Improved Kantorovich and Wielandt Operator Inequalities for Positive Linear Maps

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Abstract. This paper improves and generalizes the Kantorovich and Wielandt inequalities for positive linear maps on Hilbert space operators and presents more general and precise results compared to many recent results.

1. Introduction

Throughout this paper, we reserve M, m for real numbers and I for the identity operator. Other capital letters denote general elements of the C^*-algebra $\mathcal{B}(\mathcal{H})$ all bounded linear operators on a complex separable Hilbert space $(\langle \cdot, \cdot \rangle, \mathcal{H})$. $\|\cdot\|$ denotes the operator norm. An operator A is said to be positive (strictly positive) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ ($\langle Ax, x \rangle > 0$ for all $x \in \mathcal{H} \backslash \{0\}$) and write $A \geq 0$ ($A > 0$). $A \geq B$ ($A > B$) means $A - B \geq 0$ ($A - B > 0$). The absolute value of A is denoted by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$.

A linear map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. We say that a linear map Φ between C^*-algebras is 2-positive if whenever the 2×2 operator matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then so is $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(C) \end{bmatrix} \geq 0$.

In 1948, Kantorovich [7] introduced the well-known Kantorovich inequality. In 1990, an operator Kantorovich inequality was established by Marshall and Olkin [10]. Recently, Lin [9] proved that the operator Kantorovich inequality is order preserving under squaring. This result was further generalized by several authors (see [4, 11]), who obtained

Theorem 1.1. *[4, 11] Let $0 < m \leq A \leq M$. Then for every positive unital linear map Φ,*

$$\Phi^p(A^{-1}) \leq \frac{(m + M)^{2p}}{16 m^p M^p} \Phi(A)^{-p}, \quad p \geq 2 \tag{1}$$

and

$$\Phi^p(A^{-1}) \leq \frac{(m^2 + M^2)^p}{16 m^p M^p} \Phi(A)^{-p}, \quad p \geq 4. \tag{2}$$

When $p \geq 4$, the inequality (2) is tighter than (1). There exsits a vacancy unconsidered for $2 \leq p \leq 4$. Motivated by this, we intend to obtain some better results.

In view of positive linear map Φ, Lin [9] proved that

$$\left|\Phi(A^{-1})\Phi(A) + \Phi(A)\Phi(A^{-1})\right| \leq \frac{(M+m)^2}{2Mm} \tag{3}$$

and

$$\Phi(A^{-1})\Phi(A) + \Phi(A)\Phi(A^{-1}) \leq \frac{(M+m)^2}{2Mm}. \tag{4}$$

Fu [5] generalized the inequalities (3) and (4) to p-th power. We consider an improvement of Fu's result. Finally, we study a conjecture on Wielandt type operator inequalities and obtain some refined results.

2. Kantorovich-Type Inequalities

Firstly, we are devoted to obtain a better bound than (1) and (2). To this end, we need two important lemmas.

Lemma 2.1. *[3, Lemma 2.1] Let $A, B \geq 0$. Then the following inequality holds:*

$$\|AB\| \leq \frac{1}{4}\|A + B\|^2. \tag{5}$$

Lemma 2.2. *[1, p. 28] Let $A, B \geq 0$. Then for $1 \leq r < +\infty$,*

$$\|A^r + B^r\| \leq \|(A + B)^r\|. \tag{6}$$

We know that $\|A\| \leq 1$ is equivalent to $A^*A \leq I$. Using this fact we have the following result:

Theorem 2.3. *Let $0 < m \leq A \leq M$. Then for every positive unital linear map Φ, $1 \leq \alpha \leq 2$ and $p \geq 2\alpha$,*

$$\Phi^p(A^{-1}) \leq \frac{(m^\alpha + M^\alpha)^{\frac{2p}{\alpha}}}{16m^p M^p} \Phi(A)^{-p}. \tag{7}$$

Proof. The desired inequality is equivalent to

$$\left\|\Phi^{\frac{p}{2}}(A^{-1})\Phi^{\frac{p}{2}}(A)\right\| \leq \frac{(m^\alpha + M^\alpha)^{\frac{p}{\alpha}}}{4m^{\frac{p}{2}} M^{\frac{p}{2}}}.$$

