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Random Fixed Points for ψ-Contractions with Application to Random Differential Equations

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Abstract. In this paper, some random common fixed point and coincidence point results are proved with PPF dependence for random operators in separable Banach spaces. Our results present stochastic versions and extensions of recent results of Dhage [J. Nonlinear Sci. Appl. 5 (2012) and Differ. Equ. Appl. 2 (2012)], Kaewcharoen [J. Inequal. Appl. 2013:287] and many others. We also establish results concerning iterative approximation of PPF dependent random common fixed points. Moreover, an application to random differential equations is given here to illustrate usability of the obtained results.

1. Introduction

The topic of common fixed point for pair or families of contractive mappings in metric and abstract spaces is of great interest and has already been studied in the literature (see [2]-[27] and references therein). Recently, Bernfield et al. [4] proved some fixed point theorems for nonlinear operators in Banach spaces, where the domain and range of the operators are not same. The fixed point theorems of this kind are called fixed point theorems with the PPF (past, present and future) dependence.

The study of random fixed point theorems in abstract spaces was initiated by Spacek [28] and Hans [11] to get stochastic generalizations of the classical fixed point theorems in separable Banach spaces. The research along this line gained momentum after the results of Bharucha-Reid [5] and since then several random fixed point theorems have been proved in the literature. A common assumption among all these operators in question to map an abstract space into itself, i.e. the domain and the range of the operators are the same. The classical or deterministic fixed point theorems for the operators with respect to different domain and range spaces were studied by Bernfield et al. [4], Drici et al. [9, 10] and Dhage [6]. These results are useful for proving the existence of solutions for certain functional differential equations which may depend upon the past, present and future consideration.

In the present paper, some random common fixed point theorems with PPF dependence are proved for pair of operators in Banach spaces satisfying generalized random contractive conditions of Ćirić type. Our results are new and generalize some known results of Dhage [7, 8] and Bernfield et al. [4] under more general random contractive conditions.

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2. Preliminaries

Suppose that E is a Banach space with the norm $\|.\|_E$ and given a closed interval $I = [a, b]$ in $\mathbb{R}$, we consider the Banach space $E_0 = C(I, E)$ of continuous E-valued functions defined on I, equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|x\|_{E_0} = \sup_{t\in I} \|x(t)\|_E$$

for all $x \in E_0$. For a fixed $c \in I$, the Razumikhin class of functions [4, 6] in E_0 is defined as

$$\mathfrak{R}_c = \left\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\right\}.$$

The class $\mathfrak{R}_c$ is algebraically closed with respect to difference if $\phi - \zeta \in \mathfrak{R}_c$ whenever $\phi, \zeta \in \mathfrak{R}_c$; similarly, $\mathfrak{R}_c$ is topologically closed if it is closed w.r.t. the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Let $T : E_0 \to E$. A point $\zeta^* \in E_0$ is called a PPF fixed point of T if $T(\zeta^*) = \zeta^*(c)$ for some $c \in I$. It is known that Razumikhin class of functions play a significant role in proving the existence of PPF fixed points with different domain and range of the operators.

Definition 2.1. *An operator $T : E_0 \to E$ is called Banach type contraction if there is a real number $0 < \alpha < 1$ such that*

$$\|T(\zeta) - T(\eta)\|_E \leq \alpha \|\zeta - \eta\|_{E_0}$$

for all $\zeta, \eta \in E_0$.

The following definition is introduced in the literature on the lines of classical definition for contraction mapping given by Kannan [22].

Definition 2.2. *An operator $T : E_0 \to E$ is called strong Kannan type contraction if*

$$\|T(\zeta) - T(\eta)\|_E \leq \alpha[\|\zeta(c) - T(\zeta)\|_E + \|\eta(c) - T(\eta)\|_E]$$

for all $\zeta, \eta \in E_0$ and some $c \in I$, where $0 < \alpha < 1/2$.

Definition 2.3. *Let $S, T : E_0 \to E$ be two operators. A point $\zeta^* \in E_0$ is called*

(i) PPF dependent common fixed point of S and T if $S(\zeta^*) = T(\zeta^*) = \zeta^*(c)$ for some $c \in I$.
(ii) PPF dependent coincidence point of S and T if $S(\zeta^*) = T(\zeta^*(c))$ for some $c \in I$.

Definition 2.4. *[7]. Two operators $S, T : E_0 \to E$ are said to satisfy a condition of strong Cirić type generalized contraction if there exists a real number $0 < \lambda < 1$ satisfying*

$$\|S(\zeta) - T(\eta)\|_E \leq \lambda \max\Big\{\|\zeta(c) - \eta(c)\|_E, \|\zeta(c) - S(\zeta)\|_E, \|\eta(c) - T(\eta)\|_E, \tfrac{1}{2}[\|\zeta(c) - T(\eta)\|_E + \|\eta(c) - S(\zeta)\|_E]\Big\}$$

for all $\zeta, \eta \in E_0$ and some $c \in I$.

Definition 2.5. *[7]. The operators $A : E_0 \to E$ and $S : E_0 \to E_0$ are said to satisfy a condition of strong Cirić type generalized contraction (C) if there exists a real number $0 < \lambda < 1$ satisfying*

$$\|A(\zeta) - A(\eta)\|_E \leq \lambda \max\{\|S(\zeta(c)) - S(\eta(c))\|_E, \|S(\zeta(c)) - A(\zeta)\|_E, \|S(\eta(c)) - A(\eta)\|_E, \tfrac{1}{2}[\|S(\zeta(c)) - A(\eta)\|_E + \|S(\eta(c)) - A(\zeta)\|_E]\}$$

for all $\zeta, \eta \in E_0$ and $c \in I$.

Definition 2.6. *[7] The operators $A : E_0 \to E$ and $S : E_0 \to E_0$ are said to satisfy a condition of Cirić type generalized contraction (C) if there exists a real number $0 < \lambda < 1$ satisfying*

$$\left\|A(\zeta) - A(\eta)\right\|_E \le \lambda \max\{\left\|S(\zeta) - S(\eta)\right\|_{E_0}, \|S(\zeta(c)) - A(\zeta)\|_E, \left\|S(\eta(c)) - A(\eta)\right\|_E, \tfrac{1}{2}[\left\|S(\zeta(c)) - A(\eta)\right\|_E + \left\|S(\eta(c)) - A(\zeta)\right\|_E]\}$$

for all $\zeta, \eta \in E_0$ and $c \in I$.

Kaewcharoen [21] introduced the condition of *Cirić* type generalized ψ-contraction as follows.

Definition 2.7. *Let $S, T : E_0 \to E$. We say that S and T satisfy the condition of Cirić type generalized ψ-contraction if*

$$\left\|S\phi - T\alpha\right\|_E \le \psi(\max\{\left\|\phi - \alpha\right\|_{E_0}, \left\|\phi(c) - S\phi\right\|_E, \|\alpha(c) - T\alpha\|_E, \tfrac{1}{2}[\left\|\phi(c) - T\alpha\right\|_E + \left\|\alpha(c) - S\phi\right\|_E]\})$$

for all $\phi, \alpha \in E_0$ and for some $c \in I$.

Theorem 2.8. *[21] Suppose that $S, T : E_0 \to E$ satisfy the condition of Cirić type generalized ψ-contraction. Assume that $\mathfrak{R}_c$ is topologically closed with respect to norm topology and is algebraically closed with respect to the difference, then S and T have a unique PPF dependent common fixed point in $\mathfrak{R}_c$.*

Definition 2.9. *Let $A : E_0 \to E$ and $S : E_0 \to E_0$. We say that A and S satisfy the condition of Cirić type generalized ψ-contraction (C) if*

$$\left\|A\phi - A\alpha\right\|_E \le \psi(\max\{\left\|S\phi - S\alpha\right\|_{E_0}, \left\|S\phi(c) - A\phi\right\|_E, \|S\alpha(c) - A\alpha\|_E, \tfrac{1}{2}[\left\|S\phi(c) - A\alpha\right\|_E + \left\|S\alpha(c) - A\phi\right\|_E]\}).$$

Using the above definition Kaewcharoen proved some results under weaker condition than the condition of *Cirić* type generalized contraction (C). We shall prove stochastic versions of recently established PPF dependence fixed and common fixed point results with their application to iterative approximation and stochastic differential equations.

3. Main Results

Let (Ω, X) be a measurable space and let E be a separable Banach space with norm $\|\cdot\|_E$. We equip the Banach space E with a σ-algebra , β_E of Borel subsets of E so that (E, β_E) becomes a measurable space. A mapping $T : \Omega \to E$ is called measurable if

$$T^{-1}(B) = \{\omega \in \Omega \mid T(\omega) \in B\} \in X$$

for all Borel sets $B \in \beta_E$.

Given two Banach spaces E_1 and E_2, a mapping $T : \Omega \times E_1 \to E_2$ is called a random operator if $T(\omega, x)$ is measurable in ω for all $x \in E_1$. We also denote a random operator T on E_1 by $T(\omega)x = T(\omega, x)$. A random operator $T(\omega)$ is called continuous on E if $T(\omega, x)$ is continuous in x for each $\omega \in \Omega$. Similarly, T is called compact on $\Omega \times E_1$ if $T(\Omega \times E_1)$ is relatively compact subset of E_2. We say $T(\omega)$ is compact on E_1 if $T(\omega, E_1)$ is relatively compact subset of E_2. Finally, $T(\omega)$ is called compact on E_1 if $T(\omega, E_1)$ is a relatively compact subset of E_2 for each $\omega \in \Omega$.

Let $T : \Omega \times E_0 \to E$ be a random operator. A measurable function $\zeta^* : \Omega \to E_0$ is called a PPF dependent random fixed point of the random operator $T(\omega)$ if

$$T(\omega, \zeta^*(\omega)) = \zeta^*(c, \omega)$$

for some $c \in I$.

Definition 3.1. *A random operator $S : \Omega \times E_0 \to E$ is called a random contraction if for each $\omega \in \Omega$,*

$$\left\|S(\omega,\zeta) - S(\omega,\eta)\right\|_E \leq \lambda(\omega)\left\|\zeta - \eta\right\|_{E_0}$$

for all $\zeta, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq \lambda(\omega) < 1$ for all $\omega \in \Omega$.

Definition 3.2. *Two random operators $S, T : \Omega \times E_0 \to E$ are called a strong Ćirić type generalized random contraction if for a given $c \in I$ and for each $\omega \in \Omega$,*

$$\begin{aligned}&\left\|S(\omega,\zeta) - T(\omega,\eta)\right\|_E\\ &\leq \lambda(\omega)\max\{\left\|\zeta(c,\omega) - \eta(c,\omega)\right\|_E, \|\zeta(c,\omega) - S(\omega,\zeta)\|_E,\\ &\left\|\eta(c,\omega) - T(\omega,\eta)\right\|_E, \tfrac{1}{2}[\left\|\zeta(c,\omega) - T(\omega,\eta)\right\|_E + \left\|\eta(c,\omega) - S(\omega,\zeta)\right\|_E]\}\end{aligned} \tag{1}$$

for all $\zeta, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 < \lambda(\omega) < 1$ for all $\omega \in \Omega$.

Definition 3.3. *Two random operators $S, T : \Omega \times E_0 \to E$ are called a Ćirić type generalized random contraction if for a given $c \in I$ and for each $\omega \in \Omega$,*

$$\begin{aligned}&\left\|S(\omega,\zeta) - T(\omega,\eta)\right\|_E\\ &\leq \lambda(\omega)\max\{\left\|\zeta - \eta\right\|_{E_0}, \|\zeta(c,\omega) - S(\omega,\zeta)\|_E,\\ &\left\|\eta(c,\omega) - T(\omega,\eta)\right\|_E, \tfrac{1}{2}[\left\|\zeta(c,\omega) - T(\omega,\eta)\right\|_E + \left\|\eta(c,\omega) - S(\omega,\zeta)\right\|_E]\}\end{aligned} \tag{2}$$

for all $\zeta, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 < \lambda(\omega) < 1$ for all $\omega \in \Omega$.

It is easy to see that every strong *Ćirić* type generalized random contraction is *Ćirić* type generalized random contraction. However, the converse is not necessarily true.

Theorem 3.4. *Let $(\Omega, \aleph)$ be a measurable space and let E be a separable Banach space. If two random operators $S, T : \Omega \times E_0 \to E$ satisfy the condition of Ćirić type generalized contraction, then the following statements hold in E.*

(a) If $\mathfrak{R}_c$ is algebraically closed with respect to difference, then for a given $\zeta_0 \in E_0$ and $c \in I$, every sequence $\{\zeta_n(\omega)\}$ of measurable functions satisfying

$$S(\omega, \zeta_{2n}(\omega)) = \zeta_{2n+1}(c,\omega), T(\omega, \zeta_{2n+1}(\omega)) = \zeta_{2n+2}(c,\omega) \tag{3}$$

and

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} = \|\zeta_n(c,\omega) - \zeta_{n+1}(c,\omega)\|_E \tag{4}$$

for $n = 0, 1, 2, \cdots$ converges to a PPF dependent random common fixed point of S and T.

