Filomat 31:3 (2017), 585–596
DOI 10.2298/FIL1703585S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Revisit of Stochastic Theta Method with some Improvements

F. Soleymani[a], Ali R. Soheili[a,b]

[a]*Department of Applied Mathematics, Faculty of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran*
[b]*The Center of Excellence on Modelling and Control Systems, Ferdowsi University of Mashhad, Mashhad, Iran*

Abstract. Considering the stochastic theta method, in this work we discuss modified solvers for finding the solution of Itô stochastic differential equations in the strong sense. It is observed that the proposed variations are implicit with better stability behaviors. Finally, the derived schemes are tested numerically to confirm the efficiency.

1. Introduction

Let us take into account the numerical approximation of $\mathbb{R}^d$-valued stochastic processes, which satisfy an Itô Stochastic Differential Equation (SDE) of the form below

$$\begin{aligned} dX_t &= f(t,X_t)dt + g(t,X_t)dW_t = b^0(t,X_t)dt + \sum_{k=1}^{m} b^k(t,X_t)dW_t^k, \quad t \in [t_0,T], \\ X(t_0) &= X_0, \end{aligned} \tag{1}$$

where $W_t^k, k = 1,\ldots,m$ denote real and pairwise independent standard Brownian motions, adapted to the filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$ on the underlying probability space $(\Omega,\mathcal{F},P)$. The initial value X_0 is assumed to be $\mathcal{F}_0$-measurable and to have finite second moment. We also assume that the drift and diffusion coefficient functions $b^k : [t_0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable and fulfill the usual Lipschitz conditions such that (1) has a unique solution (see e.g. [3] and [10]).

Recently, there is an increasing interest in the stability analysis of some solvers for SDEs (see e.g. [13] and the references therein). The main reason to this interest lies in the fact that for most of the nonlinear SDEs or multi-dimensional SDEs, typical solvers with not reasonably small step size in the the time direction explode and could not capture the nonlinearity or the stiffness of the problem. Due to this many of the explicit solvers such as (explicit) Euler-Maruyama (EM) [4]

$$\begin{aligned} X_h(t_{i+1}) &= X_h(t_i) + h_{i+1}(b^0(t_i,X_h(t_i))) + \sum_{k=1}^{m} b^k(t_i,X_h(t_i))\Delta W^k(t_{i+1}), \\ X_h(0) &= X_0, \end{aligned} \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 60H35. Secondary 65L20.
Keywords. stochastic differential equations; stability; theta method; numerical schemes.
Received: 15 December 2014; Accepted: 03 February 2015
Communicated by Predrag S. Stanimirović
Email addresses: fazl_soley_bsb@yahoo.com (F. Soleymani), soheili@um.ac.ir, Corresponding author. (Ali R. Soheili)

fail to converge, where $h_{i+1} = t_{i+1} - t_i$ is the length of the $i+1$-th interval. There is an evidence that the dynamics of prices for financial instruments exhibit nonlinearity, which cannot be adequately supported solely by typical numerical solvers.

Note that only a limited class of SDEs admit explicit solutions, therefore there is a need for the development of approximate methods. Some of such computational methods can be extracted from [5], where convergence of explicit numerical methods is considered. It is worth noting that, in [8], stability of stochastic EM method has been studied.

Without loss of generality, we consider $d = 1$. Hence, the Stochastic Theta Method (STM) can be defined as

$$X_{i+1} = X_i + (1-\theta) f(t_i, X_i)\Delta t_i + \theta f(t_i, X_{i+1})\Delta t_i + g(t_i, X_i)\Delta W_i, \tag{3}$$

wherein $\theta \in [0, 1]$, $\Delta t_i = t_{i+1} - t_i$, and $\Delta W_i \sim \mathcal{N}(0, 1)\sqrt{\Delta t_i}$.

In this work, $h = \Delta t > 0$ is the constant step-size. In the deterministic case, $g \equiv 0$, (3) is called the Theta Method (TM) and the choice $\theta = 0$ gives Euler's method, $\theta = \frac{1}{2}$ gives the trapezoidal solver, and $\theta = 1$ gives the implicit, or backward Euler method.

The scheme (3) was discussed in [12], where it is called the semi-implicit Euler method. In particular, taking $\theta = 0$ gives the widely used EM method. Note that STM is a natural extension of TM.

