



Monotone Insertion of Semi-Continuous Functions on Stratifiable Spaces

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Abstract. In this paper, we consider the problem of inserting semi-continuous function above the (generalized) real-valued function in a monotone fashion. We provide some characterizations of stratifiable spaces, semi-stratifiable spaces, and k -monotonically countably metacompact spaces (k -MCM) and so on. It is established that:

- (1) A space X is k -MCM if and only if for each locally bounded real-valued function $h : X \rightarrow \mathbb{R}$, there exists a lower semi-continuous and k -upper semi-continuous function $h' : X \rightarrow \mathbb{R}$ such that (i) $|h| \leq h'$; (ii) $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$.
- (2) A space X is stratifiable if and only if for each function $h : X \rightarrow \mathbb{R}^*$ (\mathbb{R}^* is the generalized real number set), there is a lower semi-continuous function $h' : X \rightarrow \mathbb{R}^*$ such that (i) h' is locally bounded at each $x \in U_h$ with respect to \mathbb{R} , where $U_h = \{x \in X : h \text{ is locally bounded at } x \text{ with respect to } \mathbb{R}\}$; (ii) $|h| \leq h'$; (iii) $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$.

We give a negative answer to the problem posed by K.D. Li [14].

1. Introduction

People have extensively investigated a problem for a given pair of real-valued (non-continuous) functions (g, h) on a space X and $g \leq h$ ($g(x) \leq h(x)$ for each $x \in X$) under what conditions there exists a continuous function f such that $g \leq f \leq h$ since the 1920's. H. Hahn [8] first considered the particular case in which g is upper semi-continuous and h is lower-continuous. Many so called insertion results present some classic characterizations of topological spaces, such as normal spaces, stratifiable spaces and others.

M. Katětov [11] and H. Tong [21] independently proved that a space X is normal if and only if, given an upper semi-continuous function $g : X \rightarrow \mathbb{R}$ and lower semi-continuous $h : X \rightarrow \mathbb{R}$ with $g \leq h$, there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $g \leq f \leq h$. C.H. Dowker [3] proved that a space X is normal and countably paracompact if and only if, for each upper semi-continuous function $g : X \rightarrow \mathbb{R}$

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and lower semi-continuous $h : X \rightarrow \mathbb{R}$ with $g < h$, there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $g < f < h$. E. Michael [17] gave the necessary and sufficient condition for perfectly normal spaces as follows: a space X is perfectly normal if and only if, given an upper semi-continuous function $g : X \rightarrow \mathbb{R}$ and lower semi-continuous $h : X \rightarrow \mathbb{R}$ with $g \leq h$, there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $g \leq f \leq h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$. A stratifiable space can be seen as a monotonic version of a perfectly normal space. P. Nyikos and C. Pan [13] gave a characterization of stratifiable spaces by the monotonicizations of the Michael insertion properties: a space X is stratifiable if and only if, for an upper semi-continuous function $g : X \rightarrow \mathbb{R}$ and lower semi-continuous $h : X \rightarrow \mathbb{R}$ with $g \leq h$, there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $g \leq f \leq h$, $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$ and $f_1 \leq f_2$ whenever $g_1 \leq g_2$ and $h_1 \leq h_2$. C. Good and I. Stares [7] proved that the monotonicized version of Dowker's insertion property characterized stratifiability.

A space is countably paracompact (countably metacompact) if every countable open cover has a locally finite (point finite) open refinement. C. Good, R. Knight and I. Stares [6] and C. Pan [18] introduced a monotone version of countably paracompact spaces closely related to stratifiability, called monotonically countably paracompact spaces (MCP) and monotonically cp-spaces, respectively, and it was proved in [6, Proposition 14] that both these notions are equivalent.

Definition 1.1. [6] A space X is said to be monotonically countably metacompact (MCM) if there is an operator U assigning to each decreasing sequence $(D_j)_{j \in \mathbb{N}}$ of closed sets with empty intersection, a sequence of open sets $U((D_j)) = (U(n, (D_j)))_{n \in \mathbb{N}}$ such that

- (1) $D_n \subseteq U(n, (D_j))$ for each $n \in \mathbb{N}$;
- (2) $\bigcap_{n \in \mathbb{N}} U(n, (D_j)) = \emptyset$;
- (3) given two decreasing sequences of closed sets $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n, (F_j)) \subseteq U(n, (E_j))$ for each $n \in \mathbb{N}$.

X is said to be monotonically countably paracompact (MCP) if, in addition,

$$(2') \bigcap_{n \in \mathbb{N}} \overline{U(n, (D_j))} = \emptyset.$$

J. Mack [16] characterized countably paracompact spaces with locally bounded real-valued functions as follows:

Theorem 1.2. [16] A space X is countably paracompact if and only if for each locally bounded function $h : X \rightarrow \mathbb{R}$ there exists a locally bounded lower semi-continuous function $g : X \rightarrow \mathbb{R}$ such that $|h| \leq g$.

Inspired by those results, C. Good, R. Knight and I. Stares [6] characterized monotonically countably paracompact spaces by the monotone insertions of semi-continuous functions as follows:

Theorem 1.3. [6] The following are equivalent for a space X :

- (1) X is MCP,
- (2) for every locally bounded real-valued function h on X , there is a locally bounded, lower semi-continuous, real-valued $g(h)$ such that $g(h) \geq |h|$ and such that $g(h) \leq g(h')$ whenever $|h| \leq |h'|$.

C. Good, R. Knight and I. Stares proved in [6] that the MCM spaces are equivalent to β spaces. L.S. Wu [22] introduced a subclass of β spaces, i.e., $k\beta$ spaces. L.X. Peng and S. Lin [19] proved that a MCM space added another condition (for every compact subsets K of X , there exists $m \in \mathbb{N}$ such that $U(m, F_j) \cap K = \emptyset$) is $k\beta$ spaces, and they renamed $k\beta$ spaces as k -MCM. D.L.utzer [15] introduced a subclass of stratifiable spaces, i.e., k -semistratifiable spaces with the additional condition that for every compact set K of X , if $K \cap F = \emptyset$, there exists $n \in \mathbb{N}$ such that $K \cap U(n, F) = \emptyset$. P.F. Yan and E.G. Yang ([28] Theorem 3.4) have given the characterizations of k -semistratifiable spaces by the monotone insertion of semi-continuous functions.

