Filomat 31:3 (2017), 831–840
DOI 10.2298/FIL1703831W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Variety of Soliton Solutions for the Boussinesq-Burgers Equation and the Higher-Order Boussinesq-Burgers Equation

Abdul-Majid Wazwaz[a]

[a]*Department of Mathematics, Saint Xavier University, Chicago, IL 60655 USA*

Abstract. In this work we examine the Boussinesq-Burgers equation and the higher-order Boussinesq-Burgers equation. The simplified Hirota's method is used to derive multiple soliton solutions for each method. More soliton and periodic solutions are derived as well.

1. Introduction

Nonlinear evolution equations have been used as the models to describe a variety of science and engineering phenomena, such as fluid dynamics, plasma waves, chemical reactions, optical fibers, mathematical biology, and chemical kinetics and others [1–7, 10, 11]. The study of nonlinear evolution equations is a significant and interesting topic in solitary waves theory. A vast variety of systematic approaches have been established to obtain exact solutions of these equations, such as the inverse scattering method, truncated Painlevé expansion, Bäcklund transformation, Darboux transformation, Lie symmetries method [8–10, 12], and so on. The generalized symmetry method, Painlevé analysis, the inverse scattering method, the Bäcklund transformation method, the conservation law method, and the Hirota bilinear method are mostly used. The Hirota's bilinear method, and then modified the simplified form [13–17]), are rather heuristic and possesses significant features that make it ideal for the determination of multiple soliton solutions for a wide class of nonlinear evolution equations.

In this work, we will examine the Boussinesq-Burgers equation [2, 5, 12] given in the form

$$\begin{aligned} &u_t - \tfrac{1}{2}v_x + 2uu_x = 0, \\ &v_t - \tfrac{1}{2}u_{xxx} + 2(uv)_x = 0, \;\; 0 \le x \le 1, \end{aligned} \tag{1}$$

which is a nonlinear long-wave equation, where x and t represent the normalized space and time, respectively. The functions $u(x, t)$ and $v(x, t)$ represent the horizontal velocity field and the height of the water surface above a horizontal level at the bottom. The Boussinesq–Burgers equation (1) arises in the study of fluid flow and describe the propagation of shallow water waves. A good understanding of its solutions is very helpful for coastal and civil engineers to apply the nonlinear water wave model to harbor and coastal designs [5, 6, 12].

2010 *Mathematics Subject Classification*. Primary 35B35; Secondary 35Q53, 76B05
Keywords. Boussinesq-Burgers equation; simplified Hirota's method; multiple soliton solutions.
Received: 10 January 2015; Accepted: 12 March 2016
Communicated by Mića Stanković
Email address: wazwaz@sxu.edu (Abdul-Majid Wazwaz)

The Boussinesq-Burgers equation [5–7, 12] has a Lax pair, the spectral parameter

$$\phi_x = U\phi,\ U = \begin{pmatrix} \lambda + u & u_x + v \\ 1 & -\lambda - u \end{pmatrix} \tag{2}$$

and the auxiliary function

$$\phi_t = V\phi, \tag{3}$$

with

$$V = \begin{pmatrix} \lambda^2 - \frac{1}{2}u_x - u^2 & \lambda(u_x + v) + \frac{1}{2}u_{xx} - uu_x - uv + \frac{1}{2}v_x \\ \lambda - u & -\lambda^2 + \frac{1}{2}u_x + u^2 \end{pmatrix}, \tag{4}$$

where $u(x,t)$ and $v(x,t)$ are two potentials, and k is a constant spectral parameter [2–5].

The Boussinesq–Burgers equation was studied in the literature by using different methods and sometimes called the Kaup–Boussinesq equation. Darboux transformation was used by Chen and Li [2] to obtain a variety of soliton solutions. The Lie symmetry method is utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. Kaup [7] discussed the integrability for Eq.(1). Matveev et al. [10] obtained the multi-phase periodic solutions for this equation. Kamchatnov et al [6] obtained numerical solutions for by the use of the quasi–classical quantization method.