(b) If $\zeta_0, \eta_0 \in E_0$ and $\{\zeta_n(\omega)\}, \{\eta_n(\omega)\}$ are sequences defined by (3) and (4), then

$$\left\|\zeta_n(\omega) - \eta_n(\omega)\right\|_{E_0} \leq [\tfrac{1}{1-\lambda(\omega)}]\,\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \left\|\eta_0(\omega) - \eta_1(\omega)\right\|_{E_0} + \left\|\zeta_0(\omega) - \eta_0(\omega)\right\|_{E_0}.$$

If, in particular $\zeta_0 = \eta_0$ and $\{\zeta_n\} \neq \{\eta_n\}$, then

$$\left\|\zeta_n(\omega) - \eta_n(\omega)\right\|_{E_0} \leq [\tfrac{2}{1-\lambda(\omega)}]\,\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}.$$

(c) If $\mathfrak{R}_c$ is topologically closed, then for a given $\zeta_0 \in E_0$, every sequence $\{\zeta_n(\omega)\}$ of iterates of S and T constructed as in (a), converges to a unique PPF dependent random fixed point $\zeta^*(\omega)$ of S and T.

Proof. Let $\zeta_0 \in E_0$ be arbitrary. By hypothesis, $S(\omega, \zeta_0) \in E$. Suppose that $S(\omega, \zeta_0) = x_1(\omega)$, where the function $x_1 : \Omega \to E$ is measurable. Choose a measurable function $\zeta_1 : \Omega \to E_0$ such that $x_1(\omega) = \zeta_1(c, \omega)$ and that

$$\|\zeta_1(c, \omega) - \zeta_0(c)\|_E = \|\zeta_1(\omega) - \zeta_0\|_{E_0}.$$

Again, by hypothesis, $T(\omega, \zeta_1) \in E$. Suppose that $T(\omega, \zeta_1) = x_2(\omega)$. Choose $\zeta_2 \in E_0$ such that $x_2(\omega) = \zeta_2(c, \omega)$ and

$$\|\zeta_2(c, \omega) - \zeta_1(c, \omega)\|_E = \|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}.$$

Continuing in this way, by induction, we obtain

$$S(\omega, \zeta_{2n}(\omega)) = \zeta_{2n+1}(c, \omega), \; T(\omega, \zeta_{2n+1}(\omega)) = \zeta_{2n+2}(c, \omega)$$

and

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} = \|\zeta_n(c, \omega) - \zeta_{n+1}(c, \omega)\|_E$$

for all $\omega \in \Omega$ and $n = 0, 1, 2, \cdots$.

We claim that the sequence $\{\zeta_n(\omega)\}$ of measurable functions is Cauchy in E_0. For for $n = 0$, we have

$$\begin{aligned}
&\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0} = \|\zeta_1(c, \omega) - \zeta_2(c, \omega)\|_E = \|S(\omega, \zeta_0(\omega)) - T(\omega, \zeta_1(\omega))\| \\
&\leq \lambda(\omega) \max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_0(c, \omega) - S(\omega, \zeta_0(\omega))\|_E, \\
&\quad \|\zeta_1(c, \omega) - T(\omega, \zeta_1(\omega))\|_E, \tfrac{1}{2}[\|\zeta_0(c, \omega) - T(\omega, \zeta_1(\omega))\|_E + \|\zeta_1(c, \omega) - S(\omega, \zeta_0(\omega))\|_E]\} \\
&\leq \lambda(\omega) \max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_0(c, \omega) - \zeta_1(c, \omega)\|_E, \\
&\quad \|\zeta_1(c, \omega) - \zeta_2(c, \omega)\|_{E}, \tfrac{1}{2}[\|\zeta_0(c, \omega) - \zeta_2(c, \omega)\|_E + \|\zeta_1(c, \omega) - \zeta_1(c, \omega)\|_E]\} \\
&\leq \lambda(\omega) \max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \tfrac{1}{2}[\|\zeta_0(\omega) - \zeta_2(\omega)\|_{E_0} + \|\zeta_1(\omega) - \zeta_1(\omega)\|_{E_0}]\} \\
&\leq \lambda(\omega) \max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \tfrac{1}{2}\|\zeta_0(\omega) - \zeta_2(\omega)\|_{E_0}\} \\
&\leq \lambda(\omega) \max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \tfrac{1}{2}[\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}]\} \\
&\leq \lambda(\omega) \max \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0} = \|\zeta_2(c, \omega) - \zeta_3(c, \omega)\|_E = \|S(\omega, \zeta_2(\omega)) - T(\omega, \zeta_1(\omega))\| \\
&\leq \lambda(\omega) \max\{\|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_2(c, \omega) - S(\omega, \zeta_2(\omega))\|_E, \\
&\quad \|\zeta_1(c, \omega) - T(\omega, \zeta_1(\omega))\|_E, \tfrac{1}{2}[\|\zeta_2(c, \omega) - T(\omega, \zeta_1(\omega))\|_E + \|\zeta_1(c, \omega) - S(\omega, \zeta_2(\omega))\|_E]\} \\
&\leq \lambda(\omega) \max\{\|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_2(c, \omega) - \zeta_3(c, \omega)\|_E, \\
&\quad \|\zeta_1(c, \omega) - \zeta_2(c, \omega)\|_E, \tfrac{1}{2}[\|\zeta_2(c, \omega) - \zeta_2(c, \omega)\|_E + \|\zeta_1(c, \omega) - \zeta_3(c, \omega)\|_E]\} \\
&\leq \lambda(\omega) \max\{\|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0}, \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \tfrac{1}{2}[\|\zeta_2(\omega) - \zeta_2(\omega)\|_{E_0} + \|\zeta_1(\omega) - \zeta_3(\omega)\|_{E_0}]\} \\
&\leq \lambda(\omega) \max\{\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0}, \tfrac{1}{2}\|\zeta_1(\omega) - \zeta_3(\omega)\|_{E_0}\} \\
&\leq \lambda(\omega) \max\{\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \tfrac{1}{2}[\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0} + \|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0}]\} \\
&\leq \lambda(\omega) \max \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}.
\end{aligned}$$

Proceeding in this way, by induction, we obtain

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} \leq \lambda(\omega) \|\zeta_{n-1}(\omega) - \zeta_n(\omega)\|_{E_0}$$

for all $n = 1, 2, 3, \cdots$.

Hence, by repeated application of the above inequality, we have

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} \leq \lambda^n(\omega) \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}$$

for all $n = 1, 2, 3, \cdots$. If $m > n$, then by triangle inequality,

$$\begin{aligned}
&\|\zeta_n(\omega) - \zeta_m(\omega)\|_{E_0} \\
&\leq \|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} + \|\zeta_{n+1}(\omega) - \zeta_{n+2}(\omega)\|_{E_0} \\
&\quad + \cdots + \|\zeta_{m-1}(\omega) - \zeta_m(\omega)\|_{E_0} \leq [\lambda^n(\omega) + \lambda^{n+1}(\omega) + \cdots + \lambda^{m-1}(\omega)] \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} \\
&\leq [\tfrac{\lambda^n(\omega)}{1-\lambda(\omega)}] \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, $\|\zeta_n(\omega) - \zeta_m(\omega)\|_{E_0} \to 0$. This shows that $\{\zeta_n(\omega)\}$ is a Cauchy sequence of measurable functions on Ω into E_0. Since E_0 is complete and separable Banach space, there is a measurable function $\zeta^* : \Omega \to E_0$ such that $\lim_{n\to\infty} \zeta_n(\omega) = \zeta^*(\omega)$ for all $\omega \in \Omega$. Now, We prove that ζ^* is a random fixed point with PPF dependence of the random operators S and T on E_0. By inequality (2)

$$\begin{aligned}
&\|S(\omega, \zeta^*(\omega)) - \zeta^*(c, \omega)\|_E \\
&\leq \|S(\omega, \zeta^*(\omega)) - \zeta_{2n+2}(c, \omega)\|_E + \|\zeta_{2n+2}(c, \omega) - \zeta^*(c, \omega)\|_E \\
&\leq \|S(\omega, \zeta^*(\omega)) - T(\omega, \zeta_{2n+1}(\omega))\|_E + \|\zeta_{2n+2}(\omega) - \zeta^*(\omega)\|_{E_0} \\
&\leq \lambda(\omega) \max\{\|\zeta^*(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \|\zeta^*(c, \omega) - S(\omega, \zeta^*(\omega))\|_E, \|\zeta_{2n+1}(c, \omega) - T(\omega, \zeta_{2n+1}(\omega))\|_E, \\
&\quad \tfrac{1}{2}[\|\zeta^*(c, \omega) - T(\omega, \zeta_{2n+1}(\omega))\|_E + \|\zeta_{2n+1}(c, \omega) - S(\omega, \zeta^*(\omega))\|_E]\} + \|\zeta_{2n+2}(\omega) - \zeta^*(\omega)\|_{E_0} \\
&\leq \lambda(\omega) \max\{\|\zeta^*(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \|\zeta^*(c, \omega) - S(\omega, \zeta^*(\omega))\|_E, \|\zeta_{2n+1}(c, \omega) - \zeta_{2n+2}(c, \omega)\|_E, \\
&\quad \tfrac{1}{2}[\|\zeta^*(c, \omega) - \zeta_{2n+2}(c, \omega)\|_E + \|\zeta_{2n+1}(c, \omega) - S(\omega, \zeta^*(\omega))\|_E]\} + \|\zeta_{2n+2}(\omega) - \zeta^*(\omega)\|_{E_0}.
\end{aligned}$$

Taking limit superior as $n \to \infty$ in the above inequality, yields,

$$\|S(\omega, \zeta^*(\omega)) - \zeta^*(c, \omega)\|_E \leq \lambda(\omega) \|S(\omega, \zeta^*(\omega)) - \zeta^*(c, \omega)\|_E.$$

Hence, it follows that $S(\omega, \zeta^*(\omega)) = \zeta^*(c, \omega)$.Similarly, we can prove that $T(\omega, \zeta^*(\omega)) = \zeta^*(c, \omega)$.

(b) Let $\{\zeta_n(\omega)\}$ and $\{\eta_n(\omega)\}$ be two sequences of measurable functions as constructed in (a). Then for each $\omega \in \Omega$,

$$\begin{aligned}
&\left\|\zeta_n(\omega) - \eta_n(\omega)\right\|_{E_0} \leq \|\zeta_n(\omega) - \zeta_{n-1}(\omega)\|_{E_0} + \left\|\zeta_{n-1}(\omega) - \eta_{n-1}(\omega)\right\|_{E_0} + \left\|\zeta_{n-1}(\omega) - \eta_n(\omega)\right\|_{E_0} \\
&\quad \leq \lambda^n(\omega) \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \left\|\zeta_{n-1}(\omega) - \eta_{n-1}(\omega)\right\|_{E_0} + \lambda^n(\omega) \left\|\eta_0(\omega) - \eta_1(\omega)\right\|_{E_0} \\
&\quad \leq \lambda^n(\omega) [\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \left\|\eta_0(\omega) - \eta_1(\omega)\right\|_{E_0}] + \left\|\zeta_{n-1}(\omega) - \eta_{n-1}(\omega)\right\|_{E_0} \\
&\quad \leq (\lambda^n(\omega) + \cdots + 1) [\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \left\|\eta_0(\omega) - \eta_1(\omega)\right\|_{E_0}] \\
&\qquad + \left\|\zeta_0(\omega) - \eta_0(\omega)\right\|_{E_0} + \left\|\zeta_0(\omega) - \eta_0(\omega)\right\|_{E_0} \\
&\quad \leq \tfrac{1}{1-\lambda(\omega)} [\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \left\|\eta_0(\omega) - \eta_1(\omega)\right\|_{E_0}] + \left\|\zeta_0(\omega) - \eta_0(\omega)\right\|_{E_0}.
\end{aligned} \tag{5}$$

In particular, if $\zeta_0(\omega) = \eta_0(\omega)$, then $\zeta_0(c, \omega) = \eta_0(c, \omega)$ so that $S(\omega, \zeta_0) = S(\omega, \eta_0)$ which implies that $\zeta_1(c, \omega) = \eta_1(c, \omega)$. Hence, from inequality (4) it follows that

$$\left\|\zeta_n(\omega) - \eta_n(\omega)\right\|_{E_0} \leq \tfrac{2}{1-\lambda(\omega)} \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}.$$

(c) By part (a) above, the sequence $\{\zeta_n(\omega)\}$ of measurable functions as constructed in (a) converges to a random fixed point $\zeta^*(\omega)$ with PPF dependence. As $\mathfrak{R}_c$ is topologically closed, $\zeta^*(\omega) \in \mathfrak{R}_c$. Suppose that $\eta^*(\omega) \neq \zeta^*(\omega)$, $\omega \in \Omega$, be another random fixed point of T. Then

$$\begin{aligned}
&\left\|\zeta^*(\omega) - \eta^*(\omega)\right\|_{E_0} = \left\|\zeta^*(c, \omega) - \eta^*(c, \omega)\right\|_E \\
&\quad \leq \left\|S(\omega, \zeta^*(\omega)) - T(\omega, \eta^*(\omega))\right\|_E \\
&\quad \leq \lambda(\omega) \max\{\left\|\zeta^*(\omega) - \eta^*(\omega)\right\|_{E_0}, \|\zeta^*(c, \omega) - S(\omega, \zeta^*(\omega))\|_E, \left\|\eta^*(c, \omega) - T(\omega, \eta^*(\omega))\right\|_E, \\
&\quad\ \tfrac{1}{2}[\left\|\zeta^*(c, \omega) - T(\omega, \eta^*(\omega))\right\|_E + \left\|\eta^*(c, \omega) - S(\omega, \zeta^*(\omega))\right\|_E]\} \\
&\quad \leq \lambda(\omega) \max\{\left\|\zeta^*(\omega) - \eta^*(\omega)\right\|_{E_0}, 0, 0, \tfrac{1}{2}[\left\|\zeta^*(c, \omega) - \eta^*(c, \omega)\right\|_E + \left\|\eta^*(c, \omega) - \zeta^*(c, \omega)\right\|_E]\} \\
&\quad \leq \lambda(\omega) \max\{\left\|\zeta^*(\omega) - \eta^*(\omega)\right\|_{E_0}, 0, 0, \left\|\zeta^*(\omega) - \eta^*(\omega)\right\|_{E_0}\}
\end{aligned}$$

yields that $\zeta^*(\omega) = \eta^*(\omega)$, as $0 < \lambda(\omega) < 1$ for all $\omega \in \Omega$. □

On taking $S = T$ in (2), we obtain following corollary.