It is known that STM converges at least with convergence rate $\gamma = \frac{1}{2}$ in the strong sense, that is, there exists a constant $C > 0$ such that

$$\underbrace{\max}_{0\le i\le N} \left(\mathbb{E}\left(|X(t_i) - X_h(t_i)|^2\right)\right)^{\frac{1}{2}} \le C|h|^{\gamma}, \tag{4}$$

where X is the analytic solution and X_h is the numerical solution. Furthermore, this strong rate of speed has been obtained in terms of the norm

$$\|Y_h\|_{0,h} = \underbrace{\max}_{0\le i\le N} \|Y_h(t_i)\|_{L^2(\Omega)}. \tag{5}$$

It is important to have some solvers at hand in which the rate of convergence (in the strong sense) is more than 1/2 while they could overcome on the stiff problems and are not too much dependent on the step sizes along the time direction. Pursuing these two needs are the principal objectives of this paper.

The rest of this work unfolds the contents in what follows. Section 2, discussed the stability of TM to show the capability of its methods for stiff problems. Note that by considering the numerical solution of the stochastic initial value problem (1) the stiffness is a combination of problem, solution method, initial condition and local error tolerances. In Section 3, we propose a computational scheme for solving Itô-type SDEs. Some implicit variations are also brought forward. Section 4 is devoted to the application of the contributed scheme in solving some SDEs numerically. The attained results uphold the effectiveness of the proposed scheme in a computational point of view. And finally, Section 5 draws a conclusion of this paper.

2. Deterministic Case

Linear stability theory arises from the study of Dahlquist's scalar linear test equation [14]:

$$X'(t) = \lambda X(t),\ \lambda \in \mathbb{C},\ \Re(\lambda) < 0, \tag{6}$$

as a simplified model for studying the initial value problem

$$X'(t) = f(t, X(t)), \quad X(0) = X_0, \quad f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n. \tag{7}$$

Stability is described by examining a numerical method applied to (7) and obtaining the following recurrence relation

$$X_{i+1} = S(z)X_i, \ \ z \in \mathbb{C}, \tag{8}$$

wherein $z = h\lambda$ and $S(z)$ is the (rational) stability function. The boundary of absolute stability is attained via taking into account the following area:

$$|S(z)| = 1. \tag{9}$$

As a matter of fact, the fundamental idea is to study a computational scheme on the test problem (6) which is uncomplicated enough to allow analysis to be performed, but which retains features present in more general problems of interest.

Now by solving the test equation (6) via the TM, we obtain the following iterative expression:

$$X_{i+1} = \frac{(-1 + h\lambda(-1 + \theta))}{-1 + h\theta\lambda} X_i. \tag{10}$$

The recurrence relation (10) could express the stability regions of different one-step methods. For instance, for the case of explicit Euler method ($\theta = 0$), the semi-implicit trapezoidal method ($\theta = 1/2$), and the backward implicit Euler method ($\theta = 1$), we obtain, respectively:

$$X_{i+1} = (1 + h\lambda)X_i, \tag{11}$$

$$X_{i+1} = -\frac{(2 + h\lambda)}{-2 + h\lambda} X_i, \tag{12}$$

$$X_{i+1} = \frac{1}{1 - h\lambda} X_i. \tag{13}$$

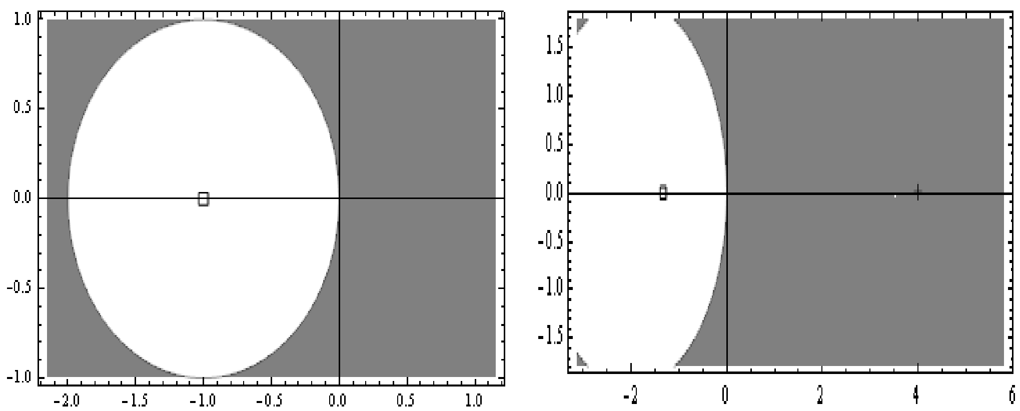

Figure 1: Stability region of TM for the cases of $\theta = 0$ (left) and $\theta = 0.25$ (right).