We point out that a higher-order Boussinesq–Burgers equation was introduced by Jin–Ming and Yao–Ming [5] and given by

$$\begin{aligned} &u_t - 3\sigma u^2 u_x + \tfrac{3}{2}\sigma(uv)_x - \tfrac{1}{4}\sigma u_{xxx} = 0, \\ &v_t + \tfrac{3}{2}\sigma v v_x - 3\sigma(u^2 v)_x + 3\sigma u_x u_{xx} + \tfrac{3}{2}\sigma u u_{xxx} - \tfrac{1}{4}\sigma v_{xxx} = 0, \end{aligned} \tag{5}$$

where σ is a non-zero arbitrary constant.

Jin–Ming and Yao–Ming [5] applied the simplified Hirota's method to derive multiple kink solutions, where they used this derivation to claim that this higher-order Boussinesq–Burgers equation is integrable although other justifications, such as Lax pair, were not given to confirm this result.

Our aim from this work is two fold. The first goal is to employ the simplified Hirota's method to derive multiple soliton solutions and other solitonic and periodic solutions for the Boussinesq–Burgers equation (1). We aim second to study the higher-order Boussinesq–Burgers equation (5) to obtain more solitonic and periodic solutions in addition to the results obtained in [5]. This difference exhibits soliton solutions for some equations and anti-soliton solutions for others. The new solutions are very helpful for coastal and civil engineers to apply the nonlinear water wave model to harbor and coastal designs.

2. The Boussinesq–Burgers Equation

In this work, we will first study the Boussinesq-Burgers equation given in the form

$$\begin{aligned} &u_t - \tfrac{1}{2}v_x + 2uu_x = 0, \\ &v_t - \tfrac{1}{2}u_{xxx} + 2(uv)_x = 0,\ 0 \le x \le 1, \end{aligned} \tag{6}$$

which is a nonlinear long-wave equation.

To determine the dispersion relation, we use the first part of (6) to obtain

$$v(x,t) = \partial_x^{-1}(2u_t) + 2u^2, \tag{7}$$

where the operator ∂_x^{-1} is the inverse operator of ∂_x, ∂_x^{-1} is a partial integration operator, where $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$. Consequently, the second part of (6) becomes

$$\left(\partial_x^{-1}(2u_t) + 2u^2\right)_t + 2u\left(2u_t + 2(u^2)_x\right) + 2u_x\left(\partial_x^{-1}(2u_t) + 2u^2\right) - \frac{1}{2}u_{xxx} = 0. \tag{8}$$

To get rid of the inverse operator of ∂_x^{-1}, we use the potential

$$u(x,t) = w_x(x,t), \tag{9}$$

that will carry out (8) to the single equation

$$\left(2w_t + 2w_x^2\right)_t + 2w_x\left(2w_{xt} + 2(w_x^2)_x\right) + 2w_{xx}\left(2w_t + 2w_x^2\right) - \frac{1}{2}w_{xxxx} = 0. \tag{10}$$

Unlike the Boussinesq equation which is characterized by quadratic nonlinearity, the fourth-order Eq. (10) contains cubic nonlinearity.

2.1. Multiple soliton solutions

To determine the dispersion relation for Eq. (10), we substitute

$$w(x,t) = e^{k_i x - c_i t}, \tag{11}$$

into the linear terms of (10) to find that the dispersion relation is given by

$$c_i = -\frac{1}{2}\,k_i^2. \tag{12}$$

Consequently, we set the dispersion variable by

$$\theta_i = k_i x + \frac{1}{2}\,k_i^2\,t. \tag{13}$$

Notice that we used Eq. (10) to determine the dispersion relation. However, to obtain the multiple soliton solutions we prefer to use the Boussinesq–Burgers equation (6). We first use the transformation

$$u(x,t) = -\frac{1}{2}(\ln f(x,t))_x, \tag{14}$$

where the auxiliary function $f(x,t)$ for the single soliton solution is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + \frac{1}{2}k_1^2 t}. \tag{15}$$

Substituting (14) into (7) gives

$$v(x,t) = u_x(x,t). \tag{16}$$

Combining (14) and (16) gives

$$\begin{aligned} u(x,t) &= -\tfrac{1}{2}(\ln f(x,t))_x, \\ v(x,t) &= -\tfrac{1}{2}(\ln f(x,t))_{xx}. \end{aligned} \tag{17}$$