Corollary 3.5. *Suppose that $T : E_0 \to E$ is a generalized random contraction. Then the following statements hold in E_0.*

(a) If $\Re_c$ is algebraically closed with respect to difference, then for a given $\zeta_0 \in E_0$ and $c \in I$, every sequence $\{\zeta_n(\omega)\}$ of measurable function satisfying

$$T(\omega, \zeta_n(\omega)) = \zeta_{n+1}(c, \omega) \tag{6}$$

and

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} = \|\zeta_n(c, \omega) - \zeta_{n+1}(c, \omega)\|_E \tag{7}$$

for $n = 0, 1, 2, \cdots$ converges to a PPF dependent random fixed point of T.

(b) If $\Re_c$ is algebraically and topologically closed, then for a given $\zeta_0 \in E_0$, every sequence $\{\zeta_n(\omega)\}$ of iterates constructed as in (a), converges to a unique PPF dependent random fixed point $\zeta^*(\omega)$ of T.

Remark 3.6. *We note that operators in Theorem 3.4 and Corollary 3.5 are not required to satisfy any continuity condition on the domains of their definition.*

Now, we prove the existence of PPF dependent random fixed point theorems for mapping satisfying the contractive condition which is weaker than the condition of *Ćirić* type generalized contraction.

Let Ψ be the set of all functions ψ where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous nondecreasing function with $\psi(t) < t$ for all $t \in (0, +\infty)$ and $\psi(0) = 0$. If $\psi \in \Psi$, then ψ is called a Ψ-map.

Definition 3.7. *Let $S, T : \Omega \times E_0 \to E$ be two random operators. We say that S and T satisfy the condition of Ćirić type generalized random ψ-contraction if for a given $c \in I$ and for each $\omega \in \Omega$,*

$$\begin{aligned}&\left\|S(\omega, \zeta) - T(\omega, \eta)\right\| \le \psi(\max\{\left\|\zeta - \eta\right\|_{E_0}, \|\zeta(c, \omega) - S(\omega, \zeta)\|_E, \\ &\left\|\eta(c, \omega) - T(\omega, \eta)\right\|_E, \tfrac{1}{2}[\left\|\zeta(c, \omega) - T(\omega, \eta)\right\|_E + \left\|\eta(c, \omega) - S(\omega, \zeta)\right\|_E]\})\end{aligned} \tag{8}$$

for all $\zeta, \eta \in E_0$.

Theorem 3.8. *Let (Ω, X) be a measurable space and let E be a separable Banach space. If two random operators $S, T : \Omega \times E_0 \to E$ satisfy the condition of Ćirić type generalized random ψ-contarction and suppose that $\Re_c$ is topologically closed with respect to norm topology and is algebraically closed with respect to the difference, then S and T have a PPF dependent random fixed point in $\Re_c$.*

Proof. Let $\zeta_0 \in E_0$ be arbitrary. Since $S(\omega, \zeta_0) \in E$, there exists a measurable function $x_1 : \Omega \to E$ such that $S(\omega, \zeta_0) = x_1(\omega)$. Choose a measurable function $\zeta_1 : \Omega \to E_0$ such that

$$x_1(\omega) = \zeta_1(c, \omega) \text{ and } \|\zeta_1 - \zeta_0\|_{E_0} = \|\zeta_1(c, \omega) - \zeta_0(c, \omega)\|_E.$$

By assumption, $T(\omega, \zeta_1) \in E$. This implies that there exists $x_2 : \Omega \to E$ such that $T(\omega, \zeta_1) = x_2(\omega)$. Therefore, we can choose $\zeta_2 \in E_0$ such that

$$x_2(\omega) = \zeta_2(c, \omega) \text{ and } \|\zeta_2 - \zeta_1\|_{E_0} = \|\zeta_2(c, \omega) - \zeta_1(c, \omega)\|_E.$$

Proceeding in this way, by induction, we have

$$\begin{aligned}&S(\omega, \zeta_{2n}(\omega)) = \zeta_{2n+1}(c, \omega), T(\omega, \zeta_{2n+1}(\omega)) = \zeta_{2n+2}(c, \omega) \\ &\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} = \|\zeta_n(c, \omega) - \zeta_{n+1}(c, \omega)\|_E\end{aligned} \tag{9}$$

for all $\omega \in \Omega$ and $n \in \mathbb{N} \cup \{0\}$.

Now, we show that the sequence $\{\zeta_n(\omega)\}$ is a Cauchy sequence in E_0. Assume that $\zeta_{N-1} = \zeta_N$ for some $N \in \mathbb{N}$. If N is even, then we have $N = 2m$ for some $m \in \mathbb{N}$. Therefore from (8) and (9), we have

$$\begin{aligned}
&\|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0} = \|\zeta_{2m}(c,\omega) - \zeta_{2m+1}(c,\omega)\|_E = \|S(\omega,\zeta_{2m}(\omega)) - T(\omega,\zeta_{2m-1}(\omega))\|_E \\
&\quad \le \psi(\max\{\|\zeta_{2m}(\omega) - \zeta_{2m-1}(\omega)\|_{E_0}, \|\zeta_{2m}(c,\omega) - S(\omega,\zeta_{2m}(\omega))\|_E, \|\zeta_{2m-1}(c,\omega) - T(\omega,\zeta_{2m-1}(\omega))\|_E, \\
&\quad\ \tfrac{1}{2}[\|\zeta_{2m}(c,\omega) - T(\omega,\zeta_{2m-1}(\omega))\|_E + \|\zeta_{2m-1}(c,\omega) - S(\omega,\zeta_{2m}(\omega))\|_E]\}) \\
&\quad \le \psi(\max\{\|\zeta_{2m}(\omega) - \zeta_{2m-1}(\omega)\|_{E_0}, \|\zeta_{2m}(c,\omega) - \zeta_{2m+1}(c,\omega)\|_E, \|\zeta_{2m-1}(c,\omega) - \zeta_{2m}(c,\omega)\|_E, \\
&\quad\ \tfrac{1}{2}[\|\zeta_{2m}(c,\omega) - \zeta_{2m}(c,\omega)\|_E + \|\zeta_{2m-1}(c,\omega) - \zeta_{2m+1}(c,\omega)\|_E]\}) \\
&\quad \le \psi(\max\{\|\zeta_{2m}(\omega) - \zeta_{2m-1}(\omega)\|_{E_0}, \|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0}, \tfrac{1}{2}\|\zeta_{2m-1}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0}\}) \\
&\quad \le \psi(\max\{\|\zeta_{2m}(\omega) - \zeta_{2m-1}(\omega)\|_{E_0}, \|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0}, \\
&\quad\ \tfrac{1}{2}[\|\zeta_{2m-1}(\omega) - \zeta_{2m}(\omega)\|_{E_0} + \|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0}]\}) \\
&\quad \le \psi(\max\{\|\zeta_{2m}(\omega) - \zeta_{2m-1}(\omega)\|_{E_0}, \|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0}\}) \\
&\quad \le \psi\left(\max\left(\|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0}\right)\right).
\end{aligned}$$

This implies that $\|\zeta_{2m}(\omega) - \zeta_{2m+1}(\omega)\|_{E_0} = 0$ and so $\zeta_{2m}(\omega) = \zeta_{2m+1}(\omega)$. Similarly, we can prove that $\zeta_{2m+1}(\omega) = \zeta_{2m+2}(\omega)$. Therefore $\zeta_N(\omega) = \zeta_{N+1}(\omega)$. By mathematical induction, we can conclude that $\zeta_{N-1}(\omega) = \zeta_{N+K}(\omega)$ for all $k \ge 0$. If N is odd, then by the same argument we also obtain that $\zeta_{N-1}(\omega) = \zeta_{N+K}(\omega)$ for all $k \ge 0$. Therefore $\{\zeta_n(\omega)\}$ is a constant sequence for all $n \ge N-1$. This implies that $\{\zeta_n(\omega)\}$ is a Cauchy sequence in E_0. Now, suppose that $\zeta_{n-1}(\omega) \ne \zeta_n(\omega)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&\|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0} = \|\zeta_{2n}(c,\omega) - \zeta_{2n+1}(c,\omega)\|_E = \|S(\omega,\zeta_{2n}(\omega)) - T(\omega,\zeta_{2n-1}(\omega))\|_E \\
&\quad \le \psi(\max\{\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}, \|\zeta_{2n}(c,\omega) - S(\omega,\zeta_{2n}(\omega))\|_E, \|\zeta_{2n-1}(c,\omega) - T(\omega,\zeta_{2n-1}(\omega))\|_E, \\
&\quad\ \tfrac{1}{2}[\|\zeta_{2n}(c,\omega) - T(\omega,\zeta_{2n-1}(\omega))\|_E + \|\zeta_{2n-1}(c,\omega) - S(\omega,\zeta_{2n}(\omega))\|_E]\}) \\
&\quad \le \psi(\max\{\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}, \|\zeta_{2n}(c,\omega) - \zeta_{2n+1}(c,\omega)\|_E, \|\zeta_{2n-1}(c,\omega) - \zeta_{2n}(c,\omega)\|_E, \\
&\quad\ \tfrac{1}{2}[\|\zeta_{2n}(c,\omega) - \zeta_{2n}(c,\omega)\|_E + \|\zeta_{2n-1}(c,\omega) - \zeta_{2n+1}(c,\omega)\|_E]\}) \\
&\quad \le \psi(\max\{\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}, \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \tfrac{1}{2}\|\zeta_{2n-1}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}\}) \\
&\quad \le \psi(\max\{\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}, \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \\
&\quad\ \tfrac{1}{2}[\|\zeta_{2n-1}(\omega) - \zeta_{2n}(\omega)\|_{E_0} + \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}]\}) \\
&\quad \le \psi(\max\{\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}, \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}\}).
\end{aligned}$$

If $\max\{\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}, \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}\} = \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}$, then

$$\begin{aligned}
&\|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0} \le \psi\left(\|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}\right) \\
&< \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}
\end{aligned}$$

a contrdiction, therefore

$$\begin{aligned}
&\|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0} \le \psi\left(\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}\right) \\
&< \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}.
\end{aligned}$$

Similarly, we can prove that

$$\|\zeta_{2n+1}(\omega) - \zeta_{2n+2}(\omega)\|_{E_0} < \|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}.$$

It follows that $\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} \le \|\zeta_{n-1}(\omega) - \zeta_n(\omega)\|_{E_0}$ for all $\mathbb{N}$. Since the sequence $\left\{\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0}\right\}$ is a nonincreasing sequence of nonnegative real numbers, we obtain that it is a convergent sequence. Suppose that

$$\lim_{n\to\infty} \|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} = \tau$$

for some real number τ. We will prove that $\tau = 0$. Assume contrary that $\tau > 0$. Since

$$\|\zeta_{2n}(\omega) - \zeta_{2n+1}(\omega)\|_{E_0} \le \psi\left(\|\zeta_{2n}(\omega) - \zeta_{2n-1}(\omega)\|_{E_0}\right)$$

for all $n \in \mathbb{N}$ and the continuity of ψ, we have

$$\tau \leq \psi(\tau) < \tau,$$

which leads a contradiction. This implies that $\tau = 0$. Now, we prove that the sequence $\{\zeta_n(\omega)\}$ is a Cauchy sequence. For this, we have to prove that the sequence $\{\zeta_{2n}(\omega)\}$ is a Cauchy sequence. Assume that $\{\zeta_{2n}(\omega)\}$ is not a Cauchy sequence. It follows that there exist $\epsilon > 0$ and two sequences of even positive integers $\{2m_k\}$ and $\{2n_k\}$ satisfying $2m_k > 2n_k > k$ for each $k \in \mathbb{N}$ and

$$\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} \geq \epsilon. \tag{10}$$

Let $\{2m_k\}$ be the sequence of least positive integers exceeding $\{2n_k\}$ which satisfies (10) and

$$\left\|\zeta_{2m_k-2}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} < \epsilon. \tag{11}$$

We will prove that $\lim\limits_{k\to\infty} \left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} = \epsilon$.