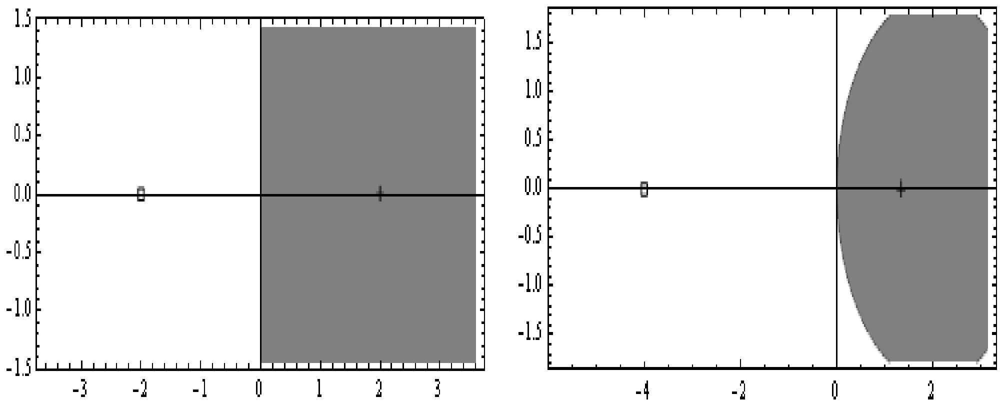

Figure 2: Stability region of TM for the cases of $\theta = 0.5$ (left) and $\theta = 0.75$ (right).

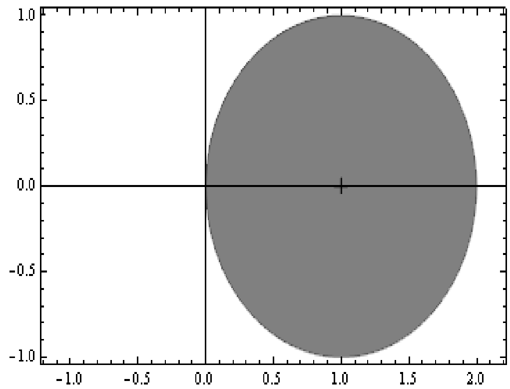

Figure 3: Stability region of TM for the case of $\theta = 1$.

The regions of stability for different methods extracted from the TM are illustrated in Figures 1-3. In the figures, the gray color refer to the instability areas. We remark that the Linear Stability Boundary, also known as LSB, is often considered as the intersection with the negative real axis. For the explicit Euler method, it is $LSB = -2$. As an instance, for an eigenvalue of $\lambda = -1$, linear stability requirements mean that the step size needs to satisfy $h < 2$, which is a very mild restriction.

It is observed that the TM for $\theta = 1$ is unconditionally stable for the entire left half-plane. The drawback is that a system of equations now has to be solved at each integration step. In fact, a root-finding algorithm must be embedded to the methods so as to extract the new approximated value X_{i+1} per cycle.

On the contrary, the low order of convergence for the TM makes its use limited in practical problems. To be more precise, it would be of obvious interest to contribute a computational scheme with higher convergence speed while its stability region is large enough to overcome the pitfalls of the higher order methods.

Accordingly, we wish to propose an algorithm for solving SDEs, which is of higher (strong) order and is implicit or consists of multi steps. This is the main motivation of this paper.

3. A Modification of the STM

Clearly, stiffness limits the effectiveness of explicit solution methods due to restrictions on the size of steps that can be taken. Hence, the purpose of this section is to present a variation of TM for solving SDEs as our main goal.

Generally speaking, there are two kinds of numerical stability for the SDE (1), namely, with respect to a sample path (strong sense), which is known as T-stability, and with respect to moments of the solution X_t (weak sense), which is called as M-stability [12].

However, the most important analysis is based on an extension of the test equation (6) to the stochastic environment. This procedure is based on applying the stochastic solver on the test equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{14}$$

which is an asset model in math-finance ($\Re(\mu) < 0$) and $X_0 = 1$, [2]. This is known as mean-square stability (MS-stability). This notion can be regarded as a stochastic version under the mean-square norm of the absolute stability for numerical methods of ordinary differential equations (ODEs).

The zero solution to (14) is said to be mean-square stable if $\lim_{t\to\infty} \mathbb{E}[|X_t|^2] = 0$. where $\mathbb{E}(\cdot)$ denotes the expected value, [9].