Using (15) and (17) gives the single kink solution for the velocity averaged over depth $u(x,t)$ by

$$u(x,t) = -\frac{k_1\,e^{k_1 x + \frac{1}{2}k_1^2\,t}}{2(1 + e^{k_1 x + \frac{1}{2}k_1^2\,t})}, \tag{18}$$

and the single soliton solution the height of the water surface above a horizontal bottom $v(x,t)$ by

$$v(x,t) = -\frac{k_1^2\,e^{k_1 x + \frac{1}{2}k_1^2\,t}}{2(1 + e^{k_1 x + \frac{1}{2}k_1^2\,t})^2}. \tag{19}$$

For the two kink solutions and the two soliton solutions we use the auxiliary function

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2}, \\ &= 1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t}. \end{aligned} \tag{20}$$

Substituting (20) into (17), the two kink solutions are given by

$$u(x,t) = -\frac{k_1\, e^{k_1 x + \frac{1}{2} k_1^2 t} + k_2\, e^{k_2 x + \frac{1}{2} k_2^2 t}}{2(1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t})}, \tag{21}$$

and the two soliton solutions for the height of the water surface above a horizontal bottom $v(x,t)$ by

$$v(x,t) = -\frac{k_1^2 e^{k_1 x + \frac{1}{2} k_1^2 t} + k_2^2 e^{k_2 x + \frac{1}{2} k_2^2 t}}{2(1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t})} + \frac{\left(k_1 e^{k_1 x + \frac{1}{2} k_1^2 t} + k_2 e^{k_2 x + \frac{1}{2} k_2^2 t}\right)^2}{2(1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t})^2}. \tag{22}$$

For the three soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \\ &= 1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t} + e^{k_3 x + \frac{1}{2} k_2^2 t}. \end{aligned} \tag{23}$$

Proceeding as before, the three kink solutions are given by

$$u(x,t) = -\frac{k_1\, e^{k_1 x + \frac{1}{2} k_1^2 t} + k_2\, e^{k_2 x + \frac{1}{2} k_2^2 t} + k_3\, e^{k_3 x + \frac{1}{2} k_3^2 t}}{2(1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t} + e^{k_3 x + \frac{1}{2} k_3^2 t})}, \tag{24}$$

and the three soliton solutions

$$\begin{aligned} v(x,t) &= -\frac{k_1^2 e^{k_1 x - \frac{1}{2} k_1^2 t} + k_2^2 e^{k_2 x - \frac{1}{2} k_2^2 t} + k_3^2 e^{k_3 x - \frac{1}{2} k_3^2 t}}{2\left(1 + e^{k_1 x - \frac{1}{2} k_1^2 t} + e^{k_2 x - \frac{1}{2} k_2^2 t} + e^{k_3 x - \frac{1}{2} k_3^2 t}\right)} \\ &+ \frac{\left(k_1 e^{k_1 x + \frac{1}{2} k_1^2 t} + k_2 e^{k_2 x + \frac{1}{2} k_2^2 t} + k_3 e^{k_3 x + \frac{1}{2} k_3^2 t}\right)^2}{2\left(1 + e^{k_1 x + \frac{1}{2} k_1^2 t} + e^{k_2 x + \frac{1}{2} k_2^2 t} + e^{k_3 x + \frac{1}{2} k_3^2 t}\right)^2}. \end{aligned} \tag{25}$$

2.2. *The general solutions*

From the analysis presented above, it is obvious that the generalized Boussinesq–Burgers equation gives multiple kink and soliton solutions, a significant feature of integrable equations. The general dispersion relations are given by

$$c_i = -\frac{1}{2} k_i^2. \tag{26}$$

Moreover, the generalized soliton solutions are given by

$$u(x,t) = -\frac{\sum_{i=1}^{N} k_i\, e^{k_i x + \frac{1}{2} k_i^2 t}}{2(1 + \sum_{i=1}^{N} e^{k_i x + \frac{1}{2} k_i^2 t})}, \tag{27}$$

and

$$v(x,t) = -\frac{\sum_{i=1}^{N} k_i^2\, e^{k_i x + \frac{1}{2} k_i^2 t}}{2(1 + \sum_{i=1}^{N} e^{k_i x - \frac{1}{2} k_i^2 t})} + \frac{\left(\sum_{i=1}^{N} k_i\, e^{k_i x - \frac{1}{2} k_i^2 t}\right)^2}{2\left(1 + \sum_{i=1}^{N} e^{k_i x - \frac{1}{2} k_i^2 t}\right)^2}. \tag{28}$$