Question 1.4. Do the k -MCM spaces have the properties similar to Theorem 1.3?

In this paper, we give the characterization of k -MCM spaces by inserting semi-continuous functions above the locally bounded real-valued functions in a monotone fashion.

C. Good and L. Haynes [5] gave a definition that a space X is said to be δ -stratifiable, if there is an operator U assigning to each regular G_δ -set F in X , a sequence of open sets $U(F) = (U(n, F))_{n \in \mathbb{N}}$ such that (i) $F \subseteq U(n, F)$ for each $n \in \mathbb{N}$; (ii) if $D \subseteq F$, then $U(n, D) \subseteq U(n, F)$ for each $n \in \mathbb{N}$; (iii) $\bigcap_{n \in \mathbb{N}} \overline{U(n, F)} = \bigcap_{n \in \mathbb{N}} U(n, F) = F$. It is a generalization of stratifiable spaces. Characterizations of semistratifiable spaces by monotone insertions of semi-continuous functions are also given in [28]. This inspires K.D. Li [14] asked whether the δ -stratifiable spaces have the similar properties.

Question 1.5. [14] *If a space X is δ -stratifiable, is there an order-preserving map $\varphi : USC(X) \rightarrow LSC(X)$ such that for any $h \in USC(X)$, $0 \leq \varphi(h) \leq h$ and $0 < \varphi(h)(x) < h(x)$ whenever $h(x) > 0$?*

One of the purposes of this paper is to attempt to give a negative answer to this question and construct a counter example.

The MCM (MCP) spaces can be seen as the real generalizations of semistratifiable (stratifiable) spaces. The semistratifiable (stratifiable) spaces are monotone versions of perfect (perfectly normal) spaces. Studying of Theorem 1.3 leads to the following question:

Question 1.6. *Do the stratifiable spaces, semistratifiable spaces and k -semistratifiable spaces have the properties similar to Theorem 1.3?*

In this article, we present some conditions similar to Theorem 1.3 and characterize the semistratifiable spaces and stratifiable spaces by means of inserting the semi-continuous functions and locally bounded functions with respect to \mathbb{R}^* above the generalize real-valued functions in a monotone fashion.

2. Basic Facts and Definitions

In this section, we recall some basic concepts and theorems about the semistratifiable spaces, stratifiable spaces, k -MCM spaces and k -semistratifiable spaces, which needed in the main results of this paper.

Definition 2.1. [2] *A space X is said to be semistratifiable, if there is an operator U assigning to each closed set F a sequence of open sets $U(F) = (U(n, F))_{n \in \mathbb{N}}$ such that*

- (1) $F \subseteq (U(n, F))$ for each $n \in \mathbb{N}$;
- (2) if $D \subseteq F$, then $U(n, D) \subseteq U(n, F)$ for each $n \in \mathbb{N}$;
- (3) $\bigcap_{n \in \mathbb{N}} U(n, F) = F$.

X is said to be stratifiable, if, in addition,

- (3') $\bigcap_{n \in \mathbb{N}} \overline{U(n, F)} = F$.

Definition 2.2. [19] *A space X is said to be k -MCM, if there is a function g on X , assigning to each sequence $\{x_n\}$, a sequence of open sets $g(n, x_n)$ such that if $g(n, x_n) \cap C \neq \emptyset$ for each $n \in \mathbb{N}$ and every compact set C on X , then the sequence $\{x_n\}$ has a cluster point on X .*

Definition 2.3. [19] *A space X is said to be k -semistratifiable, if there is an operator U assigning to each closed set F a sequence of open sets $U(F) = (U(n, F))_{n \in \mathbb{N}}$ such that*

- (1) $F = \bigcap_{n \in \mathbb{N}} U(n, F)$;
- (2) if $D \subseteq F$, then $U(n, D) \subseteq U(n, F)$ for each $n \in \mathbb{N}$;
- (3) for any compact subset K in X , if $F \cap K = \emptyset$, there is $n_0 \in \mathbb{N}$ such that $U(n_0, F) \cap K = \emptyset$.

There are many so called insertion results turning out to provide characterizations of important topological properties as theorems in [17], [28], and [23]. P.F. Yan and E.G. Yang [28] investigated the relations between the insertion of semi-continuous functions and perfect structure of spaces. Enlightened by these theorems, L.H. Xie and P.F. Yan [23] studied the relations between decreasing sequences of sets and insertion of semi-continuous functions, and present some characterizations of countably paracompact spaces and countably metacompact spaces. They also investigated the sequence of sets with some monotonicity properties which is equivalent to monotone version of the insertion properties, and obtained some known topological properties.

Theorem 2.4. [23] For a space X , the following statements are equivalent:

- (1) X is semistratifiable (resp. stratifiable);
- (2) there is an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$, a decreasing sequence of open sets $(U(n, (F_j)))_{n \in \mathbb{N}}$ such that
 - (a) $F_n \subseteq U(n, (F_j))$ for each $n \in \mathbb{N}$;
 - (b) $\bigcap_{n \in \mathbb{N}} U(n, (F_j)) = \bigcap_{n \in \mathbb{N}} F_n$ (resp. $\bigcap_{n \in \mathbb{N}} \overline{U(n, (F_j))} = \bigcap_{n \in \mathbb{N}} F_n$);
 - (c) given two decreasing sequences of closed sets $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n, (F_j)) \subseteq U(n, (E_j))$ for each $n \in \mathbb{N}$.

Theorem 2.5. [28] For a topological space X , the following statements are equivalent:¹⁾

- (1) the space X is k -semistratifiable;
- (2) there is an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$, a decreasing sequence of open sets $(U(n, (F_j)))_{n \in \mathbb{N}}$ such that
 - (i) $F_n \subseteq U(n, (F_j))$ for each $n \in \mathbb{N}$;
 - (ii) for any compact subset K in X , if $\bigcap_{n \in \omega} F_n \cap K = \emptyset$, there is $n_0 \in \omega$ such that $U(n_0, (F_j)) \cap K = \emptyset$;
 - (iii) given two decreasing sequences of closed sets (F_j) and (E_j) such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n, (F_j)) \subseteq U(n, (E_j))$ for each $n \in \mathbb{N}$.