Compute

$$\left\|m^{\frac{p}{2}} M^{\frac{p}{2}} \Phi^{\frac{p}{2}}(A^{-1})\Phi^{\frac{p}{2}}(A)\right\| \leq \frac{1}{4}\left\|m^{\frac{p}{2}} M^{\frac{p}{2}} \Phi^{\frac{p}{2}}(A^{-1}) + \Phi^{\frac{p}{2}}(A)\right\|^2 \quad \text{(by (5))}$$

$$\leq \frac{1}{4}\left\|\left(m^\alpha M^\alpha \Phi^\alpha(A^{-1}) + \Phi^\alpha(A)\right)^{\frac{p}{2\alpha}}\right\|^2 \quad \text{(by (6))}$$

$$= \frac{1}{4}\left\|m^\alpha M^\alpha \Phi^\alpha(A^{-1}) + \Phi^\alpha(A)\right\|^{\frac{p}{\alpha}}$$

$$\leq \frac{1}{4}(m^\alpha + M^\alpha)^{\frac{p}{\alpha}}.$$

The last inequality above holds as follows: The condition $0 < m \leq A \leq M$ implies that

$$M^\alpha m^\alpha A^{-\alpha} + A^\alpha \leq M^\alpha + m^\alpha, \tag{8}$$

and hence

$$M^\alpha m^\alpha \Phi(A^{-\alpha}) + \Phi(A^\alpha) \leq M^\alpha + m^\alpha, \tag{9}$$

The well-known inequality (see [1, p. 53]) says

$$\Phi^\alpha(T) \leq \Phi(T^\alpha)$$

for every positive unital linear map Φ and $T > 0$. Then it follows from (9) that

$$M^\alpha m^\alpha \Phi^\alpha(A^{-1}) + \Phi^\alpha(A) \leq M^\alpha + m^\alpha.$$

Therefore

$$\left\| \Phi^{\frac{p}{2}}(A^{-1}) \Phi^{\frac{p}{2}}(A) \right\| \leq \frac{(m^\alpha + M^\alpha)^{\frac{p}{\alpha}}}{4 m^{\frac{p}{2}} M^{\frac{p}{2}}}. \tag{10}$$

So the inequality (7) is obtained. $\square$

Remark 2.4. *Inequalities* (1) *and* (2) *are two special cases of* (7) *by taking $\alpha = 1, 2$.*
Putting $\alpha = 2$ and $p = 4$, the inequality (7) *reduces to Lin's result (see [8, Theorem 4.3]).*

We next present the generalizations of (3) and (4). The following lemma is useful in our proof of Theorem 2.6.

Lemma 2.5. *For any bounded operator X,*

$$|X| \leq tI \Leftrightarrow \|X\| \leq t \Leftrightarrow \left[\begin{array}{cc} tI & X \\ X^* & tI \end{array} \right] \geq 0 \ (t \geq 0).$$

Theorem 2.6. *Let A be a positive operator on a Hilbert space $\mathcal{H}$ with $0 < m \leq A \leq M$ and Φ be a positive linear map on $\mathcal{B}(\mathcal{H})$. Then for $1 \leq \alpha \leq 2$ and $p \geq \alpha$,*

$$\left| \Phi^p(A^{-1}) \Phi^p(A) + \Phi^p(A) \Phi^p(A^{-1}) \right| \leq \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{2 M^p m^p} \tag{11}$$

and

$$\Phi^p(A^{-1}) \Phi^p(A) + \Phi^p(A) \Phi^p(A^{-1}) \leq \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{2 M^p m^p}. \tag{12}$$

Proof. By (10) and Lemma 2.5, we deduce

$$\left[\begin{array}{cc} \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{4 M^p m^p} I & \Phi^p(A) \Phi^p(A^{-1}) \\ \Phi^p(A^{-1}) \Phi^p(A) & \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{4 M^p m^p} I \end{array} \right] \geq 0$$

and

$$\left[\begin{array}{cc} \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{4 M^p m^p} I & \Phi^p(A^{-1}) \Phi^p(A) \\ \Phi^p(A) \Phi^p(A^{-1}) & \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{4 M^p m^p} I \end{array} \right] \geq 0.$$

Summing up these two operator matrices, we have

$$\left[\begin{array}{cc} \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{2 M^p m^p} I & \Phi^p(A) \Phi^p(A^{-1}) + \Phi^p(A^{-1}) \Phi^p(A) \\ \Phi^p(A^{-1}) \Phi^p(A) + \Phi^p(A) \Phi^p(A^{-1}) & \frac{(M^\alpha + m^\alpha)^{\frac{2p}{\alpha}}}{2 M^p m^p} I \end{array} \right] \geq 0.$$

By Lemma 2.5 again, we obtain (11).

As $\Phi^p(A) \Phi^p(A^{-1}) + \Phi^p(A^{-1}) \Phi^p(A)$ is self-adjoint, (12) follows from the maximal characterization of geometric mean. $\square$

Remark 2.7. *Taking $\alpha = 1$ and $p = 1$, (11) and (12) collapse to (3) and (4), respectively. Fu showed a special case of Theorem 2.6 for $\alpha = 1$ in [5, Theorem 4].*

When $\alpha = 2$, Theorem 2.6 implies the following.