2.3. Hyperbolic solutions

To determine hyperbolic solutions for the Boussinesq–Burgers equation (6), we first assume that the solutions take the form

$$\begin{aligned} u(x,t) &= a_0 + a_1 \tanh(kx - ct), \\ v(x,t) &= b_0 + b_1 \tanh^2(kx - ct), \end{aligned} \tag{29}$$

Substituting (29) into (6) we find

$$\begin{aligned} &(-2a_1^2 k + b_1 k)\tanh^3(kx - ct) + (a_1 c - 2a_1 k a_0)\tanh^2(kx - ct) \\ &+(2a_1^2 k - b_1 k)\tanh(kx - ct) - a_1 c + 2a_1 k a_0 = 0, \\ &(-6a_1 k b_1 + 3a_1 k^3)\tanh^4(kx - ct) + (2b_1 c - 4b_1 k a_0)\tanh^3(kx - ct) \\ &+(-4a_1 k^3 + 6a_1 k b_1 - 2a_1 k b_0)\tanh^2(kx - ct) + (-2b_1 c + 4b_1 k a_0)\tanh(kx - ct) \\ &+a_1 k^3 + 2a_1 k b b_0 = 0. \end{aligned} \tag{30}$$

Equating the coefficients of $\tanh^i(kx - ct), 1 \le i \le 4$ to zero, and solving the resulting equations we find two sets of solutions given by

$$\begin{aligned} a_0 &= \tfrac{c}{2k}, \\ a_1 &= -\tfrac{k}{2}, \\ b_0 &= -\tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} a_0 &= \tfrac{c}{2k}, \\ a_1 &= \tfrac{k}{2}, \\ b_0 &= -\tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{32}$$

The first set gives the solitonic solutions

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} - \tfrac{k}{2}\tanh(kx - ct), \\ v(x,t) &= -\tfrac{k^2}{2}\operatorname{sech}^2(kx - ct), \end{aligned} \tag{33}$$

where k and c are left as free parameters. However, the second set gives the solitonic

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} + \tfrac{k}{2}\tanh(kx - ct), \\ v(x,t) &= -\tfrac{k^2}{2}\operatorname{sech}^2(kx - ct), \end{aligned} \tag{34}$$

In a like manner, we may assume more hyperbolic solutions of the form

$$\begin{aligned} u(x,t) &= a_0 + a_1 \coth(kx - ct), \\ v(x,t) &= b_0 + b_1 \coth^2(kx - ct), \end{aligned} \tag{35}$$

Proceeding as before gives the same two sets of coefficients, hence we find singular hyperbolic solutions given by

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} + \tfrac{k}{2}\coth(kx - ct), \\ v(x,t) &= \tfrac{k^2}{2}\operatorname{csch}^2(kx - ct), \end{aligned} \tag{36}$$

with k and c are left as free parameters.

2.4. Periodic solutions

To determine periodic solutions for the Boussinesq–Burgers equation (6), we first assume that the solutions take the form

$$\begin{aligned} u(x,t) &= a_0 + a_1 \tan(kx - ct), \\ v(x,t) &= b_0 + b_1 \tan^2(kx - ct), \end{aligned} \tag{37}$$

Substituting (37) into (6) and proceeding as before we find two sets of solutions given by

$$\begin{aligned} a_0 &= \tfrac{c}{2k}, \\ a_1 &= -\tfrac{k}{2}, \\ b_0 &= \tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{38}$$

and

$$\begin{aligned} a_0 &= \tfrac{c}{2k}, \\ a_1 &= \tfrac{k}{2}, \\ b_0 &= \tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{39}$$

The first set gives the periodic solutions

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} - \tfrac{k}{2} \tan(kx - ct), \\ v(x,t) &= \tfrac{k^2}{2} \sec^2(kx - ct), \end{aligned} \tag{40}$$

where k and c are left as free parameters. However, the second set gives the solitonic

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} + \tfrac{k}{2} \tan(kx - ct), \\ v(x,t) &= -\tfrac{k^2}{2} \sec^2(kx - ct), \end{aligned} \tag{41}$$