L.X. Peng and S. Lin [19] have given the characterization of k -MCM space by investigating the decreasing sequences of closed sets.

Theorem 2.6. [19] For any topological space X , the following statements are equivalent:

- (1) X is k -MCM;
- (2) there is an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$ with $\bigcap_{j \in \mathbb{N}} F_j = \emptyset$, a decreasing sequence of open sets $(U(n, (F_j)))_{n \in \mathbb{N}}$ such that
 - (i) $F_n \subseteq U(n, (F_j))$ for each $n \in \mathbb{N}$;
 - (ii) given two decreasing sequences of closed sets $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$ and that $\bigcap_{j \in \mathbb{N}} F_j = \bigcap_{j \in \mathbb{N}} E_j = \emptyset$, then $U(n, (F_j)) \subseteq U(n, (E_j))$, for each $n \in \mathbb{N}$;
 - (iii) for any compact subset K in X , there is $n_0 \in \mathbb{N}$ such that $U(n_0, (F_j)) \cap K = \emptyset$.

Before stating the main results of this paper, we shall introduce some notions. Throughout this paper, a space X means a topological space and all spaces in this paper are assumed to be T_1 . $\mathbb{R}(X)$ represents the set of all real-valued functions on X . A real-valued function f defined on a space X is lower (upper) semi-continuous if for any real number r , the set $\{x \in X : f(x) > r\}$ (the set $\{x \in X : f(x) < r\}$) is open. We write $LSC(X)$ ($USC(X)$) for the set of all real-valued lower (upper) semi-continuous functions on X into \mathbb{R} .

¹⁾E.G. Yang also independently proved this result in [29].

3. The Characterization of k -MCM Spaces

P.F. Yan and E.G. Yang [28] introduced a concept of k -semi-continuous function to give the characterization of k -semistratifiable spaces as follows:

Definition 3.1. [28] A real-valued function f defined on a space X is k -lower (k -upper) semi-continuous if for every compact set K , f has a minimum (maximum) value on K .

Theorem 3.2. A space X is k -MCM if and only if for each locally bounded real-valued function $h : X \rightarrow \mathbb{R}$, there exists a lower semi-continuous and k -upper semi-continuous function $h' : X \rightarrow \mathbb{R}$ such that (i) $|h| \leq h'$; (ii) $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$.

Proof. Suppose that X is k -MCM. Take any locally bounded real-valued function $h : X \rightarrow \mathbb{R}$. Let $F_j^h = \overline{\{x \in X : |h(x)| \geq j\}}$ for each $j \in \mathbb{N}$. Then one can easily verify that $(F_j^h)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets such that $\bigcap_{j \in \mathbb{N}} F_j^h = \emptyset$ by h being locally bounded. By hypothesis, there is an operator U defined by $U(F_j^h) = (U(n, (F_j^h)))_{n \in \mathbb{N}}$ satisfying (i)-(iii) in Theorem 2.6. Thus we can define a lower semi-continuous function $h' : X \rightarrow \mathbb{R}$ as follows:

$$h'(x) = \begin{cases} 1 & x \in X \setminus U(1, (F_1^h)) \\ j+1 & x \in U(j, (F_j^h)) \setminus U(j+1, (F_{j+1}^h)) \end{cases} \quad (3.2.1)$$

We assert that $|h| < h'$ for each locally bounded real-valued function h . Take any $x \in X$, since $\bigcap_{j \in \mathbb{N}} U(j, (F_j^h)) = \emptyset$, there exists $n \in \mathbb{N}$ such that $x \in U(n, (F_n^h)) \setminus U(n+1, (F_{n+1}^h))$. Then $h'(x) = n+1 > |h(x)|$ by $x \notin U(n+1, (F_{n+1}^h)) \supset F_{n+1}^h = \overline{\{x \in X : |h(x)| \geq n+1\}} \supseteq \{x \in X : |h(x)| \geq n+1\}$.

We shall show that if $|h_1| \leq |h_2|$, then $h'_1 \leq h'_2$. Take two locally bounded real-valued functions h_1 and h_2 such that $|h_1| \leq |h_2|$. Then $F_j^{h_1} \subseteq F_j^{h_2}$ for each $j \in \mathbb{N}$, where $F_j^{h_i} = \overline{\{x \in X : |h_i(x)| \geq j\}}$, $(i = 1, 2)$. By hypothesis, $U(n, (F_j^{h_1})) \subseteq U(n, (F_j^{h_2}))$. For each $x \in X$, there exists $n_0 \in \mathbb{N}$ such that $x \in U(n_0, (F_{j_1}^{h_1})) \setminus U(n_0+1, (F_{j_1}^{h_1}))$ (Let $U(0, (F_{j_1}^{h_1})) = X$) by hypothesis $\bigcap_{n \in \mathbb{N}} U(n, (F_{j_1}^{h_1})) = \emptyset$. That is $x \notin U(n_0+1, (F_{j_1}^{h_1})) \supseteq U(n_0+1, (F_{j_1}^{h_2})) = \{x \in X : h'_1(x) > n_0+1\}$, and so $h'_2(x) = n_0+1 \geq h'_1(x)$. This implies that $h'_1 \leq h'_2$.

Then we need only to show that h' is k -upper semi-continuous. Suppose that K is a compact set. If $K \cap U(n, (F_j)) = \emptyset$ for some $n \in \mathbb{N}$, let $N = \min\{n : K \cap U(n, (F_j)) = \emptyset\}$. Then $K \cap U(n, (F_j)) = \emptyset$ for all $n \geq N$, and $K \cap U(n, (F_j)) \neq \emptyset$ for all $n < N$. Thus $K \cap (\bigcap_{n < N} U(n, (F_j))) = K \cap U(N-1, (F_j)) \neq \emptyset$. Take $x_0 \in K \cap (\bigcap_{n < N} U(n, (F_j)))$. Then for each $x \in K$, $h'(x) \leq N = h'(x_0)$. Then h' has a maximum value on K . Therefore, h' is k -upper semi-continuous.