Since $\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} \geq \epsilon$ for all $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} \geq \epsilon.$$

For each $k \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} \leq \left\|\zeta_{2m_k}(\omega) - \zeta_{2m_k-1}(\omega)\right\|_{E_0} \\
&+ \left\|\zeta_{2m_k-1}(\omega) - \zeta_{2m_k-2}(\omega)\right\|_{E_0} + \left\|\zeta_{2m_k-2}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} \\
&\leq \left\|\zeta_{2m_k}(\omega) - \zeta_{2m_k-1}(\omega)\right\|_{E_0} + \left\|\zeta_{2m_k-1}(\omega) - \zeta_{2m_k-2}(\omega)\right\|_{E_0} + \epsilon.
\end{aligned}$$

This implies that $\lim\limits_{k\to\infty} \left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} \leq \epsilon$. Therefore

$$\lim_{k\to\infty} \left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} = \epsilon.$$

Similarly, we can prove that

$$\begin{aligned}
&\lim_{k\to\infty} \left\|\zeta_{2m_k+1}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} = \epsilon, \\
&\lim_{k\to\infty} \left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k-1}(\omega)\right\|_{E_0} = \epsilon
\end{aligned}$$

and

$$\lim_{k\to\infty} \left\|\zeta_{2m_k+1}(\omega) - \zeta_{2n_k-1}(\omega)\right\|_{E_0} = \epsilon.$$

For each $k \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&\left\|\zeta_{2m_k+1}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} = \left\|\zeta_{2m_k+1}(c,\omega) - \zeta_{2n_k}(c,\omega)\right\|_E \\
&\leq \left\|S(\omega, \zeta_{2m_k}(\omega)) - T(\omega, \zeta_{2n_k-1}(\omega))\right\|_E \\
&\leq \psi(\max\{\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k-1}(\omega)\right\|_{E_0}, \left\|\zeta_{2m_k}(c,\omega) - S(\omega, \zeta_{2m_k}(\omega))\right\|_E, \left\|\zeta_{2n_k-1}(c,\omega) - T(\omega, \zeta_{2n_k-1}(\omega))\right\|_E, \\
&\quad \tfrac{1}{2}[\left\|\zeta_{2m_k}(c,\omega) - T(\omega, \zeta_{2n_k-1}(\omega))\right\|_E + \left\|\zeta_{2n_k-1}(c,\omega) - S(\omega, \zeta_{2n_k}(\omega))\right\|_E]\}) \\
&\leq \psi(\max\{\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k-1}(\omega)\right\|_{E_0}, \left\|\zeta_{2m_k}(c,\omega) - \zeta_{2m_k+1}(c,\omega)\right\|_E, \left\|\zeta_{2n_k-1}(c,\omega) - \zeta_{2n_k}(c,\omega)\right\|_E, \\
&\quad \tfrac{1}{2}[\left\|\zeta_{2m_k}(c,\omega) - \zeta_{2n_k}(c,\omega)\right\|_E + \left\|\zeta_{2n_k-1}(c,\omega) - \zeta_{2n_k+1}(c,\omega)\right\|_E]\}) \\
&\leq \psi(\max\{\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k-1}(\omega)\right\|_{E_0}, \left\|\zeta_{2m_k}(\omega) - \zeta_{2m_k+1}(\omega)\right\|_{E_0}, \left\|\zeta_{2n_k-1}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0}, \\
&\quad \tfrac{1}{2}[\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} + \left\|\zeta_{2n_k-1}(\omega) - \zeta_{2n_k+1}(\omega)\right\|_{E_0}]\}) \\
&\leq \psi(\max\{\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k-1}(\omega)\right\|_{E_0}, \left\|\zeta_{2m_k}(\omega) - \zeta_{2m_k+1}(\omega)\right\|_{E_0}, \left\|\zeta_{2n_k-1}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0}, \\
&\quad \tfrac{1}{2}[\left\|\zeta_{2m_k}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} + \left\|\zeta_{2n_k-1}(\omega) - \zeta_{2n_k}(\omega)\right\|_{E_0} + \left\|\zeta_{2n_k}(\omega) - \zeta_{2n_k+1}(\omega)\right\|_{E_0}]\}).
\end{aligned}$$

By taking limit of both sides, we have

$$\epsilon \leq \psi(\epsilon) < \epsilon,$$

which is a contradiction. It follows that the sequence $\{\zeta_{2n}(\omega)\}$ is a Cauchy sequence and so $\{\zeta_n(\omega)\}$ is a Cauchy sequence. Since E_0 is complete, therefore $\{\zeta_n(\omega)\}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} \zeta_n(\omega) = \zeta(\omega)$ for some $\zeta(\omega) \in E_0$. Since $\mathfrak{R}_c$ is algebraically closed with respect to the norm topology, we have $\zeta(\omega) \in \mathfrak{R}_c$. Moreover, we also obtain that

$$\lim_{n\to\infty} \zeta_{2n+1}(\omega) = \zeta(\omega) = \lim_{n\to\infty} \zeta_{2n+2}(\omega).$$

Now, we will prove that $\zeta(\omega)$ is a PPF dependent random fixed point of S. Using (8), we obtain that

$$\begin{aligned}
&\|S(\zeta(\omega)) - \zeta(c,\omega)\|_E \\
&\quad\leq \|S(\zeta(\omega)) - \zeta_{2n+2}(c,\omega)\|_E + \|\zeta_{2n+2}(c,\omega) + \zeta(c,\omega)\|_E \\
&\quad\leq \|S(\zeta(\omega)) - T(\zeta_{2n+1}(\omega))\|_E + \|\zeta_{2n+2}(\omega) + \zeta(\omega)\|_{E_0} \\
&\quad\leq \psi(\max\{\|\zeta(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \|\zeta(c,\omega) - S(\zeta(\omega))\|_E, \|\zeta_{2n+1}(c,\omega) - T(\zeta_{2n+1}(\omega))\|_E, \\
&\quad\quad \tfrac{1}{2}[\|\zeta(c,\omega) - T(\zeta_{2n+1}(\omega))\| + \|\zeta_{2n+1}(c,\omega) - S(\zeta(\omega))\|_E]\}) + \|\zeta_{2n+2}(\omega) + \zeta(\omega)\|_{E_0} \\
&\quad\leq \psi(\max\{\|\zeta(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \|\zeta(c,\omega) - S(\zeta(\omega))\|_E, \|\zeta_{2n+1}(c,\omega) - \zeta_{2n+2}(c,\omega)\|_E, \\
&\quad\quad \tfrac{1}{2}[\|\zeta(c,\omega) - \zeta_{2n+2}(c,\omega)\|_E + \|\zeta_{2n+1}(c,\omega) - S(\zeta(\omega))\|_E]\}) + \|\zeta_{2n+2}(\omega) + \zeta(\omega)\|_{E_0} \\
&\quad\leq \psi(\max\{\|\zeta(\omega) - \zeta_{2n+1}(\omega)\|_{E_0}, \|\zeta(c,\omega) - S(\zeta(\omega))\|_E, \|\zeta_{2n+1}(\omega) - \zeta_{2n+2}(\omega)\|_{E_0}, \\
&\quad\quad \tfrac{1}{2}[\|\zeta(\omega) - \zeta_{2n+2}(\omega)\|_{E_0} + \|\zeta_{2n+1}(c,\omega) - S(\zeta(\omega))\|_E]\}) + \|\zeta_{2n+2}(\omega) + \zeta(\omega)\|_{E_0}.
\end{aligned}$$

Letting $n \to \infty$, we obtain that $\|S(\zeta(\omega)) - \zeta(c,\omega)\|_E = 0$. Therefore $S(\zeta(\omega)) = \zeta(c,\omega)$. Similarly, we can prove that $T(\zeta(\omega)) = \zeta(c,\omega)$. This implies that $\zeta(\omega)$ is a PPF dependent random fixed point of S and T. □

Remark 3.9. *Define a function* $\psi : [0,+\infty) \to [0,+\infty)$ *by* $\psi(t) = \lambda t$ *for all* $t \in [0,+\infty)$ *and* $0 < \lambda < 1$. *Therefore* ψ *is a continuous nondecreasing function and* $\psi(t) < t$ *for all* $t \in [0,+\infty)$ *and* $\psi(0) = 0$. *Then, Theorem 3.4. is a special case of Theorem 3.8.*

4. Random Coincidence Points with PPF Dependence

Definition 4.1. *Let* $A : \Omega \times E_0 \to E$ *and* $S : \Omega \times E_0 \to E_0$ *be two random operators. A point* $\zeta^*(\omega) \in E_0$ *is called a PPF dependent random coincidence point of* A *and* S *if* $A(\omega, \zeta^*(\omega)) = S(\omega, \zeta^*(c,\omega))$ *for some* $c \in I$ *and* $\omega \in \Omega$. *Any statement that guarantees the existence of such a random coincidence point is called a random coincidence point theorem with PPF dependence.*

Definition 4.2. *The random operators* $A : \Omega \times E_0 \to E$ *and* $S : \Omega \times E_0 \to E_0$ *are said to satisfy a condition of strong Ćirić type generalized random contraction (C) if for a given* $c \in I$ *and for each* $\omega \in \Omega$,

$$\begin{aligned}
\|A(\omega,\zeta) - A(\omega,\eta)\|_E \leq \lambda(\omega)\max\{&\|S(\omega,\zeta(c,\omega)) - S(\omega,\eta(c,\omega))\|_E, \|S(\omega,\zeta(c,\omega)) - A(\omega,\zeta(\omega))\|_E, \\
&\|S(\omega,\eta(c,\omega)) - A(\omega,\eta(\omega))\|_E, \tfrac{1}{2}[\|S(\omega,\zeta(c,\omega)) - A(\omega,\eta(\omega))\|_E + \|S(\omega,\eta(c,\omega)) - A(\omega,\zeta(\omega))\|_E]\}
\end{aligned} \tag{12}$$

for all $\zeta, \eta \in E_0$, *where* $\lambda : \Omega \to \mathbb{R}_+$ *is a measurable function satisfying* $0 < \lambda(\omega) < 1$ *for all* $\omega \in \Omega$.

Definition 4.3. *The random operators* $A : \Omega \times E_0 \to E$ *and* $S : \Omega \times E_0 \to E_0$ *are said to satisfy a condition of Ćirić type generalized random contraction (C) if for a given* $c \in I$ *and for each* $\omega \in \Omega$,

$$\begin{aligned}
\|A(\omega,\zeta) - A(\omega,\eta)\|_E \leq \lambda(\omega)\max\{&\|S(\omega,\zeta(\omega)) - S(\omega,\eta(\omega))\|_{E_0}, \|S(\omega,\zeta(c,\omega)) - A(\omega,\zeta(\omega))\|_E, \\
&\|S(\omega,\eta(c,\omega)) - A(\omega,\eta(\omega))\|_E, \tfrac{1}{2}[\|S(\omega,\zeta(c,\omega)) - A(\omega,\eta(\omega))\|_E + \|S(\omega,\eta(c,\omega)) - A(\omega,\zeta(\omega))\|_E]\}
\end{aligned} \tag{13}$$

for all $\zeta, \eta \in E_0$, *where* $\lambda : \Omega \to \mathbb{R}_+$ *is a measurable function satisfying* $0 < \lambda(\omega) < 1$ *for all* $\omega \in \Omega$.

Theorem 4.4. *Let $A : \Omega \times E_0 \to E$ and $S : \Omega \times E_0 \to E_0$ be two random operators satisfying a Ćirić type generalized random contraction (C). Further suppose that*

(a) $A(\Omega \times E_0) \subset S(\Omega \times E_0)(c)$

(b) $S(E_0)$ is complete, and

(c) $S(\omega, \cdot)$ is continuous.

If $\mathfrak{R}_c$ is algebraically and topologically closed with respect to the difference, then A and S have a PPF dependent random coincidence point in $\mathfrak{R}_c$.

Proof. Let $\mu_0 \in E_0$ be arbitrary. By hypothesis, $A(\omega, \mu_0) \in E$. Suppose that $A(\omega, \mu_0) = x_1(\omega)$, where the function $x_1 : \Omega \to E$ is measurable. Since $A(\Omega \times E_0) \subset S(\Omega \times E_0)(c)$, choose a measurable function $\mu_1 : \Omega \to E_0$ such that $x_1(\omega) = S(\omega, \mu_1(c, \omega)) = \zeta_1(c, \omega)$ and

$$\|\zeta_1(c,\omega) - \zeta_0(c)\|_E = \|\zeta_1(\omega) - \zeta_0\|_{E_0}.$$

Again, by hypothesis, $A(\omega, \mu_1) \in E$. Suppose that $A(\omega, \mu_1) = x_2(\omega)$. Again since $A(\Omega \times E_0) \subset S(\Omega \times E_0)(c)$, choose a measurable function $\mu_2 \in E_0$ such that $x_2(\omega) = S(\omega, \mu_2(c, \omega)) = \zeta_2(c, \omega)$ and

$$\|\zeta_2(c,\omega) - \zeta_1(c)\|_E = \|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}.$$

Continuing in this way, by induction, we obtain

$$A(\omega, \mu_n(\omega)) = S(\omega, \mu_{n+1}(c,\omega)),\ S(\omega, \mu_{n+1}(\omega)) = \zeta_{n+1}(\omega) \tag{14}$$

and

$$\|\zeta_n(c,\omega) - \zeta_{n+1}(c)\|_E = \|\zeta_n(\omega) - \zeta_{n+1}\|_{E_0}. \tag{15}$$

for all $n = 0, 1, 2, \cdots$.