In a like manner, we may assume more singular solutions of the form

$$\begin{aligned} u(x,t) &= a_0 + a_1 \cot(kx - ct), \\ v(x,t) &= b_0 + b_1 \cot^2(kx - ct), \end{aligned} \tag{42}$$

Proceeding as before gives the same two sets of coefficients, hence we find singular trigonometric solutions given by

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} - \tfrac{k}{2} \cot(kx - ct), \\ v(x,t) &= \tfrac{k^2}{2} \csc^2(kx - ct), \end{aligned} \tag{43}$$

and

$$\begin{aligned} u(x,t) &= \tfrac{c}{2k} + \tfrac{k}{2} \cot(kx - ct), \\ v(x,t) &= \tfrac{k^2}{2} \csc^2(kx - ct), \end{aligned} \tag{44}$$

with k and c are left as free parameters.

3. The Higher-Order Boussinesq–Burgers Equation

A higher-order Boussinesq–Burgers equation was introduced by Jin–Ming and Yao–Ming (2011) in the form

$$\begin{aligned} &u_t - 3\sigma u^2 u_x + \tfrac{3}{2}\sigma(uv)_x - \tfrac{1}{4}\sigma u_{xxx} = 0,\\ &v_t + \tfrac{3}{2}\sigma v v_x - 3\sigma(u^2 v)_x + 3\sigma u_x u_{xx} + \tfrac{3}{2}\sigma u u_{xxx} - \tfrac{1}{4}\sigma v_{xxx} = 0, \end{aligned} \tag{45}$$

where σ is a non-zero arbitrary constant. As stated before, the simplified Hirota's method was applied to derive multiple kink solutions for this equation. Jin–Ming and Yao–Ming (2011) used the transformations

$$\begin{aligned} u(x,t) &= \pm\tfrac{1}{2}(\ln(f))_x,\\ v(x,t) &= -\tfrac{1}{2}(\ln(f))_{xx}, \end{aligned} \tag{46}$$

where the auxiliary function $f(x,t)$ reads

$$f(x,t) = 1 + e^{k_1 x + \frac{\sigma k_1^3}{4}t}, \tag{47}$$

where the dispersion relation was derived as

$$\omega_i = -\frac{\sigma k_i^3}{4}, i = 1,2,3. \tag{48}$$

Based on this and using the simplified Hirota's method, multiple soliton solutions were obtained.

We aim in this work to develop new hyperbolic and periodic solutions to the higher-order Boussinesq–Burgers equation (45). For simplicity we set $\sigma = 1$.

3.1. Hyperbolic solutions

To determine hyperbolic solutions for the Boussinesq–Burgers equation (45), we first assume that the solutions take the form

$$\begin{aligned} u(x,t) &= a_0 + a_1\tanh(kx - ct),\\ v(x,t) &= b_0 + b_1\tanh^2(kx - ct), \end{aligned} \tag{49}$$

Substituting (49) into (45) we find

$$\begin{aligned} &(3a_1^3 k - \tfrac{9}{2}a_1 k b_1 + \tfrac{3}{2}a_1 k^3)\tanh^4(kx-ct) + (6a_1^2 k a_0 - 3b_1 k a_0)\tanh^3(kx-ct)\\ &+(3a_1 k a_0^2 + \tfrac{9}{2}a_1 k b_1 - 3a_1^3 k - 2a_1 k^3 - \tfrac{3}{2}a_1 k b_0 + a_1 c)\tanh^2(kx-ct)\\ &(-6a_1^2 k a_0 + 3b_1 k a_0)\tanh(kx-ct) - a_1 c + \tfrac{1}{2}a_1 k^3 - 3a_1 k a_0^2 + \tfrac{3}{2}a_1 k b_0 = 0,\\ &(6b_1 k^3 - 15a_1^2 k^3 + 12a_1^2 k b_1 - 3b_1^2 k)\tanh^5(kx-ct)\\ &+(-9a_0 a_1 k^3 + 18a_1 k a_0 b_1)\tanh^4(kx-ct)\\ &+(6b_1 k a_0^2 - 12a_1^2 k b_1 - 10b_1 k^3 + 24a_1^2 * k^3 + 6a_1^2 k b_0 + 2b_1 c + 3b_1^2 k - 3b_1 k b_0)\\ &\times\tanh^3(kx-ct)\\ &+(6a_1 k a_0 b_0 - 18a_1 k a_0 b_1 + 12a_0 a_1 k^3)\tanh^2(kx-ct)\\ &+(-6a_1^2 k b_0 + 4b_1 k^3 - 9a_1^2 k^3 + 3b_1 k b_0 - 2b_1 c - 6b_1 k a_0^2)\tanh(kx-ct)\\ &-6a_1 k a_0 b_0 - 3a_0 a_1 k^3. \end{aligned} \tag{50}$$