Conversely, let $(F_j)_{j \in \mathbb{N}}$ be a decreasing sequence of closed sets with empty intersection in X , We define an upper semi-continuous function $h_{(F_j)} : X \rightarrow \mathbb{R}$ by

$$h_{(F_j)}(x) = \begin{cases} 1 & x \in X \setminus F_1 \\ n+1 & x \in F_n \setminus F_{n+1} \end{cases} \quad (3.2.2)$$

Clearly, $h_{(F_j)}$ is also locally bounded. By hypothesis, there exists a lower semi-continuous and k -upper semi-continuous function $h' : X \rightarrow \mathbb{R}$ such that (i) $h' > |h_{(F_j)}|$; (ii) $h'_1 \geq h'_2$ whenever $|h_1| \geq |h_2|$. Set $U_j = \{x \in X : h'(x) > j\}$ for each $j \in \mathbb{N}$, thus we can define an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$ by $U((F_j)) = (U(n, (F_j)))_{n \in \mathbb{N}}$ where $U(n, (F_j)) = U_n$ for each $n \in \mathbb{N}$. We assert that U satisfies (i)-(iii) of Theorem 2.6. By $h' > |h_{(F_j)}|$, it is easy to see that $U_n \supseteq F_n$.

Take any two decreasing sequences of closed sets $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ in X such that $F_j \subseteq E_j$ for each $j \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} F_n = \bigcap_{n \in \mathbb{N}} E_n = \emptyset$, then one can easily obtain $|h_{(E_j)}| \geq |h_{(F_j)}|$, where $h_{(E_j)}$ and $h_{(F_j)}$ are defined by (3.2.2). By hypothesis there are functions $h'_{(E_j)}, h'_{(F_j)} : X \rightarrow \mathbb{R}$ such that $h'_{(E_j)} > |h_{(E_j)}|$ and $h'_{(F_j)} > |h_{(F_j)}|$. And we also have $h'_{(E_j)} \geq h'_{(F_j)}$, then we can get $U(n, (E_j)) = \{x \in X : h'_{(E_j)}(x) > n\} \supseteq \{x \in X : h'_{(F_j)}(x) > n\} = U(n, (F_j))$ for each $n \in \mathbb{N}$.

Suppose C is a compact subset of X . Then h' is k -upper semi-continuous, and so there exists $x_0 \in C$ such that $|h_{(F_j)}(x)| < h'(x) \leq h'(x_0)$ for all $x \in C$. Fix $n_0 \in \mathbb{N}$ such that $h'(x_0) \leq n_0$. Then $h'(x) \leq n_0$ and so $x \in \{x \in X : h'(x) \leq n_0\}$, which implies that $x \notin U_{n_0} = \{x \in X : h'(x) > n_0\}$. Then we have $C \cap U_{n_0} = \emptyset$. This concludes the proof. \square

The Question 1.5 has a negative answer. We prove that the spaces satisfying the conditions are zero-dimensional spaces. Example 3.3 provides a counterexample in this case.

Definition 3.3. [4] A topological space X is zero-dimensional ($\text{ind } X = 0$) if it is Hausdorff and has a basis consisting of clopen (i.e., open and closed) sets.

For example, the Cantor space C is a zero-dimensional space, but \mathbb{R}^n , and \mathbb{C}^n are connected.

Definition 3.4. [28] A map $\varphi : \mathbb{R}(X) \rightarrow \mathbb{R}(X)$ is called order-preserving if $\varphi(g) \leq \varphi(h)$ for every pair g, h of elements of $\mathbb{R}(X)$ satisfying $g \leq h$.

Theorem 3.5. Let X be a topological space. If there is an order-preserving map $\varphi : \text{USC}(X) \rightarrow \text{LSC}(X)$ such that for any $h \in \text{USC}(X)$, $0 \leq \varphi(h) \leq h$ and $0 < \varphi(h)(x) < h(x)$ whenever $h(x) > 0$, then X is a zero-dimensional space ($\text{ind } X = 0$).

Proof. Let U be an open set and $\phi_0 : \text{USC}(X) \rightarrow \text{LSC}(X)$ be an operator satisfying the conditions given in the theorem. We define an upper semi-continuous function $h_{(U)} : X \rightarrow \mathbb{R}$ as follows:

$$h_{(U)}(x) = \begin{cases} 1 & x \in X \setminus U \\ 0 & x \in U \end{cases}$$

By hypothesis, $\phi_0(h_{(U)}) : X \rightarrow \mathbb{R}$ is lower semi-continuous. Let $F_j = \{x \in X : \phi_0(h_{(U)})(x) \leq \frac{1}{j}\}$ for each $j \in \mathbb{N}$. Clearly, $(F_j)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets. By the definition of $h_{(U)}$ and $0 \leq \phi_0(h_{(U)}) \leq h_{(U)}$, $U = \{x \in X : h_{(U)}(x) = 0\} \subseteq \{x \in X : \phi_0(h_{(U)})(x) = 0\} = \bigcap_{n \in \mathbb{N}} F_n$, for each $j \in \mathbb{N}$. We assert that $U = \bigcap_{n \in \mathbb{N}} F_n$. Take $x \in \bigcap_{n \in \mathbb{N}} F_n$, because of $\phi_0(h_{(U)})(x) = 0$, $h_{(U)}(x) = 0$ by $0 < \phi_0(h_{(U)})(x) < h_{(U)}(x)$ whenever $h_{(U)}(x) > 0$. Then $x \in U$. We get $U = \bigcap_{n \in \mathbb{N}} F_n$. This implies that U is also closed.

In addition, for any closed set F , there exists a closed sequence $(E_j)_{j \in \mathbb{N}}$ such that $X \setminus F = \bigcap_{n \in \mathbb{N}} E_n$. Then $F = \bigcup_{n \in \mathbb{N}} (X \setminus E_n)$. This means F is also open. This means $\text{ind } X = 0$.