We claim that $\{\zeta_n(\omega)\}$ is a Cauchy sequence in E_0. Now for $n = 0$, we have

$$\begin{aligned}
&\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0} = \|\zeta_1(c,\omega) - \zeta_2(c)\|_E = \left\|A(\omega,\mu_0(\omega)) - A(\omega,\mu_1(\omega))\right\|_E \\
&\quad \le \lambda(\omega)\max\{\left\|S(\omega,\mu_0(\omega)) - S(\omega,\mu_1(\omega))\right\|_{E_0}, \left\|S(\omega,\mu_0(c,\omega)) - A(\omega,\mu_0(\omega))\right\|_E, \\
&\quad \left\|S(\omega,\mu_1(c,\omega)) - A(\omega,\mu_1(\omega))\right\|_E, \tfrac{1}{2}[\left\|S(\omega,\mu_0(c,\omega)) - A(\omega,\mu_1(\omega))\right\|_E + \left\|S(\omega,\mu_1(c,\omega)) - A(\omega,\mu_0(\omega))\right\|_E]\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_0(c,\omega) - \zeta_1(c,\omega)\|_E, \left\|\mu_1(c,\omega) - \mu_2(c,\omega)\right\|_E, \\
&\quad \tfrac{1}{2}[\|\zeta_0(c,\omega) - \zeta_2(c,\omega)\|_E + \|\zeta_1(c,\omega) - \zeta_1(c,\omega)\|]\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \\
&\quad \tfrac{1}{2}[\|\zeta_0(\omega) - \zeta_2(\omega)\|_{E_0} + \|\zeta_1(\omega) - \zeta_1(\omega)\|_{E_0}]\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \tfrac{1}{2}\|\zeta_0(\omega) - \zeta_2(\omega)\|_{E_0}\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}, \tfrac{1}{2}[\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}]\} \\
&\quad \le \lambda(\omega)\|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0} = \left\|S(\omega,\mu_2(c,\omega)) - S(\omega,\mu_3(c,\omega))\right\|_E = \left\|A(\omega,\mu_2(\omega)) - A(\omega,\mu_1(\omega))\right\|_E \\
&\quad \le \lambda(\omega)\max\{\left\|S(\omega,\mu_2(\omega)) - S(\omega,\mu_1(\omega))\right\|_{E_0}, \left\|S(\omega,\mu_2(c,\omega)) - A(\omega,\mu_2(\omega))\right\|_E, \\
&\quad \left\|S(\omega,\mu_1(c,\omega)) - A(\omega,\mu_1(\omega))\right\|_E, \tfrac{1}{2}[\left\|S(\omega,\mu_2(c,\omega)) - A(\omega,\mu_1(\omega))\right\|_E \\
&\quad + \left\|S(\omega,\mu_1(c,\omega)) - A(\omega,\mu_2(\omega))\right\|_E]\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_2(c,\omega) - \zeta_3(c,\omega)\|_E, \|\zeta_1(c,\omega) - \zeta_2(c,\omega)\|_E, \\
&\quad \tfrac{1}{2}[\|\zeta_2(c,\omega) - \zeta_2(c,\omega)\|_E + \|\zeta_1(c,\omega) - \zeta_3(c,\omega)\|]\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_2(\omega) - \zeta_1(\omega)\|_{E_0}, \|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0}, \|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \\
&\quad \tfrac{1}{2}[\|\zeta_2(\omega) - \zeta_2(\omega)\|_{E_0} + \|\zeta_1(\omega) - \zeta_3(\omega)\|_{E_0}]\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0}, \tfrac{1}{2}\|\zeta_1(\omega) - \zeta_3(\omega)\|_{E_0}\} \\
&\quad \le \lambda(\omega)\max\{\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}, \tfrac{1}{2}[\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0} + \|\zeta_2(\omega) - \zeta_3(\omega)\|_{E_0}]\} \\
&\quad \le \lambda(\omega)\|\zeta_1(\omega) - \zeta_2(\omega)\|_{E_0}.
\end{aligned}$$

Proceeding in this way, by induction, we obtain

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} \leq \lambda \|\zeta_{n-1}(\omega) - \zeta_n(\omega)\|_{E_0}$$

for all $n = 1, 2, 3, \cdots$.

Hence, repeated application of the above inequality yields

$$\|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} \leq \lambda^n \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}$$

for all $n = 1, 2, 3, \cdots$.If $m > n$, by triangle inequality, we obtain

$$\begin{aligned}
&\|\zeta_n(\omega) - \zeta_m(\omega)\|_{E_0} \leq \|\zeta_n(\omega) - \zeta_{n+1}(\omega)\|_{E_0} + \cdots + \|\zeta_{m-1}(\omega) - \zeta_m(\omega)\|_{E_0} \\
&\leq \lambda^n \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} + \cdots + \lambda^{m-1} \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} \\
&\leq (\lambda^n + \cdots + \lambda^{m-1}) \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0} \\
&\leq [\tfrac{\lambda^n}{1-\lambda}] \|\zeta_0(\omega) - \zeta_1(\omega)\|_{E_0}.
\end{aligned}$$

Hence,

$$\lim_{m>n\to\infty} \|\zeta_n(\omega) - \zeta_m(\omega)\|_{E_0} = 0.$$

This shows that $\{\zeta_n(\omega)\}$ is a Cauchy sequence of measurable functions on Ω into E_0.Since E_0 is complete, $\{\zeta_n(\omega)\}$ and every subsequence of it converges to a limit point $\zeta^*(\omega)$ in E_0, that is,

$$\begin{aligned}
\lim_{n\to\infty} \zeta_n(\omega) &= \lim_{n\to\infty} S(\omega, \mu_n(\omega)) = \zeta^*(\omega), \\
\lim_{n\to\infty} \zeta_n(c, \omega) &= \lim_{n\to\infty} A(\omega, \mu_n(\omega)) = \zeta^*(c, \omega).
\end{aligned}$$

From continuity of S it follows that

$$\zeta^*(\omega) = \lim_{n\to\infty} \zeta_n(\omega) = \lim_{n\to\infty} S(\omega, \mu_n(\omega)) = S\Big(\omega, \lim_{n\to\infty} \mu_n(\omega)\Big) = S(\omega, \mu^*(\omega)).$$

Now, we prove that $\mu^*(\omega)$ is a PPF dependent random coincidence point of A and S. Assume contrary that $\mu^*(\omega)$ is not a random coincidence point of A and S. Then, by (13), we obtain

$$\begin{aligned}
&\left\|A(\omega, \mu^*(\omega)) - S(\omega, \mu^*(c, \omega))\right\|_E \\
&= \left\|A(\omega, \mu^*(\omega)) - A(\omega, \mu_n(\omega))\right\|_E + \left\|A(\omega, \mu_n(\omega)) - S(\omega, \mu^*(c, \omega))\right\|_E \\
&\leq \left\|A(\omega, \mu^*(\omega)) - A(\omega, \mu_n(\omega))\right\|_E + \left\|S(\omega, \mu_n(c, \omega)) - S(\omega, \mu^*(c, \omega))\right\|_E \\
&\leq \lambda(\omega) \max\{\left\|S(\omega, \mu^*(\omega)) - S(\omega, \mu_n(\omega))\right\|_{E_0}, \left\|S(\omega, \mu^*(c, \omega)) - A(\omega, \mu^*(\omega))\right\|_E, \\
&\left\|S(\omega, \mu_n(c, \omega)) - A(\omega, \mu_n(\omega))\right\|_E, \tfrac{1}{2}[\left\|S(\omega, \mu^*(c, \omega)) - A(\omega, \mu_n(\omega))\right\|_E \\
&+ \left\|S(\omega, \mu_n(c, \omega)) - A(\omega, \mu^*(\omega))\right\|_E]\} + \left\|S(\omega, \mu_n(c, \omega)) - S(\omega, \mu^*(c, \omega))\right\|_E \\
&\leq \lambda(\omega) \max\{\|\zeta^*(\omega) - \zeta_n(\omega)\|_{E_0}, \left\|S(\omega, \mu^*(c, \omega)) - A(\omega, \mu^*(\omega))\right\|_E, \\
&S(\omega, \mu_n(c, \omega)) - A(\omega, \mu^*(\omega))_E, \tfrac{1}{2}[\left\|S(\omega, \mu^*(c, \omega)) - S(\omega, \mu_n(c, \omega))\right\|_E \\
&+ \left\|S(\omega, \mu_n(c, \omega)) - A(\omega, \mu^*(\omega))\right\|_E\} + \left\|S(\omega, \mu_n(c, \omega)) - S(\omega, \mu^*(c, \omega))\right\|_E
\end{aligned}$$

By taking limit as $n \to \infty$, we have

$$\begin{aligned}
&\left\|A(\omega, \mu^*(\omega)) - S(\omega, \mu^*(c, \omega))\right\|_E \\
&\leq \lambda(\omega) \max\{0, \left\|S(\omega, \mu^*(c, \omega)) - A(\omega, \mu^*(\omega))\right\|_E, 0, \\
&\tfrac{1}{2}[0 + \left\|S(\omega, \mu^*(c, \omega)) - A(\omega, \mu^*(\omega))\right\|_E]\} \\
&= \lambda(\omega) \left\|A(\omega, \mu^*(\omega)) - S(\omega, \mu^*(c, \omega))\right\|_E,
\end{aligned}$$

which leads to a contradiction since $0 < \lambda(\omega) < 1$. Hence $A(\omega, \mu^*(\omega)) = S(\omega, \mu^*(c, \omega))$. Thus $\mu^*(\omega)$ is a PPF dependent random coincidence point of A and S. □

We now prove the existence of PPF dependent coincidence points for mapping satisfying a weaker contractive condition than defined in (12) (without using topological closedness of the Razumikhin class.

Definition 4.5. *The random operators $A:\Omega\times E_0\to E$ and $S:\Omega\times E_0\to E_0$ are said to satisfy a condition of Ćirić type generalized random ψ-contraction (C) if for a given $c\in I$ and for each $\omega\in\Omega$,*

$$\begin{aligned}
&\left\|A(\omega,\zeta)-A(\omega,\eta)\right\|_E\\
&\le\psi(\max\{\left\|S(\omega,\zeta(\omega))-S(\omega,\eta(\omega))\right\|_{E_0},\\
&\|S(\omega,\zeta(c,\omega))-A(\omega,\zeta(\omega))\|_E,\left\|S(\omega,\eta(c,\omega))-A(\omega,\eta(\omega))\right\|_E,\\
&\tfrac{1}{2}[\left\|S(\omega,\zeta(c,\omega))-A(\omega,\eta(\omega))\right\|_E+\left\|S(\omega,\eta(c,\omega))-A(\omega,\zeta(\omega))\right\|_E]\}.
\end{aligned}\tag{16}$$

Theorem 4.6. *Let $A:\Omega\times E_0\to E$ and $S:\Omega\times E_0\to E_0$ be two random operators satisfying a Ćirić type generalized random ψ-contraction (C). Further suppose that*

(a) $A(\Omega\times E_0)\subset S(\Omega\times E_0)(c)$
(b) $S(E_0)$ is complete, and
(c) $S(\omega,\cdot)$ is continuous.

If $\mathfrak{R}_c$ is algebraically closed with respect to the difference, then A and S have a PPF dependent random coincidence point in $\mathfrak{R}_c$.

Proof. Let $\mu_0\in E_0$ be arbitrary. By hypothesis, $A(\omega,\mu_0)\in E$. Suppose that $A(\omega,\mu_0)=x_1(\omega)$, where the function $x_1:\Omega\to E$ is measurable. Since $A(\Omega\times E_0)\subset S(\Omega\times E_0)(c)$, choose a measurable function $\mu_1:\Omega\to E_0$ such that $x_1(\omega)=S(\omega,\mu_1(c,\omega))=\alpha_1(c,\omega)$ and $\|\alpha_1(w)-\alpha_0(\omega)\|_{E_0}=\|\alpha_1(c,w)-\alpha_0(c,\omega)\|_E$. Since $A(\omega,\mu_1)\in E_0$ and by assumption, we have $A(\omega,\mu_1)=x_2(\omega)$ for some $x_2(\omega)\in E$. Again since $A(\Omega\times E_0)\subset S(\Omega\times E_0)(c)$, choose $\mu_2\in E_0$ such that $x_2(\omega)=S(\omega,\mu_2(c,\omega))=\alpha_2(c,\omega)$ and $\|\alpha_2(w)-\alpha_1(\omega)\|_{E_0}=\|\alpha_2(c,w)-\alpha_1(c,\omega)\|_E$. By continuing the process, we can construct the sequence $\{\alpha_n(\omega)\}$ such that

$$A(\omega,\mu_n(\omega))=S(\omega,\mu_{n+1}(c,\omega)),\,S(\omega,\mu_{n+1}(\omega))=\alpha_{n+1}(\omega)\tag{17}$$

and

$$\|\alpha_n(w)-\alpha_{n+1}(\omega)\|_{E_0}=\|\alpha_n(c,w)-\alpha_{n+1}(c,\omega)\|_E\tag{18}$$

for all $n\in\mathbb{N}\cup\{0\}$. We will show that $\{\alpha_n(\omega)\}$ is a Cauchy sequence in E_0.If $\alpha_N=\alpha_{N+1}$ for some $N\in\mathbb{N}$, then, by (16), (17) and (18) we have

$$\begin{aligned}
&\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0}=\|\alpha_{N+1}(c,\omega)-\alpha_{N+2}(c,\omega)\|_E\\
&=\left\|A(\omega,\mu_N(\omega))-A(\omega,\mu_{N+1}(\omega))\right\|_E\\
&\le\psi(\max\{\left\|S(\omega,\mu_N(\omega))-S(\omega,\mu_{N+1}(\omega))\right\|_{E_0},\left\|S(\omega,\mu_N(c,\omega))-A(\omega,\mu_N(\omega))\right\|_E,\\
&\left\|S(\omega,\mu_{N+1}(c,\omega))-A(\omega,\mu_{N+1}(\omega))\right\|_E,\tfrac{1}{2}[\left\|S(\omega,\mu_N(c,\omega))-A(\omega,\mu_{N+1}(\omega))\right\|_E\\
&+\left\|S(\omega,\mu_{N+1}(c,\omega))-A(\omega,\mu_N(\omega))\right\|_E]\})\\
&\le\psi(\max\{\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0},\|\alpha_N(c,\omega)-\alpha_{N+1}(c,\omega)\|_E,\\
&\|\alpha_{N+1}(c,\omega)-\alpha_{N+2}(c,\omega)\|_E,\tfrac{1}{2}[\|\alpha_N(c,\omega)-\alpha_{N+2}(c,\omega)\|_E\\
&+\|\alpha_{N+1}(c,\omega)-\alpha_{N+2}(c,\omega)\|_E]\})\\
&\le\psi(\max\{\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0},\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0},\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0},\\
&\tfrac{1}{2}\|\alpha_N(\omega)-\alpha_{N+2}(\omega)\|_{E_0}\})\\
&\le\psi(\max\{\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0},\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0},\\
&\tfrac{1}{2}\|\alpha_N(\omega)-\alpha_{N+2}(\omega)\|_{E_0}\})\\
&\le\psi(\max\{\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0},\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0},\\
&\tfrac{1}{2}[\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0}+\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0}]\})\\
&\le\psi(\max\{\|\alpha_N(\omega)-\alpha_{N+1}(\omega)\|_{E_0},\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0}\})\\
&\le\psi(\|\alpha_{N+1}(\omega)-\alpha_{N+2}(\omega)\|_{E_0}).
\end{aligned}$$