Equating the coefficients of $\tanh^i(kx - ct), 1 \le i \le 5$ to zero, and solving the resulting equations we find two sets of solutions given by

$$\begin{aligned} a_0 &= \pm\sqrt{\tfrac{-(4c+k^3)}{12k}}, \tfrac{(4c+k^3)}{k} < 0\\ a_1 &= -\tfrac{k}{2},\\ b_0 &= -\tfrac{k^2}{2},\\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} a_0 &= \pm\sqrt{\tfrac{-(4c+k^3)}{12k}}, \tfrac{(4c+k^3)}{k} < 0, \\ a_1 &= \tfrac{k}{2}, \\ b_0 &= -\tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{52}$$

The first set gives the solitonic solutions

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c+k^3)}{12k}}, \tfrac{(4c+k^3)}{k} < 0 \\ &-\tfrac{k}{2}\tanh(kx-ct), \\ v(x,t) &= -\tfrac{k^2}{2}\operatorname{sech}^2(kx-ct), \end{aligned} \tag{53}$$

where k and c are left as free parameters. However, the second set gives the solitonic

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c+k^3)}{12k}}, \tfrac{(4c+k^3)}{k} < 0 \\ &+\tfrac{k}{2}\tanh(kx-ct), \\ v(x,t) &= -\tfrac{k^2}{2}\operatorname{sech}^2(kx-ct), \end{aligned} \tag{54}$$

In a like manner, we may assume more hyperbolic solutions of the form

$$\begin{aligned} u(x,t) &= a_0 + a_1\coth(kx-ct), \\ v(x,t) &= b_0 + b_1\coth^2(kx-ct), \end{aligned} \tag{55}$$

Proceeding as before gives the same two sets of coefficients, hence we find singular hyperbolic solutions given by

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c+k^3)}{12k}} + \tfrac{k}{2}\coth(kx-ct), \\ v(x,t) &= \tfrac{k^2}{2}\operatorname{csch}^2(kx-ct), \end{aligned} \tag{56}$$

with k and c are left as free parameters.

3.2. *Periodic solutions*

To determine periodic solutions for the Boussinesq–Burgers equation (45), we first assume that the solutions take the form

$$\begin{aligned} u(x,t) &= a_0 + a_1\tan(kx-ct), \\ v(x,t) &= b_0 + b_1\tan^2(kx-ct), \end{aligned} \tag{57}$$

Substituting (57) into (45) and proceeding as before we find two sets of solutions given by

$$\begin{aligned} a_0 &= \pm\sqrt{\tfrac{-(4c-k^3)}{12k}}, \tfrac{(4c-k^3)}{k} < 0, \\ a_1 &= -\tfrac{k}{2}, \\ b_0 &= \tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{58}$$

and

$$\begin{aligned} a_0 &= \tfrac{c}{2k}, \\ a_1 &= \tfrac{k}{2}, \\ b_0 &= \tfrac{k^2}{2}, \\ b_1 &= \tfrac{k^2}{2}, \end{aligned} \tag{59}$$

The first set gives the periodic solutions

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c-k^3)}{12k}} - \tfrac{k}{2}\tan(kx-ct), \\ v(x,t) &= \tfrac{k^2}{2}\sec^2(kx-ct), \end{aligned} \tag{60}$$

where k and c are left as free parameters. However, the second set gives the solitonic

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c-k^3)}{12k}} + \tfrac{k}{2}\tan(kx-ct), \\ v(x,t) &= -\tfrac{k^2}{2}\sec^2(kx-ct), \end{aligned} \tag{61}$$

In a like manner, we may assume more singular solutions of the form

$$\begin{aligned} u(x,t) &= a_0 + a_1\cot(kx-ct), \\ v(x,t) &= b_0 + b_1\cot^2(kx-ct), \end{aligned} \tag{62}$$

