gamma_1^{-1}(\hat{x}))$, then $y \in A(y)$ and $y \notin \Gamma_1^{-1}(\hat{x})$. In other words, $y \in A(y)$ and $\hat{x} \notin \Gamma_1(y)$; i.e.

$$(\omega^* \circ f)^\circ (\hat{x}, \eta(y, \hat{x})) \geq 0, \quad \text{for some } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

This is a contradiction to the assumption that

$$(\omega^* \circ f)^\circ (\hat{x}, \eta(y, \hat{x})) < 0, \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

If $y \in K \setminus A^{-1}(\hat{x})$, then $\hat{x} \notin A(y)$; contradicting the assumption that $\hat{x} \in A(y)$ for all $y \in K \setminus D$. Hence $\Theta_1(\hat{x}) \subseteq D$.

Another claim we make is that $\bigcap_{x \in K} \mathrm{cl}\, \Theta_1(x) \neq \emptyset$. Indeed, for all $x \in K$, $\mathrm{cl}\, \Theta_1(x)$ is compact because D is compact. Furthermore, $\mathrm{co}\,\{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^n \Theta_1(x_i) \subseteq \bigcup_{i=1}^n \mathrm{cl}\, \Theta_1(x_i)$ for each finite subset $\{x_1, \cdots, x_n\}$ of K. Therefore by Lemma 2.7, we have $\bigcap_{x \in K} \mathrm{cl}\, \Theta_1(x) \neq \emptyset$.

Next we show that $\bigcap_{x \in K_0} \mathrm{cl}\, \Theta_1(x) \subseteq \bigcap_{x \in K_0} \Theta_2(x)$. So let $y \in \Theta_1(x)$ for all $x \in K_0$. Let us fix $x_0 \in K_0$, then $y \in \mathrm{cl}\, \Theta_1(x_0)$ and hence there exists a net $\{y_t\}_{t \in I} \subseteq \Theta_1(x_0)$, with I being some index set, that converges to y. So we have for all $t \in I$

$$y_t \in \left[E \cap (K \setminus \Gamma_1^{-1}(x_0))\right] \cup \left[K \setminus A^{-1}(x_0)\right].$$

If $y_t \in E \cap (K \setminus \Gamma_1^{-1}(x_0))$, then $y \in E$ because E is assumed to be closed and $y_t \to y$. Furthermore, $y_t \notin \Gamma_1^{-1}(x_0))$ or equivalently, $x_0 \notin \Gamma_1(y_t)$; i.e.

$$(\omega^* \circ f)^\circ (x_0, \eta(y_t, x_0)) \geq 0, \quad \text{for some } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

But f is densely C^*−pseudomonotone which means that

$$(\omega^* \circ f)^\circ (y_t, \eta(y_t, x_0)) \geq 0, \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

Since both $(\omega^* \circ f)^\circ (\cdot, \cdot)$ and $\eta(\cdot, \cdot)$ are upper semicontinuous, we have for all $\omega^* \in C^* \setminus \{\mathbf{0}\}$

$$0 \leq \limsup_{y_t \to y} (\omega^* \circ f)^\circ (y_t, \eta(y_t, x_0)) \leq (\omega^* \circ f)^\circ (y, \eta(y, x_0)),$$

which shows that $x_0 \notin \Gamma_2(x_0)$ or $y \notin \Gamma_2^{-1}(x_0)$. So $y \in \Theta_2(x_0)$ and hence $y \in \bigcap_{x \in K_0} \Theta_2(x)$. On the other hand, if $y_t \in K \setminus A^{-1}(x_0)$ and since $K \setminus A^{-1}(x_0)$ is closed because by assumption $A^{-1}(x)$ is open for all $x \in K_0$, we see that $y \in K \setminus A^{-1}(x_0)$ or $x_0 \notin A(y)$; implying by definition of Ψ_2 that $x_0 \notin \Psi_2(y)$. Thus, $y \notin \Psi_2^{-1}(x_0)$ or equivalently $y \in \Theta_2(x_0)$. So, in this case also, $y \in \bigcap_{x \in K_0} \Theta_2(x)$. Because K_0 is segment-dense in K and A has closed values, we have

$$\bigcap_{x \in K_0} \Theta_2(x) = \bigcap_{x \in K} \Theta_2(x).$$

In summary, the following statement is true

$$\bigcap_{x \in K} \Theta_1(x) \subseteq \bigcap_{x \in K_0} \mathrm{cl}\, \Theta_1(x) \subseteq \bigcap_{x \in K_0} \Theta_2(x) = \bigcap_{x \in K} \Theta_2(x).$$

Since $\bigcap_{x \in K} \mathrm{cl}\, \Theta_1(x)$ is not empty, we conclude that $\bigcap_{x \in K} \Theta_2(x)$ is also not empty. Therefore, there exists $\bar{x} \in K$ such that

$$\bar{x} \in \bigcap_{x \in K} \Theta_2(x) = \bigcap_{x \in K} K \setminus \Psi_2^{-1}(x) = K \setminus \bigcap_{x \in K} \Psi_2^{-1}(x),$$

which implies that for all $x \in K$, $\bar{x} \notin \Psi_2^{-1}(x)$ or equivalently $x \notin \Psi_2(\bar{x})$. This means that $\Psi_2(\bar{x}) = \emptyset$. Now only two cases are possible. The first is if $\bar{x} \in K \setminus E$, then

$$\emptyset = \Psi_2(\bar{x}) = A(\bar{x});$$

which contradicts the the fact that $A(x) \neq \emptyset$ for all $x \in K$. In the second case, if $\bar{x} \in E$, we get

$$\emptyset = \Psi_2(\bar{x}) = A(\bar{x}) \cap \Gamma_2(\bar{x});$$

meaning that $\bar{x} \in A(\bar{x})$ and $\bar{x} \notin \Gamma_2(y)$ for all $y \in A(\bar{x})$. Therefore $\bar{x} \in A(\bar{x})$ and for all $y \in A(\bar{x})$, we have

$$(\omega^* \circ f)^\circ (y, \eta(y, \bar{x})) \geq 0, \quad \text{for all } \omega^* \in C^* \setminus \{\mathbf{0}\}.$$

This shows that $\bar{x}$ is a solution to (NMQVLIP) concluding the proof. □

Remark 4.5.

(a) *In our proof of both Theorems 4.2 and 4.4, we do not require that $x \in A(x)$ for all $x \in K$. Therefore Harker condition [25] is not satisfied. So our results are not straight forward.*

(b) *Theorem 4.4 generalizes and improves Theorem 9 in [38] and Theorem 2 in [3].*

(c) *The assumption on A in Theorem 4.4 is milder than the one in Theorem 4.2 in the sense that $A^{-1}(x)$ is required to be open on K_0 rather than on the whole set K.*

Remark 4.6. *In Theorem 4.2 and Theorem 4.4, if we assume more that η is affine in the first argument and f is C^*-pseudoinvex, then by Proposition 3.12, $\bar{x}$ is a solution of (NSQVLIP) which is a weakly efficient solution of (VQOP) by Proposition 3.5.*

5. Concluding Remarks

We studied the problem of vector quasi-optimization and established two existence theorems for its solutions. We introduced scalarized nonsmooth quasi-variational-like inequalities through Clarke generalized directional derivative and discussed different connections between them from one side and between them and the vector quasi-optimization problem and its scalarized version from another. The existence results were introduced under C^*-pseudomonotonicity and then under densely C^*-pseudomonotonicity. The results we obtained are new in their approach to quasi-variational inequalities and, furthermore, more general than the existence results by Santos et al [38] and Alshahrani et al [3].

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