Corollary 2.8. *Let A be a positive operator on a Hilbert space $\mathcal{H}$ with $0 < m \leq A \leq M$ and Φ be a positive linear map on $\mathcal{B}(\mathcal{H})$. Then for $p \geq 2$,*

$$\left|\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1})\right| \leq \frac{(M^2 + m^2)^p}{2M^p m^p}, \tag{13}$$

and

$$\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1}) \leq \frac{(M^2 + m^2)^p}{2M^p m^p}. \tag{14}$$

Remark 2.9. *When $p \geq 2$, the inequalities (13) and (14) is tighter than that of Fu [5, Theorem 4], respectively.*

3. Wielandt-Type Inequalities

In 2000, Bhatia and Davis [2] proved an operator Wielandt inequality which states that if $0 < m \leq A \leq M$ and X, Y are two partial isometries on $\mathcal{H}$ whose final spaces are orthogonal to each other, then for every 2-positive linear map Φ on $\mathcal{B}(\mathcal{H})$,

$$\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \leq \left(\frac{M-m}{M+m}\right)^2 \Phi(X^*AX).$$

Lin [9, Conjecture 3.4] conjecture that the following assertion could be true:

$$\left\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\right\| \leq \left(\frac{M-m}{M+m}\right)^2. \tag{15}$$

Recently, Fu and He [4] attempt to solve the conjecture and get a step closer to the conjecture. But Gumus [6] obtains a better upper bound to approximate the right side of (15) based on

$$\left\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\right\| \leq \frac{(M-m)^2}{2(M+m)\sqrt{Mm}},$$

which is equivalent to

$$\left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\right)^2 \leq \frac{(M-m)^4}{4(M+m)^2 Mm}\Phi^2(X^*AX). \tag{16}$$

Soon after, Zhang [11] proved two generalized inequalities

$$\left\|\left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\right)^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\right\| \leq \frac{1}{4}\left(\left(\frac{M-m}{M+m}\right)^2 M + \frac{1}{m}\right)^p, \quad p \geq 2 \tag{17}$$

and

$$\left\|\left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\right)^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}\right\| \leq \left(\frac{M-m}{M+m}\right)^p \left(\frac{M}{m}\right)^{\frac{p}{2}}, \quad p \geq 1. \tag{18}$$

Now, Let us give improvements of (17) and (18).

Theorem 3.1. *Let $0 < m \leq A \leq M$, X and Y be two isometries in $\mathcal{H}$ whose final spaces are orthogonal to each other and Φ be a 2-positive linear map on $\mathcal{B}(\mathcal{H})$. Then for $1 \leq \alpha \leq 2$ and $p \geq 2\alpha$,*

$$\left\| \left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \right\| \leq \frac{(M-m)^p(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{2^{2+\frac{p}{2}}M^{\frac{3p}{4}}m^{\frac{3p}{4}}(M+m)^{\frac{p}{2}}}. \tag{19}$$

Proof. By using (16), we have

$$\left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^\alpha \leq \frac{(M-m)^{2\alpha}}{2^\alpha(M+m)^\alpha M^{\frac{\alpha}{2}}m^{\frac{\alpha}{2}}}\Phi^\alpha(X^*AX). \tag{20}$$

Combining (20) with Lemma 2.1 and Lemma 2.2, we get

$$\left\| \frac{(M-m)^p}{2^{\frac{p}{2}}(M+m)^{\frac{p}{2}}}M^{\frac{p}{4}}m^{\frac{p}{4}}\left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \right\|$$

$$\leq \frac{1}{4}\left\| \left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} + \left(\frac{(M-m)^2}{2(M+m)}\sqrt{Mm}\Phi(X^*AX)^{-1} \right)^{\frac{p}{2}} \right\|^2$$

$$\leq \frac{1}{4}\left\| \left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^\alpha + \frac{(M-m)^{2\alpha}}{2^\alpha(M+m)^\alpha}(Mm)^{\frac{\alpha}{2}}\Phi(X^*AX)^{-\alpha} \right\|^{\frac{p}{\alpha}}$$

$$\leq \frac{1}{4}\left\| \frac{(M-m)^{2\alpha}}{2^\alpha(Mm)^{\frac{\alpha}{2}}(M+m)^\alpha}\Phi(X^*AX)^\alpha + \frac{(M-m)^{2\alpha}}{2^\alpha(M+m)^\alpha}(Mm)^{\frac{\alpha}{2}}\Phi(X^*AX)^{-\alpha} \right\|^{\frac{p}{\alpha}}$$

$$= \frac{(M-m)^{2p}}{2^{2+p}M^{\frac{p}{2}}m^{\frac{p}{2}}(M+m)^p}\left\| \Phi(X^*AX)^\alpha + M^\alpha m^\alpha\Phi(X^*AX)^{-\alpha} \right\|^{\frac{p}{\alpha}}$$

$$\leq \frac{(M-m)^{2p}(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{2^{2+p}M^{\frac{p}{2}}m^{\frac{p}{2}}(M+m)^p}.$$