And we complete the proof. \square

From the proofs we can see that X satisfying the conditions in Theorem 3.5 has the properties: every open set is closed and every closed set is open.

Example 3.6. \mathbb{R} with the usual topology is stratifiable (hence δ -stratifiable). For \mathbb{R} is connected, $\text{Ind } \mathbb{R} = 1$.

4. The Characterization of Semistratifiable Spaces and k -Semistratifiable Spaces

In this section, we refer to the proof of C. Good [6] (Theorem 1.3) and get similar results. We give another equivalent conditions for semi-stratifiable spaces and stratifiable spaces. But the real-valued function is not enough to describe semistratifiable spaces. We introduce the definitions as follows:

Definition 4.1. [9] The symbols $-\infty$ and $+\infty$ are called generalized real numbers. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$, which is homeomorphic to $[0, 1]$. \mathbb{R}^* is called generalized real number set. It satisfies the order relation: $-\infty < r < +\infty$, for any $r \in \mathbb{R}$. The function $f : X \rightarrow \mathbb{R}^*$ is called generalized real-valued function.

Definition 4.2. [9] A generalized real-valued function $f : X \rightarrow \mathbb{R}^*$ is called lower (upper) semi-continuous, if for any $r \in \mathbb{R}$, the set $\{x \in X : f(x) > r\}$ ($\{x \in X : f(x) < r\}$) is open.

Definition 4.3. For a topological space X , $h : X \rightarrow \mathbb{R}^*$ is called locally bounded at x with respect to \mathbb{R} , if there is a positive real number r , and an open neighborhood U of x such that $h(U) \subseteq (-r, r)$; if h is locally bounded at each $x \in X$ with respect to \mathbb{R} , then h is called locally bounded on X with respect to \mathbb{R} .

For a function $h : X \rightarrow \mathbb{R}^*$, define

$$U_h = \{x \in X : h \text{ is locally bounded at } x \text{ with respect to } \mathbb{R}\}.$$

Clearly, U_h is an open set in X .

Definition 4.4. A generalized real-valued function $f : X \rightarrow \mathbb{R}^*$ is called k -lower (k -upper) semi-continuous, if for every compact set K , there exists $x_0 \in K$ such that $f(x_0), f(x_0) \leq f(x)$ ($f(x_0) \geq f(x)$) for any $x \in K$.

In this section, a function represents a generalized real-valued function unless otherwise specified. We can verify that for a decreasing sequence of closed (open) sets, it can define an upper (lower) semi-continuous function with respect to \mathbb{R} as following.

Lemma 4.5. If $(F_j)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets in X , then there is a function $h_{(F_j)} : X \rightarrow \mathbb{R}^*$ defined by

$$h_{(F_j)} = \begin{cases} 1 & x \in X \setminus F_1 \\ n + 1 & x \in F_n \setminus F_{n+1} \\ +\infty & x \in \bigcap_{j \in \mathbb{N}} F_j \end{cases} \quad (4.5.1)$$

such that $h_{(F_j)}$ is upper semi-continuous and locally bounded in $X \setminus \bigcap_{j \in \mathbb{N}} F_j$ with respect to \mathbb{R} . Given two decreasing sequences of closed sets $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ such that $F_j \subseteq E_j$ for each $j \in \mathbb{N}$, then $h_{(F_j)} \leq h_{(E_j)}$.

Proof. To prove $h_{(F_j)}$ is upper semi-continuous, it is enough to show that the set $\{x \in X : h_{(F_j)}(x) \geq r\}$ is closed for any real number r . Without loss of generality, we assume $r \in [1, +\infty)$. Thus, (i) if $r \in \mathbb{N}$, then $\{x \in X : h_{(F_j)}(x) \geq r\} = F_{r-1}$ (let $X = F_0$); (ii) if $r \notin \mathbb{N}$, we have $\{x \in X : h_{(F_j)}(x) \geq r\} = F_{[r]}$. Since $(F_j)_{j \in \mathbb{N}}$ is a sequence of closed sets, the set $\{x \in X : h_{(F_j)}(x) \geq r\}$ is closed, which implies that $h_{(F_j)}$ is upper semi-continuous.

If $x \in \bigcap_{j \in \mathbb{N}} E_j$, then $h_{(E_j)} = +\infty \geq h_{(F_j)}$. On the other hand, if $x \notin \bigcap_{j \in \mathbb{N}} E_j$, there exists $n \in \mathbb{N}$ such that $x \in E_n \setminus E_{n+1}$. If $x \notin F_{n+1}$, then $h_{(F_j)} \leq n + 1$. Then we get $h_{(E_j)} = n + 1 \geq h_{(F_j)}$, since $x \notin E_{n+1} \supseteq F_{n+1}$. \square

Lemma 4.6. If $(U_j)_{j \in \mathbb{N}}$ is a decreasing sequence of open sets in X , then there is a function $h_{(U_j)} : X \rightarrow \mathbb{R}^*$ defined by

$$h_{(U_j)} = \begin{cases} 1 & x \in X \setminus U_1 \\ n + 1 & x \in U_n \setminus U_{n+1} \\ +\infty & x \in \bigcap_{j \in \mathbb{N}} U_j \end{cases} \quad (4.6.1)$$

such that $h_{(U_j)}$ is lower semi-continuous. Given two decreasing sequences of open sets $(U_j)_{j \in \mathbb{N}}$ and $(G_j)_{j \in \mathbb{N}}$ such that $U_j \subseteq G_j$ for each $j \in \mathbb{N}$, then $h_{(U_j)} \leq h_{(G_j)}$.

Proof. To prove $h_{(U_j)}$ is lower semi-continuous, it is enough to show that the set $\{x \in X : h_{(U_j)}(x) > r\}$ is open for any real number r . Without loss of generality, we assume $r \in [1, +\infty)$. Thus, (i) if $r \in \mathbb{N}$, then $\{x \in X : h_{(U_j)}(x) > r\} = U_r$; (ii) if $r \notin \mathbb{N}$, we have $\{x \in X : h_{(U_j)}(x) > r\} = U_{[r]}$. Since $(U_j)_{j \in \mathbb{N}}$ is a sequence of open sets, the set $\{x \in X : h_{(U_j)}(x) > r\}$ is open, which implies that $h_{(U_j)}$ is lower semi-continuous.