Therefore $\alpha_{N+1}(\omega) = \alpha_{N+2}(\omega)$. By mathematical induction, we obtain that $\alpha_N(\omega) = \alpha_{N+k}(\omega)$ for all $k \in \mathbb{N}$. This implies that $\{\alpha_n(\omega)\}$ is a constant sequence for $n \geq N$. Thus $\{\alpha_n(\omega)\}$ is a Cauchy sequence in E_0. Suppose that $\alpha_n(\omega) \neq \alpha_{n+1}(\omega)$ for all $n \in \mathbb{N}$. Again, by (16), (17) and (18), we have for each $n \in \mathbb{N}$

$$\begin{aligned}
&\|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0} = \|\alpha_{n+1}(c,\omega) - \alpha_{n+2}(c,\omega)\|_E \\
&= \left\|A(\omega,\mu_n(\omega)) - A(\omega,\mu_{n+1}(\omega))\right\|_E \\
&\leq \psi(\max\{\left\|S(\omega,\mu_n(\omega)) - S(\omega,\mu_{n+1}(\omega))\right\|_{E_0}, \left\|S(\omega,\mu_n(c,\omega)) - A(\omega,\mu_n(\omega))\right\|_E, \\
&\left\|S(\omega,\mu_{n+1}(c,\omega)) - A(\omega,\mu_{n+1}(\omega))\right\|_E, \tfrac{1}{2}[\left\|S(\omega,\mu_n(c,\omega)) - A(\omega,\mu_{n+1}(\omega))\right\|_E \\
&+ \left\|S(\omega,\mu_{n+1}(c,\omega)) - A(\omega,\mu_n(\omega))\right\|_E]\}) \\
&\leq \psi(\max\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_n(c,\omega) - \alpha_{n+1}(c,\omega)\|_E, \\
&\|\alpha_{n+1}(c,\omega) - \alpha_{n+2}(c,\omega)\|_E, \tfrac{1}{2}[\|\alpha_n(c,\omega) - \alpha_{n+2}(c,\omega)\|_E \\
&+ \|\alpha_{n+1}(c,\omega) - \alpha_{n+1}(c,\omega)\|_E]\}) \\
&\leq \psi(\max\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}, \\
&\tfrac{1}{2}\|\alpha_n(\omega) - \alpha_{n+2}(\omega)\|_{E_0}\}) \\
&\leq \psi(\max\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}, \\
&\tfrac{1}{2}\|\alpha_n(\omega) - \alpha_{n+2}(\omega)\|_{E_0}\}) \\
&\leq \psi(\max\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}, \\
&\tfrac{1}{2}[\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0} + \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}]\}) \\
&\leq \psi(\max\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}\});
\end{aligned}$$

if

$$\begin{aligned}
&\max\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}, \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}\} \\
&= \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0},
\end{aligned}$$

then

$$\begin{aligned}
&\|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0} \leq \psi\left(\|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}\right) \\
&< \|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0}.
\end{aligned}$$

This is a contradiction. Therefore

$$\begin{aligned}
&\|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0} \leq \psi\left(\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}\right) \\
&< \|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}.
\end{aligned}$$

It follows that $\|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0} \leq \|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}$ for all $n \in \mathbb{N}$. Since the sequence $\left\{\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}\right\}$ is a nonincreasing sequence of real numbers, we obtain that it is a convergent sequence. Suppose that $\lim_{n\to\infty} \|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0} = \alpha$ for some nonnegative real number α. Now, we have to show that $\alpha = 0$. Assume contrary that $\alpha > 0$. Since

$$\|\alpha_{n+1}(\omega) - \alpha_{n+2}(\omega)\|_{E_0} \leq \psi\left(\|\alpha_n(\omega) - \alpha_{n+1}(\omega)\|_{E_0}\right)$$

for all $n \in \mathbb{N}$ and the continuity of ψ, we have

$$\alpha \leq \psi(\alpha) < \alpha,$$

which leads a contradiction. Thus $\alpha = 0$. We will prove that $\{\alpha_n(\omega)\}$ is a Cauchy sequence in E_0. It suffices to prove that the sequence $\{\alpha_{2n}(\omega)\}$ is a Cauchy sequence in E_0. Assume contrary that $\{\alpha_{2n}(\omega)\}$ is not a Cauchy sequence. It follows that there exist $\epsilon > 0$ and two sequences of even positive integers $\{2m_k\}$ and $\{2n_k\}$ satisfying $2m_k > 2n_k > k$ for each $k \in \mathbb{N}$ and

$$\left\|\alpha_{2m_k}(\omega) - \alpha_{2n_k}(\omega)\right\|_{E_0} \geq \epsilon. \tag{19}$$

Let $\{2m_k\}$ be the sequence of least positive integers exceeding $\{2n_k\}$ which satisfies equation (19) and

$$\left\|\alpha_{2m_k-2}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}<\epsilon. \tag{20}$$

Now, we will show that $\lim\limits_{k\to\infty}\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}=\epsilon$. Since $\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}\geq\epsilon$ for all $k\in\mathbb{N}$, we have $\lim\limits_{k\to\infty}\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}\geq\epsilon$. For each $k\in\mathbb{N}$, we obtain that

$$\begin{aligned}
&\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}\\
&\leq\left\|\alpha_{2m_k}(\omega)-\alpha_{2m_k-1}(\omega)\right\|_{E_0}+\left\|\alpha_{2m_k-1}(\omega)-\alpha_{2m_k-2}(\omega)\right\|_{E_0}+\left\|\alpha_{2m_k-2}(\omega)-\alpha_{2n_k(\omega)}\right\|_{E_0}\\
&\leq\left\|\alpha_{2m_k}(\omega)-\alpha_{2m_k-1}(\omega)\right\|_{E_0}+\left\|\alpha_{2m_k-1}(\omega)-\alpha_{2m_k-2}(\omega)\right\|_{E_0}+\epsilon.
\end{aligned}$$

This implies that $\lim\limits_{k\to\infty}\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}\leq\epsilon$. Therefore

$$\lim_{k\to\infty}\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}=\epsilon.$$

Similarly, we can prove that

$$\lim_{k\to\infty}\left\|\alpha_{2m_k+1}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}=\epsilon,\ \lim_{k\to\infty}\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k-1}(\omega)\right\|_{E_0}=\epsilon$$

and

$$\lim_{k\to\infty}\left\|\alpha_{2m_k+1}(\omega)-\alpha_{2n_{k-1}}(\omega)\right\|_{E_0}=\epsilon.$$

Now,

$$\begin{aligned}
&\left\|\alpha_{2n_k}(\omega)-\alpha_{2m_k+1}(\omega)\right\|_{E_0}=\left\|\alpha_{2n_k}(c,\omega)-\alpha_{2m_k+1}(c,\omega)\right\|_{E}\\
&=\left\|A(\omega,\alpha_{2n_k-1}(\omega))-A(\omega,\alpha_{2m_k}(\omega))\right\|\\
&\leq\psi(\max\{\left\|S(\omega,\alpha_{2n_k-1}(\omega))-S((\omega,\alpha_{2m_k}(\omega)))\right\|_{E_0},\\
&\left\|S(\omega,\alpha_{2n_k-1}(c,\omega))-A((\omega,\alpha_{2n_k-1}(\omega)))\right\|_{E},\\
&\left\|S((\omega,\alpha_{2m_k}(c,\omega)))-A((\omega,\alpha_{2m_k}(\omega)))\right\|_{E},\\
&\tfrac{1}{2}[\left\|S(\omega,\alpha_{2n_k-1}(c,\omega))-A((\omega,\alpha_{2m_k}(\omega)))\right\|_{E}\\
&+\left\|S((\omega,\alpha_{2m_k}(c,\omega)))-A((\omega,\alpha_{2n_k-1}(\omega)))\right\|_{E}]\})\\
&\leq\psi(\max\{\left\|\alpha_{2n_k-1}(\omega)-\alpha_{2m_k}(\omega)\right\|_{E_0},\left\|\alpha_{2n_k-1}(c,\omega)-\alpha_{2n_k}(c,\omega)\right\|_{E},\\
&\left\|\alpha_{2m_k}(c,\omega)-\alpha_{2m_k+1}(c,\omega)\right\|_{E},\tfrac{1}{2}[\left\|\alpha_{2n_k-1}(c,\omega)-\alpha_{2m_k+1}(c,\omega)\right\|_{E}\\
&+\left\|\alpha_{2m_k}(c,\omega)-\alpha_{2n_k}(c,\omega)\right\|_{E}]\})\\
&\leq\psi(\max\{\left\|\alpha_{2n_k-1}(\omega)-\alpha_{2m_k}(\omega)\right\|_{E_0},\left\|\alpha_{2n_k-1}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0},\\
&\left\|\alpha_{2m_k}(\omega)-\alpha_{2m_k+1}(\omega)\right\|_{E_0},\tfrac{1}{2}[\left\|\alpha_{2n_k-1}(\omega)-\alpha_{2m_k+1}(\omega)\right\|_{E_0}\\
&+\left\|\alpha_{2m_k}(\omega)-\alpha_{2n_k}(\omega)\right\|_{E_0}]\}),
\end{aligned}$$

gives by taking limit on both sides,

$$\epsilon\leq\psi(\epsilon)<\epsilon,$$

a contradiction. It follows that the sequence $\{\alpha_{2n}(\omega)\}$ is a Cauchy sequence and so for $\{\alpha_n(\omega)\}$ is a Cauchy sequence in E_0. Therefore, $\{S(\omega,\alpha_n(\omega))\}$ is a Cauchy sequence in $S(\mathfrak{R}_c)$. By the completeness of $S(\mathfrak{R}_c)$, we have $\{S(\omega,\alpha_n(\omega))\}$ is a convergent sequence. Suppose that $\lim\limits_{n\to\infty}S(\omega,\alpha_n(\omega))=\overset{*}{\alpha}(\omega)$ for some $\overset{*}{\alpha}$ in $S(\mathfrak{R}_c)$. Therefore $\overset{*}{\alpha}(\omega)=S(\omega,\alpha(\omega))$ for some $\alpha\in\mathfrak{R}_c$. Moreover, we have

$$\lim_{n\to\infty}A(\omega,\alpha_n(\omega))=\lim_{n\to\infty}S(\omega,\alpha_{n+1}(\omega))=S(\omega,\alpha(c,\omega))$$

Now, we shall prove that $\alpha(\omega)$ is a PPF dependent random coincidence fixed point of A and S. By using (16), we obtain that

$$\begin{aligned}&\|A(\omega,\alpha(\omega)) - S(\omega,\alpha(c,\omega))\|_E = \|A(\omega,\alpha(\omega)) - A(\omega,\alpha_n(\omega))\|_E \\ &+ \|A(\omega,\alpha_n(\omega)) - S(\omega,\alpha(c,\omega))\|_E \\ &\le \|A(\omega,\alpha(\omega)) - A(\omega,\alpha_n(\omega))\|_E + \|S(\omega,\alpha_{n+1}(\omega)) - S(\omega,\alpha(c,\omega))\|_E \\ &\le \psi(\max\{\|S(\omega,\alpha(\omega)) - S(\omega,\alpha_n(\omega))\|_{E_0}, \|S(\omega,\alpha(c,\omega)) - A(\omega,\alpha(\omega))\|_E, \\ &\|S(\omega,\alpha_n(\omega)) - A(\omega,\alpha_n(\omega))\|_E, \tfrac{1}{2}[\|S(\omega,\alpha(c,\omega)) - A(\omega,\alpha_n(\omega))\|_E \\ &+ \|S(\omega,\alpha_n(\omega)) - A(\omega,\alpha(\omega))\|_E]\}) + \|S(\omega,\alpha_{n+1}(\omega)) - S(\omega,\alpha(c,\omega))\|_E.\end{aligned}$$

By taking $\lim_{n\to\infty}$, we have $A(\omega,\alpha(\omega)) = S(\omega,\alpha(c,\omega))$. Hence $\alpha(\omega)$ is a PPF dependent random coincidence point of A and S. □

Remark 4.7. *Define a function $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = \lambda t$ for all $t \in [0,+\infty)$ and $0 < \lambda < 1$. Therefore ψ is a continuous nondecreasing function and $\psi(t) < t$ for all $t \in [0,+\infty)$ and $\psi(0) = 0$. Then, Theorem **4.4**. is a special case of Theorem 4.6.*

5. Iterative Approximation of PPF Dependent Random Common Fixed Points

Let two random operators $S, T : \Omega \times E_0 \to E$, let $CF(\omega,(S,T))$ denote the class of all PPF dependent random common fixed points of S and T in E_0, that is,

$$CF(\omega,(S,T)) = \{\zeta \in E_0 : S(\omega,\zeta^*(\omega)) = \zeta^*(c,\omega) = T(\omega,\zeta^*(\omega))\}.$$

Definition 5.1. *Two random operators $S, T : \Omega \times E_0 \to E$ are generalized nonexpansive if*

$$\begin{aligned}&\left\|S(\omega,\zeta(\omega)) - T(\omega,\eta(\omega))\right\| \\ &\le \max\{\left\|\zeta(\omega) - \eta(\omega)\right\|_{E_0}, \tfrac{1}{2}[\|\zeta(c,\omega) - S(\omega,\zeta(\omega))\|_E \\ &+ \left\|\eta(c,\omega) - T(\omega,\eta(\omega))\right\|_E], \tfrac{1}{2}[\left\|\zeta(c,\omega) - T(\omega,\eta(\omega))\right\|_E \\ &+ \left\|\eta(c,\omega) - S(\omega,\zeta(\omega))\right\|_E]\}\end{aligned} \tag{21}$$

for all $\zeta, \eta \in E_0$.