The last inequality follows from (8) and $0 < m \leq X^*AX \leq M$. This proves the inequality (19). □

Putting $\alpha = 1, 2$ in Theorem 3.1, we have

Corollary 3.2. *Under the same conditions as in Theorem 3.1, then*

$$\left\| \left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \right\| \leq \frac{(M-m)^p(M+m)^{\frac{p}{2}}}{2^{2+\frac{p}{2}}M^{\frac{3p}{4}}m^{\frac{3p}{4}}}, \quad p \geq 2 \tag{21}$$

and

$$\left\| \left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \right\| \leq \frac{(M-m)^p(M^2+m^2)^{\frac{p}{2}}}{2^{2+\frac{p}{2}}M^{\frac{3p}{4}}m^{\frac{3p}{4}}(M+m)^{\frac{p}{2}}}, \quad p \geq 4. \tag{22}$$

Remark 3.3. *(22) is better than (21) as $p \geq 4$. When $\frac{M}{m} \leq 57.7341$, a simple computation shows that*

$$\frac{(M-m)^2(M+m)}{8M^{\frac{3}{2}}m^{\frac{3}{2}}} \leq \left(\frac{M-m}{M+m} \right)^2\frac{M}{m} \leq \frac{1}{4}\left(\left(\frac{M-m}{M+m} \right)^2 M + \frac{1}{m} \right)^2,$$

which points out that the right side of (21) is a better bound than that of (17) for $p \geq 2$ and that of (18) for $p = 2$, respectively.

Following from (19) and the line of the prove of Theorem 2.6, we have

Theorem 3.4. *Under the same conditions as in Theorem 3.1, denote*

$$\Gamma = \left(\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}},$$

then

$$|\Gamma + \Gamma^*| \leq \frac{(M-m)^p(M^\alpha + m^\alpha)^{\frac{p}{\alpha}}}{2^{1+\frac{p}{2}}M^{\frac{3p}{4}}m^{\frac{3p}{4}}(M+m)^{\frac{p}{2}}}$$

and

$$\Gamma + \Gamma^* \leq \frac{(M-m)^p(M^\alpha + m^\alpha)^{\frac{p}{\alpha}}}{2^{1+\frac{p}{2}}M^{\frac{3p}{4}}m^{\frac{3p}{4}}(M+m)^{\frac{p}{2}}}.$$

Acknowledgement

References

[1] R. Bhatia, Positive definite matrices, Princeton (NJ): Princeton University Press, 2007.
[2] R. Bhatia, C. Davis, More operator versions of the Schwarz inequality, Comm. Math. Phys. 215 (2000) 239–244.
[3] R. Bhatia, F. Kittaneh, Notes on matrix arithmetic-geometric mean inequalities, Linear Algebra Appl. 308 (2000) 203–211.
[4] X. Fu, C. He, Some operator inequalities for positive linear maps, Linear Multilinear Algebra. 63 (2015), no. 3, 571–577.
[5] X. Fu, Some generalizations of operator inequalities, J. Math. Inequal. 9 (2015), no. 1, 101–105.
[6] I. H. Gumus, A note on a conjecture about Wielandt's inequality, Linear Multilinear Algebra. 63 (2015), no. 9, 1909–1913.
[7] L. V. Kantorovich, Functional analysis and applied mathematics (in Russian), Uspehi Matem. Nauk (N.S.) 3 (1948) 89–185.
[8] M. Lin, Squaring a reverse AM-GM inequality, Studia Math. 215 (2013) 187–194.
[9] M. Lin, On an operator Kantorovich inequality for positive linear maps, J. Math. Anal. Appl. 402 (2013) 127–132.
[10] A.W. Marshall, I. Olkin, Matrix versions of the Cauchy and Kantorovich inequalities, Aequationes Math. 40 (1990) 89–93.
[11] P. Zhang, More operator inequalities for positive linear maps, Banach J. Math. Anal. 9 (2015), no. 1, 166–172.