If $x \in \bigcap_{j \in \mathbb{N}} G_j$, then $h_{(G_j)} = +\infty \geq h_{(U_j)}$. On the other hand, if $x \notin \bigcap_{j \in \mathbb{N}} G_j$, there exists $n \in \mathbb{N}$ such that $x \in G_n \setminus G_{n+1}$. If $x \notin U_{n+1}$, then $h_{(U_j)} \leq n + 1$. Then we get $h_{(G_j)} = n + 1 \geq h_{(U_j)}$, since $x \notin G_{n+1} \supseteq U_{n+1}$. \square

Theorem 4.7. A space X is semistratifiable if and only if for each function $h : X \rightarrow \mathbb{R}^*$, there is a lower semi-continuous function $h' : X \rightarrow \mathbb{R}^*$, such that (i) $h'(U_h) \subseteq \mathbb{R}$; (ii) $|h| \leq h'$; (iii) $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$.

Proof. Suppose that X is semistratifiable. Take any function $h : X \rightarrow \mathbb{R}^*$. Let $F_j^h = \overline{\{x \in X : |h(x)| \geq j\}}$ for each $j \in \mathbb{N}$. Then one can easily verify that $(F_j^h)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets such that $\bigcap_{j \in \mathbb{N}} F_j^h = X \setminus U_h$ by h being locally bounded at each $x \in U_h$ with respect to \mathbb{R} . According to Theorem 2.4, there exists an operator U satisfies the conditions of (i)-(iii).

Define a function $h' : X \rightarrow \mathbb{R}^*$ as follows:

$$h'(x) = \begin{cases} 1 & x \in X \setminus U(1, (F_j^h)) \\ j + 1 & x \in U(j, (F_j^h)) \setminus U(j + 1, (F_j^h)) \\ +\infty & x \in X \setminus U_h \end{cases} \quad (4.7.1)$$

Then h' is lower semi-continuous. We prove that $h' \geq |h|$ for each $h : X \rightarrow \mathbb{R}^*$. Take any $x \in X$. If $x \in X \setminus U_h$, then $h'(x) = +\infty = |h(x)|$ by $\bigcap_{n \in \mathbb{N}} U(n, (F_j^h)) = \bigcap_{n \in \mathbb{N}} F_n^h = X \setminus U_h$. Otherwise $x \in U_h$, for $\bigcap_{n \in \mathbb{N}} U(n, (F_j^h)) = X \setminus U_h$, there exists $n \in \mathbb{N}$ such that $x \in U_n^h \setminus U_{n+1}^h$. Then $h'(x) = n + 1 > |h(x)|$ by $x \notin U_{n+1}^h \supseteq F_{n+1}^h = \overline{\{x \in X : |h(x)| \geq n + 1\}} \supseteq \{x \in X : |h(x)| \geq n + 1\}$.

Secondly, we prove that if $|h_1| \leq |h_2|$, then $h'_1 \leq h'_2$. It is known that $F_j^{h_1} \subseteq F_j^{h_2}$ for $|h_1| \leq |h_2|$. Then we have $\bigcap_{j \in \mathbb{N}} F_j^{h_1} \subseteq \bigcap_{j \in \mathbb{N}} F_j^{h_2}$, $X \setminus U_{h_1} \subseteq X \setminus U_{h_2}$, and $U(n, (F_j^{h_1})) \subseteq U(n, (F_j^{h_2}))_{n \in \mathbb{N}}$. If $x \in U_{h_2}$, there exists $n_0 \in \mathbb{N}$ such that $x \in U(n_0, (F_j^{h_2})) \setminus U(n_0 + 1, (F_j^{h_2}))$ (let $X = U(0, F_j^{h_2})$) by $\bigcap_{n \in \mathbb{N}} U(n, (F_j^{h_2})) = X \setminus U_{h_2}$. Furthermore, $x \notin U(n_0 + 1, (F_j^{h_2})) \supseteq U(n_0 + 1, (F_j^{h_1})) = \{x \in X : h'_1(x) > n_0 + 1\}$. This implies $h'_2(x) = n_0 + 1 \geq h'_1(x)$. That is $h'_2 \geq h'_1$. Otherwise if $x \in U_{h_1} \setminus U_{h_2}$, there exists $n_1 \in \mathbb{N}$ such that $x \in U(n_1, (F_j^{h_1})) \setminus U(n_1 + 1, (F_j^{h_1}))$ (let $X = U(0, F_j^{h_1})$) by $\bigcap_{n \in \mathbb{N}} U(n, (F_j^{h_1})) = X \setminus U_{h_1}$. Since $h'_1(x) = n_1 + 1$, and $h'_2(x) = +\infty$, then we have $h'_1 \leq h'_2$. Finally, if $x \in X \setminus U_{h_1}$, then $h'_1(x) = h'_2(x) = +\infty$. The result is obvious.

Conversely, take any decreasing sequence $(F_j)_{j \in \mathbb{N}}$ of closed sets in X . Then define a function $h_{(F_j)} : X \rightarrow \mathbb{R}^*$ by (4.5.1). Clearly, $h_{(F_j)}$ is upper semi-continuous and $U_{h_{(F_j)}} = X \setminus \bigcap_{j \in \mathbb{N}} F_j$. By hypothesis, there is a lower semi-continuous function $h' : X \rightarrow \mathbb{R}^*$, such that $h'(U_h) \subseteq \mathbb{R}$, $|h| \leq h'$ and $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$. Set $U_n = \{x \in X : h'_{(F_j)}(x) > n\}$. Then we can define an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$ by $U((F_j)) = (U(n, (F_j)))_{n \in \mathbb{N}}$, where $U(n, (F_j)) = U_n$ for each $n \in \mathbb{N}$. To show X is semistratifiable, it need prove that the operator U satisfies (a)-(c) in Theorem 2.4.