Proceeding as before gives the same two sets of coefficients, hence we find singular trigonometric solutions given by

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c-k^3)}{12k}} - \tfrac{k}{2}\cot(kx-ct), \\ v(x,t) &= \tfrac{k^2}{2}\csc^2(kx-ct), \end{aligned} \tag{63}$$

and

$$\begin{aligned} u(x,t) &= \pm\sqrt{\tfrac{-(4c-k^3)}{12k}} + \tfrac{k}{2}\cot(kx-ct), \\ v(x,t) &= \tfrac{k^2}{2}\csc^2(kx-ct), \end{aligned} \tag{64}$$

with k and c are left as free parameters.

4. Discussion

The Boussinesq-Burgers equation and the higher-order Boussinesq-Burgers equation were investigated. We used the simplified Hirota's method and other hyperbolic and trigonometric ansatze to derive multiple soliton solutions, solitons and periodic solutions. We also obtained solitons and periodic solutions for the higher-order Boussinesq-Burgers equation. We showed that these equations, that represent phenomena for coastal and harbor applications, possess a variety of solitonic and periodic solutions.

References

[1] R. Abedian, H. Adibi, M. Dehghan, A high–order symmetrical weighted hybrid ENO–flux limiter scheme for hyperbolic conservation laws, Computer Physics Communications, 185 (2014) 106–127.

[2] A. Chen, X. Li, Darboux transformation and soliton solutions for Boussinesq-Burgers equation, Chaos, Solitons and Fractals 27 (2006) 43-49.

[3] M. Dehghan, A. Ghesmati, A Numerical simulation of two–dimensional sine–Gordon solitons via a local weak meshless technique based on the radial point interpolation method (RPIM), Computer Physics Communications, 181 (2010) 772–786.

[4] M. Dehghan, A. Shokri, A numerical method for solution of the two-dimensional sine-Gordon equation using the radial basis functions, Comput. Math. Simulation 79 (2008) 700-715.

[5] Z. Jin-Ming, Z. Yao-Ming, The Hirota bilinear method for the coupled Burgers equation and the high-order Boussinesq–Burgers equation, Chinese Physics B, 20 (2011) 010205.

[6] A. Kamchatnov, A. Kraenkel, B. Umarov, Asymptotic soliton train solutions of Kaup-Boussinesq equations, Wave Motion 38 (2003) 355–365.

[7] D. Kaup, A higher–order water–wave equation and the method for solving it, Prog. Theor. Phys. 54 (1975) 396–408.

[8] C. M. Khalique, Exact solutions and conservation laws of a coupled integrable dispersionless system, Filomat 26 (2012) 957-964.

[9] S. Kumar, L. Wei, Y. He, Numerical study based on an implicit fully discreate local discontinuous Galerkin method for time fractional KdV- Burgers Kuramoto equation, JAMM Journal of Applied Mathematics and Mechanics 93 (2013) 14–28.
[10] V. Matveev, M. Salle, Darboux Transformation and Solitons Springer, Berlin, (1991).
[11] I. Mhlanga, C. M. Khalique, Exact solutions of generalized Boussinesq-Burgers equations and (2+1)-Dimensional Davey-Stewartson equations, Journal of Applied Mathematics, 2012 (2012) 1–9.
[12] Z. Wang, A. Chen, Explicit solutions of Boussinesq-Burgers equation, Chinese Physics 16 (2007) 1233–1238.
[13] A. M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Higher Education Press and Springe, Beijing and Berlin, (2009).
[14] A. M. Wazwaz, Two integrable extensions of the Kadomtsev-Petviashvili equation, Central European Journal of Physics 91 (2011) 49–56.
[15] A. M. Wazwaz, Solitons and singular solitons for a variety of Boussinesq-like equations, Ocean Engineering 53 (2012) 1-5.
[16] A. M. Wazwaz, Multiple soliton solutions and rational solutions for the (2+1)-dimensional dispersive long water-wave system, Ocean Engineering 60 (2013) 95–98.
[17] A. M. Wazwaz, A variety of (3+1)-dimensional mKdV equations derived by using the mKdV recursion operator, Computers and Fluids, 93 (2014) 41–45.