Theorem 5.2. *Suppose that $S, T : \Omega \times E_0 \to E$ are generalized nonexpansive and that $CF(\omega,(S,T)) \neq \emptyset$. Suppose that $\mathfrak{R}_c$ is topologically and algebraically closed with respect to the difference and $\{\zeta_n(\omega)\}$ is a sequence of iterates of S and T defined as in Theorem 3.4 satisfying for some $c \in I$,*

$$\|\zeta_n(\omega) - \zeta(\omega)\|_{E_0} = \|\zeta_n(c,\omega) - \zeta(c,\omega)\|_E \tag{22}$$

for all $\zeta(\omega) \in CF(\omega,(S,T))$. Then $\{\zeta_n(\omega)\}$ converges to a PPF dependent random common fixed point of S and T if and only if

$$\lim_{n\to\infty} d_{E_0}(\zeta_n(\omega), CF(\omega,(S,T))) = 0. \tag{23}$$

Proof. First consider that $\lim_{n\to\infty} d_{E_0}(\zeta_n(\omega), CF(\omega,(S,T))) = 0$. Then

$$\lim_{n\to\infty} d_{E_0}(\zeta_n(\omega), CF(\omega,(S,T))) = 0 \text{ and } \lim_{n\to\infty} d_{E_0}(\zeta_{2n+2}(\omega), CF(\omega,(S,T))) = 0. \tag{24}$$

Similarly, for $\zeta^*(\omega) \in CF(\omega,(S,T))$,

$$\begin{aligned}&\|S(\omega,\zeta(\omega)) - \zeta^*(c,\omega)\|_E \le \|\zeta(\omega) - \zeta^*(c,\omega)\|_E, \\ &\|T(\omega,\zeta(\omega)) - \zeta^*(c,\omega)\|_E \le \|\zeta(\omega) - \zeta^*(c,\omega)\|_E\end{aligned} \tag{25}$$

for all $\zeta \in E_0$. Suppose that $\zeta_n(\omega) \to \zeta^*(\omega)$ for some $\zeta^*(\omega) \in CF(\omega,(S,T))$. Then,

$$\lim_{n\to\infty} d_{E_0}(\zeta_n(\omega), CF(\omega,(S,T))) = \lim_{n\to\infty}[\inf_{\zeta(\omega)\in CF(\omega,(S,T))} \|\zeta_n(\omega)-\zeta(\omega)\|_{E_0}]$$
$$\leq \lim_{n\to\infty} \|\zeta_n(\omega)-\zeta^*(\omega)\|_{E_0} = 0.$$

Now assume that $\lim_{n\to\infty} d_{E_0}(\zeta_n(\omega), CF(\omega,(S,T))) = 0$. Then for $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$d_{E_0}(\zeta_n(\omega), CF(\omega,(S,T))) < \frac{\epsilon}{2} \tag{26}$$

for all $n \geq n_0$. We claim that $\{\zeta_n(\omega)\}$ is a Cauchy sequence in E_0.Now, for any $m > n \geq n_0$ one has

$$\|\zeta_m(\omega)-\zeta_n(\omega)\|_{E_0} \leq \|\zeta_m(\omega)-\zeta(\omega)\|_{E_0} + \|\zeta(\omega)-\zeta_n(\omega)\|_{E_0} \tag{27}$$

for all $\zeta(\omega) \in CF(\omega,(S,T))$. Now, consider

$$\begin{aligned}
&\|\zeta_{2m+1}(\omega)-\zeta(\omega)\|_{E_0} = \|\zeta_{2m+1}(c,\omega)-\zeta(c,\omega)\|_E \\
&= \|S(\omega,\zeta_{2m}(\omega))-\zeta(c,\omega)\|_E \\
&\leq \|\zeta_{2m}(c,\omega)-\zeta(c,\omega)\|_E \\
&= \|T(\omega,\zeta_{2m-1}(\omega))-\zeta(c,\omega)\|_E \\
&\leq \|\zeta_{2m-1}(\omega)-\zeta(\omega)\|_{E_0} \\
&\vdots \\
&\leq \left\|\zeta_{n_0}(\omega)-\zeta(\omega)\right\|_{E_0}.
\end{aligned}$$

Again,

$$\|\zeta_{2m+2}(\omega)-\zeta(\omega)\|_{E_0} \leq \left\|\zeta_{n_0}(\omega)-\zeta(\omega)\right\|_{E_0}.$$

Since m is arbitrary, one has

$$\|\zeta_m(\omega)-\zeta(\omega)\|_{E_0} \leq \left\|\zeta_{n_0}(\omega)-\zeta(\omega)\right\|_{E_0}. \tag{28}$$

Similarly,

$$\|\zeta_n(\omega)-\zeta(\omega)\|_{E_0} \leq \left\|\zeta_{n_0}(\omega)-\zeta(\omega)\right\|_{E_0} \tag{29}$$

for all $\zeta(\omega) \in CF(\omega,(S,T))$. Hence, from (27), (28) and (29) it follows that

$$\|\zeta_m(\omega)-\zeta_n(\omega)\|_{E_0} \leq 2\left\|\zeta_{n_0}(\omega)-\zeta(\omega)\right\|_{E_0}$$

for all $\zeta(\omega) \in CF(\omega,(S,T))$. Taking infimum over $CF(\omega,(S,T))$, we obtain

$$\begin{aligned}
&\|\zeta_m(\omega)-\zeta_n(\omega)\|_{E_0} \leq 2\inf_{\zeta(\omega)\in CF(\omega,(S,T))} \left\|\zeta_{n_0}(\omega)-\zeta(\omega)\right\|_{E_0} \\
&= 2d_{E_0}(\zeta_{n_0}(\omega), CF(\omega,(S,T))) < \epsilon.
\end{aligned}$$

Hence, $\{\zeta_n(\omega)\}$ is a Cauchy sequence in E_0. Since E_0 is complete, $\{\zeta_n(\omega)\}$ and every subsequence of it converges to a unique limit point, say $\zeta^*(\omega) \in E_0$. Now it can be prove as in Theorem 3.4 that $S(\omega,\zeta^*(\omega)) = \zeta^*(c,\omega) = T(\omega,\zeta^*(\omega))$. Thus $\zeta^*(\omega) \in CF(\omega,(S,T))$. □

6. Application to Random Differential and Integral Equations

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 12, 13, 25] and references therein). In this section, we prove the existence of PPF dependent random solutions under some Lipschitz and compactness type conditions. Given the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$, for some reals $r > 0, T > 0$, let ϱ denote the space of continuous real-valued functions defined on I_0. We equip the space ϱ with the supremum norm $\|\cdot\|_{\varrho}$ defined by

$$\|\xi\|_{\varrho} = \sup_{\vartheta \in I_0} |\xi_{\vartheta}| .$$

It is clear that ϱ is a Banach space with this norm called the history space of the problem under consideration.For each $t \in I = [0, T]$, define a function $t \to x_t \in \varrho$ by

$$x_t(\vartheta) = x(t + \vartheta), \ \vartheta \in I_0,$$

where the argument ϑ represents the delay in the argument of solutions.

Let (Ω, X) be a measurable space. Define a mapping $x : \Omega \to C(J, \mathbb{R})$, we denote a function $x(t, \omega)$, which is continuous in the variable t for each $\omega \to \Omega$. In this case, we also write $x(t, \omega) = x(\omega)(t)$. Given the measurable functions $\varphi : \Omega \to \varrho$ and $x : \Omega \to C(I, \mathbb{R})$, consider an initial value problem of functional random differential equations of delay type (in short FRDE),

$$\begin{aligned} x^{/}(t, \omega) &= f(t, x_t(\omega), \omega) \\ x_0(\omega) &= \varphi(\omega) \end{aligned} \tag{30}$$

for all $t \in I$ and $\omega \in \Omega$, where $f : I \times \varrho \times \Omega \to \mathbb{R}$. By a random solution x of FRDE (30) we mean a measurable function $x : \Omega \to C(J, \mathbb{R})$ that satisfies the equations in (30) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

In this section, we will prove the existence of random solutions with PPF dependence for the FRDE (30) defined on J with the condition of Theorem **3.4.** We consider the following hypothesis in what follows.

(H_1) The function $\omega \longmapsto f(t, x, \omega)$ is measurable for each $t \in I$ and $x \in \varrho$ and the function $(t, x) \longmapsto f(t, x, \omega)$ is jointly continuous for each $\omega \in \Omega$.

(H_2) There exists a real number $M_f > 0$ such that for each $\omega \in \Omega$,

$$\left| f(t, x, \omega) \right| \leq M_f$$

for all $t \in I$ and $x \in \varrho$.

(H_3) There exists real number $L > 0$ such that for each $\omega \in \Omega$,

$$\left| f(t, x, \omega) - f(t, y, \omega) \right| \leq L \left\| x - y \right\|_{\varrho}$$

for all $t \in I$ and $x, y \in \varrho$.

Theorem 6.1. *Suppose that the hypotheses (H_1)–(H_3) hold. If $LT < 1$, then the FRDE (30) has a unique PPF dependent random solution defined on J.*

Proof. Consider $E = C(J, \mathbb{R})$ which is a separable Banach space. Given a function $x \in C(J, \mathbb{R})$, define a mapping $\hat{x} : I \to \varrho$ by $\hat{x}(t) = x_t \in \varrho$ so that $\hat{x}(t)(0) = x_t(0) = x(t)$, $t \in I$ and $\hat{x}(0) = x_0$. Define a set $\hat{E}$ of functions by

$$\hat{E} = \{\hat{x} = (x_t)_{t \in I} : x_t \in \varrho, x \in C(J, \mathbb{R}) \text{ and } x_0 = \varphi\} . \tag{31}$$

We define a norm on $\hat{E}$ by

$$\|\hat{x}\|_{\hat{E}} = \sup_{t \in I} \|x_t\|_{\varrho} . \tag{32}$$

Clearly, $\hat{x} \in C(I_0, \mathbb{R}) = \varrho$. Now, we show that $\hat{E}$ is a Banach space. Let $\{\hat{x}_n\}$ be a Cauchy sequence in $\hat{E}$ and $\hat{x}_n(t) = x_t^n$, then $\left\{ \left(x_t^n \right)_{t \in I} \right\}$ is a Cauchy sequence in ϱ for each $t \in I$. This further implies that $\left\{ x_t^m(s) \right\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\left\{ x_t^m(s) \right\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\left\{ x_t^n \right\}$ is a sequence of uniformly continuous functions for a fixed $t \in I, x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. Since $\hat{E}$ is complete, moreover, $\hat{E}$ is a separable Banach space. The FRDE (30) is equivalent to the nonlinear random integral equation

$$x(t,\omega) = \begin{cases} \varphi(0,\omega) + \int_0^t f(s, x_s(\omega), \omega)\, ds, & \text{if } \quad t \in I \\ \varphi(t,\omega), & \text{if } t \in I_0. \end{cases}$$

Given a measurable function $\hat{x} : \Omega \to \hat{E}$, the operators $S, T : \Omega \times \hat{E} \to \mathbb{R}$ defined by

$$\begin{aligned} & S(\omega, \hat{x}(t,\omega)) = S(\omega, x_t(\omega)) \\ = \; & \begin{matrix} \varphi(0,\omega) + \int_0^t f(s, x_s(\omega), \omega) ds & \text{if } t \in I \\ \varphi(t,\omega) & \text{if } t \in I_0 \end{matrix} \end{aligned} \tag{33}$$

and

$$\begin{aligned} & T(\omega, \hat{y}(t,\omega)) = T(\omega, y_t(\omega)) \\ = \; & \begin{matrix} \varphi(0,\omega) + \int_0^t g(s, y_s(\omega), \omega) ds & \text{if } t \in I \\ \varphi(t,\omega) & \text{if } t \in I_0 \end{matrix} \end{aligned} \tag{34}$$

satisfy the *Ćirić* type generalized random contraction. Then (31) is equivalent to the random operator equations,

$$\begin{aligned} S(\omega, \hat{x}(\omega)) &= \hat{x}(0,\omega) = \hat{x}(\omega)(0) \\ T(\omega, \hat{y}(\omega)) &= \hat{y}(0,\omega) = \hat{y}(\omega)(0). \end{aligned} \tag{35}$$

Define a sequence $\{\hat{x}_n(\omega)\}$ of measurable functions by

$$\begin{aligned} S(\omega, \hat{x}_{2n}(\omega)) = \hat{x}_{2n+1}(\omega)(0), T(\omega, \hat{x}_{2n+1}(\omega)) = \hat{x}_{n+2}(\omega)(0) \\ \|\hat{x}_n(\omega) - \hat{x}_{n+1}(\omega)\|_{E_0} = \|\hat{x}_n(\omega)(0) - \hat{x}_{n+1}(\omega)(0)\|_E, \end{aligned} \tag{36}$$

for $n = 1, 2 \cdots$. Now, we shall show that the operators S and T satisfy the condition (a) of Theorem **3.4** on $\Omega \times \hat{E}$. First, we show that the operators S and T are random operators on $\Omega \times \hat{E}$. Since hypothesis (H_1) holds, by Caratheodory theorem, the function $\omega \to f(t, x, \omega)$ is measurable for all $t \in I$ and $x \in \varrho$. As integral is the limit of the finite sum of measurable functions, the map

$$\omega \longmapsto \int_0^t f(s, x_s(\omega), \omega)\, ds$$

is measurable. Hence, the operators $S(\omega, \hat{x})$ and $T(\omega, \hat{x})$ are measurable in ω for each $\hat{x} \in \hat{E}$. Thus, we have the operators S and T are random operators on $\hat{E}$ into E. Secondly, we show that the random operators S and T are continuous on $\hat{E}$. Let $\omega \in \Omega$ be fixed. We show that the continuity of the random operators S and T in the following two cases.