By $|h_{(F_j)}| \leq h'_{(F_j)}$, it is easy to see that $U_n \supseteq F_n$ for each $n \in \mathbb{N}$ and therefore, the operator U satisfies (a) in Theorem 2.4.

For two decreasing sequences $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ of closed sets in X such that $F_j \subseteq E_j$, it is obvious that $h_{(F_j)} \geq h_{(E_j)}$. Then $U(n, (F_j)) = \{x \in X : h'_{(F_j)}(x) > n\} \subseteq \{x \in X : h'_{(E_j)}(x) > n\} = U(n, (E_j))$ for each $n \in \mathbb{N}$ and therefore, the operator U satisfies (c) in Theorem 2.4.

Clearly, $\bigcap_{j \in \mathbb{N}} U(n, (F_j)) \supseteq \bigcap_{n \in \mathbb{N}} F_n$. Take any $x \notin \bigcap_{j \in \mathbb{N}} F_j$. By hypothesis, there is $r \in \mathbb{R}$ such that $h'_{(F_j)}(x) = r$. Hence, $x \notin U(n_0, (F_j))$ whenever $n_0 > r$. This implies that $\bigcap_{j \in \mathbb{N}} U(n, (F_j)) = \bigcap_{n \in \mathbb{N}} F_n$. This shows that the operator U satisfies (b) in Theorem 2.4. \square

Theorem 4.8. *A space X is stratifiable if and only if for each function $h : X \rightarrow \mathbb{R}^*$, there is a lower semi-continuous function $h' : X \rightarrow \mathbb{R}^*$ such that (i) h' is locally bounded at each $x \in U_h$ with respect to \mathbb{R} ; (ii) $|h| \leq h'$; (iii) $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$.*

Proof. Suppose that X is stratifiable. Take any function $h : X \rightarrow \mathbb{R}^*$. Let $F_j^h = \overline{\{x \in X : |h(x)| \geq j\}}$ for each $j \in \mathbb{N}$. Then one can easily verify that $(F_j^h)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets such that $\bigcap_{j \in \mathbb{N}} F_j^h = X \setminus U_h$ by h being locally bounded in U_h . According to Theorem 2.4, there exists an operator U which satisfies the conditions of (i)-(iii).

Define a function $h' : X \rightarrow \mathbb{R}^*$ by

$$h'(x) = \begin{cases} 1 & x \in X \setminus U(1, (F_j^h)) \\ j + 1 & x \in U(j, (F_j^h)) \setminus U(j + 1, (F_j^h)) \\ +\infty & x \in X \setminus U_h \end{cases} \quad (4.8.1)$$

It is easy to see that h' is lower semi-continuous.

To show h' is locally bounded at each $x \in U_h$ with respect to \mathbb{R} . Take any $x \in U_h$, since $\bigcap_{n \in \mathbb{N}} \overline{U(n, (F_j^h))} = \bigcap_{n \in \mathbb{N}} (F_n^h) = X \setminus U_h$, there exists $n_0 \in \mathbb{N}$ such that $x \in \overline{U(n_0, (F_j^h))} \setminus \overline{U(n_0 + 1, (F_j^h))}$, and an open neighborhood O of x such that $O \cap U((n_0 + 1, (F_j^h))) = \emptyset$. It implies that $|h(O)| \leq n_0 + 1$. This completes the proof that h' is locally bounded at each $x \in U_h$.

We prove $|h| \leq h'$. Take any $x \in U_h$, since $\bigcap_{n \in \mathbb{N}} U(n, (F_j^h)) = X \setminus U_h$, there exists $n \in \mathbb{N}$ such that $x \in U(n, (F_j^h)) \setminus U(n + 1, (F_j^h))$. Then $h'(x) = n + 1 > |h(x)|$ by $x \notin U(n + 1, (F_j^h)) \supseteq F_{n+1}^h = \overline{\{x \in X : |h(x)| \geq n + 1\}} \supseteq \{x \in X : |h(x)| \geq n + 1\}$. Otherwise if $x \in X \setminus U_h$, then $h'(x) = |h(x)| = +\infty$. We obtain the results $|h| \leq h'$.

The proof of the property (iii) is similar to the proof of Theorem 4.7.

Conversely, take any decreasing sequence $(F_j)_{j \in \mathbb{N}}$ of closed sets in X . Then define a function $h_{(F_j)} : X \rightarrow \mathbb{R}^*$ by (4.5.1). Clearly, $h_{(F_j)}$ is upper semi-continuous such that $U_{h_{(F_j)}} = X \setminus \bigcap_{j \in \mathbb{N}} F_j$. By hypothesis, there is a function $h'_{(F_j)} : X \rightarrow \mathbb{R}^*$ satisfying the properties (i) through (iii). Set $U_n = \{x \in X : h'_{(F_j)}(x) > n\}$. Then we can define an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$ by $U((F_j)) = (U(n, (F_j)))_{n \in \mathbb{N}}$, where $U(n, (F_j)) = U_n$ for each $n \in \mathbb{N}$. To show X is stratifiable, it need prove that the operator U satisfies (a)-(c) in Theorem 2.4.

By $|h_{(F_j)}| \leq h'_{(F_j)}$, it is easy to see that $U_n \supseteq F_n$ for each $n \in \mathbb{N}$ and therefore, the operator U satisfies (a) in Theorem 2.4.

For two decreasing sequences $(F_j)_{j \in \mathbb{N}}$ and $(E_j)_{j \in \mathbb{N}}$ of closed sets in X such that $F_j \subseteq E_j$, it is obvious that $h_{(F_j)} \geq h_{(E_j)}$. Then $U(n, (F_j)) = \{x \in X : h'_{(F_j)}(x) > n\} \subseteq \{x \in X : h'_{(E_j)}(x) > n\} = U(n, (E_j))$ for each $n \in \mathbb{N}$ and therefore, the operator U satisfies (c) in Theorem 2.4.

Clearly, $\bigcap_{j \in \mathbb{N}} \overline{U(n, (F_j))} \supseteq \bigcap_{n \in \mathbb{N}} F_n$. Take any $x \notin \bigcap_{j \in \mathbb{N}} F_j$. By hypothesis, $h'_{(F_j)}$ is locally bounded at x with respect to \mathbb{R} , so there is $r \in \mathbb{R}$ and a neighborhood O of x such that $h'_{(F_j)}(O) \subseteq (-r, r)$. Hence, $x \notin \overline{U(n_0, (F_j))}$ whenever $n_0 > r$. This implies that $\bigcap_{j \in \mathbb{N}} \overline{U(n, (F_j))} = \bigcap_{n \in \mathbb{N}} F_n$. This shows that the operator U satisfies (b) in Theorem 2.4. \square

Recall that the stratifiable (semistratifiable) spaces is the monotone versions of the perfectly normal (perfect) spaces. We get the similar results for perfectly normal (perfect) spaces as follows.

Theorem 4.9. *A space X is perfect if and only if for each function $h : X \rightarrow \mathbb{R}^*$, there is a lower semi-continuous function $h' : X \rightarrow \mathbb{R}^*$, such that (i) $h'(U_h) \subseteq \mathbb{R}$; (ii) $|h| \leq h'$.*

Theorem 4.10. *A space X is perfectly normal if and only if for each function $h : X \rightarrow \mathbb{R}^*$, there is a lower semi-continuous function $h' : X \rightarrow \mathbb{R}^*$ such that (i) h' is locally bounded at each $x \in U_h$ with respect to \mathbb{R} ; (ii) $|h| \leq h'$.*

Theorem 4.11. *A space X is k -semistratifiable if and only if for each function $h : X \rightarrow \mathbb{R}^*$, there is a lower semi-continuous and k -upper semi-continuous function $h' : X \rightarrow \mathbb{R}^*$, such that (i) $h'(U_h) \subseteq \mathbb{R}$; (ii) $|h| \leq h'$; (iii) $h'_1 \leq h'_2$ whenever $|h_1| \leq |h_2|$.*

Proof. Suppose that X is k -semistratifiable. Define a function h' as that in the proof of Theorem 4.7 (necessary) with U satisfying the additional condition (ii) in Theorem 2.5 for k -semistratifiable. Then we need only to show that for each function $h : X \rightarrow \mathbb{R}^*$, $h' : X \rightarrow \mathbb{R}^*$ is k -upper semi-continuous.

Suppose that K is a compact set in X . If $K \cap U(n, (F_j^h)) = \emptyset$ for some $n \in \mathbb{N}$, let $m = \min\{n : K \cap U(n, (F_j^h)) = \emptyset\}$. Then we have $K \cap U(n, (F_j^h)) = \emptyset$ for all $n \geq m$ and $K \cap U(n, (F_j^h)) \neq \emptyset$ for all $n < m$. Thus $K \cap (\bigcap_{n < m} U(n, (F_j^h))) = K \cap U(m - 1, (F_j^h)) \neq \emptyset$. Take $x_0 \in K \cap (\bigcap_{n < m} U(n, (F_j^h)))$. Then for each $x \in K$, $h'(x) \leq h'(x_0) = m$.

If $K \cap U(n, (F_j^h)) \neq \emptyset$ for each $n \in \mathbb{N}$, then, by condition (ii) in Theorem 2.5, $K \cap F_n \neq \emptyset$ for each $n \in \mathbb{N}$, and so $K \cap (\bigcap_{n \in \mathbb{N}} F_n) \neq \emptyset$ because of the compactness of K . Thus by the equality $\bigcap_{n \in \mathbb{N}} F_n = \bigcap_{n \in \mathbb{N}} U(n, (F_j^h))$, we have $K \cap (\bigcap_{n \in \mathbb{N}} U(n, (F_j^h))) \neq \emptyset$. Take $x_0 \in K \cap (\bigcap_{n \in \mathbb{N}} U(n, (F_j^h)))$. Then for each $x \in K$, $h'(x) \leq h'(x_0) = +\infty$.

From the discussion above, we see that h' have a maximum value on K . Therefore, for each h , h' is k -upper semi-continuous.

Conversely, take any decreasing sequence $(F_j)_{j \in \mathbb{N}}$ of closed sets in X . Then define a function $h_{(F_j)} : X \rightarrow \mathbb{R}^*$ by (4.5.1). Clearly, $h_{(F_j)}$ is upper semi-continuous such that $U_{h_{(F_j)}} = X \setminus \bigcap_{j \in \mathbb{N}} F_j$. By hypothesis, there is a lower semi-continuous and k -upper semi-continuous function $h'_{(F_j)} : X \rightarrow \mathbb{R}^*$ satisfying the properties (i) through (iii). Set $U_n = \{x \in X : h'_{(F_j)}(x) > n\}$. Then we can define an operator U assigning to each decreasing sequence of closed sets $(F_j)_{j \in \mathbb{N}}$ by $U((F_j)) = (U(n, (F_j)))_{n \in \mathbb{N}}$, where $U(n, (F_j)) = U_n$ for each $n \in \mathbb{N}$. Then the operator U satisfies (i) and (iii) of Theorem 2.5 as shown in Theorem 4.7 (sufficiency). Thus we need only to show that U also satisfies condition (ii) of Theorem 2.5.

Suppose K is an arbitrary compact set of X and $\bigcap_{n \in \omega} F_n \cap K = \emptyset$. Then $K \subseteq U_{h_{(F_j)}}$, and so $h'_{(F_j)}(K) \subseteq \mathbb{R}$. There exists $x_0 \in K$ such that $h'_{(F_j)}(x) \leq h'_{(F_j)}(x_0)$ for all $x \in K$. Take $n_0 \in \mathbb{N}$ such that $h'_{(F_j)}(x) \leq h'_{(F_j)}(x_0) \leq n_0$. Then $x \in \{x \in X : h'_{(F_j)}(x) \leq n_0\}$, which implies that $x \notin U_{n_0} = \{x \in X : h'_{(F_j)}(x) > n_0\}$. Then we have $K \cap U_{n_0} = \emptyset$. This concludes the proof. \square

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