Case 1: Let $t \in [0, T]$ and let $\{\hat{x}_n(\omega)\}$ be a sequence of points in $\hat{E}$ such that $\hat{x}_n(\omega) \to \hat{x}(\omega)$ as $n \to \infty$. Then, by dominated convergence theorem, we have

$$\begin{aligned} & \lim_{n\to\infty} S(\omega, \hat{x}_n(t,\omega)) = \lim_{n\to\infty} (\varphi(0,\omega) + \int_0^t f(s, x_s^n(\omega), \omega)\, ds) \\ & = \varphi(0,\omega) + \lim_{n\to\infty} (\int_0^t f(s, x_s^n(\omega), \omega)\, ds) \\ & = \varphi(0,\omega) + (\int_0^t \lim_{n\to\infty} f(s, x_s^n(\omega), \omega)\, ds) \\ & = \varphi(0,\omega) + \int_0^t f(s, x_s(\omega), \omega)\, ds \\ & = S(\omega, \hat{x}(t,\omega)) \end{aligned}$$

for all $t \in [0, T]$ and for each fixed $\omega \in \Omega$.

Case II: Suppose that $t \in [-r, 0]$. Then we have

$$|S(\omega, \hat{x}_n(\omega)) - S(\omega, \hat{x}(\omega))| = |\varphi(t, \omega) - \varphi(t, \omega)| = 0$$

for each fixed $\omega \in \Omega$. Hence,

$$\lim_{n\to\infty} S(\omega), \hat{x}_n(t, \omega) = S(\omega)\,\hat{x}(t, \omega)$$

for all $t \in [-r, 0]$ and $\omega \in \Omega$. From case I and case II, we conclude that $S(\omega)$ is a pointwise continuous random operator on $\hat{E}$ into itself. Similarly, we can prove that $T(\omega)$ is pointwise continuous on $\hat{E}$.

Now, we show that the families of functions $\{S(\omega, \hat{x}_n(\omega))\}$ and $\{T(\omega, \hat{x}_n(\omega))\}$ are uniformly continuous sets in E for a fixed $\omega \in \Omega$. We consider the following three cases.

Case I: Let $\varepsilon > 0$ and $t_1, t_2 \in [0, T]$ be arbitrary. Then, we have

$$\begin{aligned}
&\left|S\left(\omega, x^n_{t_1}(\omega)\right) - S\left(\omega, x^n_{t_2}(\omega)\right)\right| \le \left|\int_0^{t_1} f(s, x^n_s(\omega), \omega)\, ds - \int_0^{t_2} f(s, x^n_s(\omega), \omega)\, ds\right| \\
&\le \left|\int_{t_2}^{t_1} \left|f(s, x^n_s(\omega), \omega)\, ds\right|\right| \\
&\le M|t_1 - t_2|.
\end{aligned}$$

Choose $\tau_1 = \frac{\epsilon}{2(M_f+1)} > 0$. Then, if $|t_1 - t_2| < \tau_1$ implies

$$\left|S\left(\omega, x^n_{t_1}(\omega)\right) - S\left(\omega, x^n_{t_2}(\omega)\right)\right| < \frac{M_f \epsilon}{2(M_f+1)}$$

uniformly for $x^n_t = \hat{x}_n \in E_0$.

Case II: Let $t_1, t_2 \in [-r, 0]$ be arbitrary. Since $t \longmapsto \varphi(\omega, t)$, is continuous on a compact interval $[-r, 0]$, it is uniformly continuous there. Hence, for above $\epsilon > 0$ there exists a $\tau_2 > 0$ such that $|t_1 - t_2| < \tau_1$ implies

$$\begin{aligned}
&\left|S\left(\omega, x^n_{t_1}(\omega)\right) - S\left(\omega, x^n_{t_2}(\omega)\right)\right| = \left|\varphi(t_1, \omega) - \varphi(t_2, \omega)\right| \\
&\le \frac{\epsilon}{2(M_f+1)}
\end{aligned}$$

uniformly for $\hat{x}_n \in E_0$.

Case III: Let $t_1 \in [-r, 0]$ and $t_2 \in [0, T]$ be arbitrary. Choose $\tau = \min\{\tau_1, \tau_2\}$. Then, $|t_1 - t_2| < \tau$ implies

$$\begin{aligned}
&\left|S\left(\omega, x^n_{t_1}(\omega)\right) - S\left(\omega, x^n_{t_2}(\omega)\right)\right| \\
&\le \left|S\left(\omega, x^n_{t_1}(\omega)\right) - S\left(\omega, x^n_{0}(\omega)\right)\right| + \left|S\left(\omega, x^n_{0}(\omega)\right) - S\left(\omega, x^n_{t_2}(\omega)\right)\right| \\
&< \frac{M_f \epsilon}{2(M_f+1)} + \frac{\epsilon}{2(M_f+1)} = \epsilon
\end{aligned}$$

uniformly for $\hat{x}_n \in E_0$. Thus, in all three cases, $|t_1 - t_2| < \tau$ implies

$$\left|S\left(\omega, x^n_{t_1}(\omega)\right) - S\left(\omega, x^n_{t_2}(\omega)\right)\right| < \epsilon$$

uniformly for all $t_1, t_2 \in J$ and $\hat{x}_n \in E_0$. This shows that $\{S(\omega, \hat{x}_n(\omega))\}$ is a sequence of uniformly continuous functions on J. Hence, it converges uniformly on J. Hence, $S(\omega, \hat{x})$ is a continuous random operator on $\hat{E}$ for a fixed $\omega \in \Omega$. Similarly, we can prove that $T(\omega, \hat{x})$ is a continuous random operator on $\hat{E}$ for a fixed $\omega \in \Omega$. Now, we show that S and T are random contractions on $\hat{E}$. Let $\omega \in \Omega$ be fixed. Then, we have

$$\begin{aligned}
&\left\|S(\omega, \hat{x}(\omega)) - S(\omega, \hat{y}(\omega))\right\|_E = \left\|S(\omega, x_t(\omega)) - S(\omega, y_t(\omega))\right\|_E \\
&= \sup_{t\in I}\left|\int_0^t f(s, x_s(\omega), \omega)\, ds - \int_0^t f(s, y_s(\omega), \omega)\, ds\right| \\
&\le \int_0^T L\left\|x_s(\omega) - y_s(\omega)\right\|_{\varrho} ds \\
&\le \int_0^T L\left\|\hat{x}(\omega) - \hat{y}(\omega)\right\|_{\hat{E}} ds \\
&\le LT\left\|\hat{x}(\omega) - \hat{y}(\omega)\right\|_{\hat{E}}
\end{aligned}$$

for all $\hat{x}(\omega), \hat{y}(\omega) \in \hat{E}$. Hence, S is a random contraction on $\hat{E}$ with contraction constant $\alpha = LT < 1$. Similarly, we can prove that T is a random contraction. Thus, the condition (a) of Theorem **3.4** is satisfied. Hence, an application of Theorem **3.4** (a) yields that the functional random integral equation (35) has a random solution with PPF dependence defined on J which implies that the FRDE (30) has a PPF dependent random solution ζ^* defined on J and the sequence $\{\zeta_n(\omega)\}$ of measurable functions constructed as in (36) converges to ζ^*. Moreover, here the Razumikhin class $\Re_0$, $0 \in [-r, T]$ is $C([0,T], \mathbb{R})$ which is topologically and algebraically closed with respect to difference, so by Theorem **3.4** (c), ζ^* is unique random solution with PPF dependence for the FRDE (30) defined on J. □

References

[1] R. P. Agarwal, N. Hussain, M. A. Taoudi, Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations, *Abstr. Appl. Anal.*, Art. ID 245872, (2012) 1–15.

[2] R. P. Agarwal, P. Kumam, W. Sintunavarat, PPF dependent fixed point theorems for an α_c-admissible non-self mapping in the Razumikhin class, *Fixed Point Theory and Appl.* 280 (2013) 1-14:

[3] I. Beg, M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, *Fixed Point Theory and Appl.*, Art. ID 74503 (2006) 1–7.

[4] S. R. Bernfield, V. Lakshmikatham, Y. M. Reddy, Fixed point theorems of operators with PPF dependence in Banach spaces, *Applicable Anal.* 6 (4) (1977) 271–280.

[5] A. T. Bharucha-Reid, Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc.* 82 (5) (1976) 641–657.

[6] B. C. Dhage, Fixed point theorems with PPF dependence and functional differential equations, *Fixed Point Theory*, 13 (2) (2012) 439–452.

[7] B. C. Dhage, On some common fixed point theorems with PPF dependence in Banach spaces, *J. Nonlinear Sci. Appl.* 5 (3) (2012) 220–232.

[8] B. C. Dhage, Some basic random fixed point theorems with PPF dependence and functional random differential equations, *Differ. Equ. Appl.* 4 (2) (2012) 181–195.

[9] Z. Drici, F. A. Mcrae, J. Vasundhara Devi, Fixed point theorems for mixed monotone operators with PPF dependence, *Nonlinear Anal.* 69 (2) (2008) 632–636.

[10] Z. Drici, F. A. Mcrae, J. Vasundhara Devi, Fixed point theorems in partially ordered metric spaces for operators with PPF dependence, *Nonlinear Anal.* 67 (2) (2007) 641–647.

[11] P. Hans, Random fixed point theorems, *Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30,* (1957) 105–125.

[12] N. Hussain, A. R. Khan, R. P. Agarwal, Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces, *J. Nonlinear Convex Anal.* 11 (3) (2010) 475–489.

[13] N. Hussain, M. A. Taoudi, Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations, *Fixed Point Theory Appl.* 2013:196 (2013) 1–16.

[14] N. Hussain, Common fixed points in best approximation for Banach operator pairs with Ćirić type *I*-contractions, *J. Math. Anal. Appl.* 338 (2) (2008) 1351–1363.

[15] N. Hussain, S. Khaleghizadeh, P. Salimi, F. Akbar, New fixed point results with PPF dependence in Banach spaces endowed with a graph, *Abstr. Appl. Anal.*, Art. ID 827205 (2013) 1–9.

[16] N. Hussain, A. Latif, N. Shafqat, Weak contractive inequalities and compatible mixed monotone random operators in ordered metric spaces, *J. Inequal. Appl.* 2012:257 (2012) 1–20.

[17] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.* 9 (4) (1986) 771–779.

[18] G. Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.* 103 (1988) 977–983.

[19] G. Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces, *Far East J. Math. Sci.* 4 (2) (1996) 199–215.

[20] G. Jungck, N. Hussain, Compatible maps and invariant approximations, *J. Math. Modal Algorithms* 325 (2007) 1003–1012.

[21] A. Kaewcharoen, PPF dependent common fixed point theorems for mappings in Banach spaces, *J. Inequal. Appl.* 287 (2013) 1–14.

[22] R. Kannan, Some results on fixed points II, *Amer. Math. Monthly* 76 (4) (1969) 405–408.

[23] A. R. Khan, N. Hussain, Random coincidence point theorem in Frechet spaces with applications, *Stochastic Anal. Appl.* 22 (1) (2004) 155–167.

[24] M. A. Kutbi, N. Hussain, S. Khaleghizadeh, New *PPF* dependent fixed point theorems for Suzuki type *GF*-contractions, J. Function Spaces, Vol. 2015, Article ID 136306, 13 pp.

[25] J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 2005, **22**: 223–239.

[26] R. P. Pant, Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.* 188 (1994) 436–440.

[27] B. E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, 226 (1977), 257–290.

[28] A. Spacek, Zufallige Gleichungen, *Czechoslaviak Math. J.*, 5 (1955), 462–466.