Recently, a deep study has been carried out for the STM on the test equation (14) in [15] and showed the relationship of the choice of θ and mean-square exponential stability in the STM. In fact, one could observe from [15] that under the linear growth condition for the drift term, when $\theta \in [0, 1/2)$, the STM may preserve the mean-square exponential stability of the exact solution, but the counterexample shows that the STM cannot reproduce this stability without this condition. In the meantime, once $\theta \in (1/2, 1)$, without the linear growth condition for the drift term, the STM can create the mean-square exponential stability of the exact solution, but the bound of the Lyapunov exponent cannot be preserved. And lastly, with $\theta = 1$, the STM can reproduce not only the mean-square exponential stability, but also the bound of the Lyapunov exponent.

Accordingly, we now study the following computational solver [1]

$$\begin{aligned} X_{i+1} &= X_i + (1-\theta) f(t_i, X_i)\Delta t_i + \theta f(t_i, X_{i+1})\Delta t_i + g(t_i, X_i)\Delta W_i \\ &\quad + \frac{1}{2} g(t_i, X_i) \frac{\partial g}{\partial x}(t_i, X_i)\left(\Delta W_i^2 - \Delta t_i\right), \end{aligned} \tag{15}$$

where $\theta \in [0, 1]$.

The interesting point in this computational scheme is that for the cases $\theta \in (0, 1]$, the extracted method is implicit and possess larger stability regions. Note that a similar reasoning as in Section 2 is valid for the application of (15) on the test equation (14).

Now, we have a revisit of the STM which has two strong points. First, it is implicit with a better stability region for the case $\theta = 1$ and second it is of higher computational order $\gamma = 1$ in the strong sense. This is recognizable since for the case of $\theta = 0$, (15) is simplified to the well-known method of Milstein discussed in [11].

The scheme (15) with $\theta = 1$ obeys of the dynamical (MS-stability) question that does $\lim_{i\to\infty} X_i$ look like $\lim_{t\to\infty} X_t$?

Before going on, we recall that if $Z_i \sim N(0, 1)$, then

$$\mathbb{E}[Z_i^n] = \begin{cases} 0, & n \text{ is odd}, \\ (n-1)\times(n-3)\times\cdots\times 1, & n \text{ is even}. \end{cases} \tag{16}$$

Lemma 3.1. *Let the drift and diffusion terms of an SDE are sufficiently smooth, measurable and satisfy the (usual) Lipshictiz condition with linear growth condition and finite second moment. Choosing $\theta \in [0, 1]$ in (15), the method is MS-stable when*

$$|y|(|y| + 2) < 2(\theta x - 1)((x\theta)^* - 1) - 2((\theta - 1)x - 1)((\theta^* - 1)x^* - 1), \tag{17}$$

where $x = \Delta t\mu$, and $y = \Delta t\sigma^2$ and $*$ stands for the conjugate of a complex number.

Proof. To provide the MS-stability region of (15) analytically, we first recall that the SDE (14) with $X_0 = 1$ is MS-stable if

$$\lim_{t\to\infty} E[(X_t)^2] = 0 \Leftrightarrow 2\Re(\mu) + |\sigma|^2 < 0. \tag{18}$$

On the other hand, applying (15) on (14) results in the following recurrence relation

$$X_{i+1} = \frac{\left(-2 + 2\Delta(-1+\theta)\mu - 2Z_i\sqrt{\Delta}\sigma - \left(-1 + Z_i^2\right)\Delta\sigma^2\right)}{-2 + 2\Delta\theta\mu} X_i, \tag{19}$$

wherein $Z_i \sim N(0,1)$. Now, the (Itô) stability function for the stochastic computational scheme can be deduced as

$$S(Z_i) = \frac{\left(-2 + 2\Delta(-1+\theta)\mu - 2Z_i\sqrt{\Delta}\sigma - \left(-1 + Z_i^2\right)\Delta\sigma^2\right)}{-2 + 2\Delta\theta\mu}. \tag{20}$$

Now we choose

$$a = \frac{-2 - 2x + 2\,\theta x + y}{-2 + 2\,\theta x}, \tag{21}$$

$$b = \frac{-2\,y^{1/2}}{-2 + 2\,\theta x}, \tag{22}$$

and

$$c = \frac{-y}{-2 + 2\,\theta x}, \tag{23}$$

where $x = \Delta t\mu$, and $y = \Delta t\sigma^2$ based on (20). Accordingly for the MS-stability, we could write

$$\begin{aligned} \mathbb{E}[|X_{i+1}|^2] &= \mathbb{E}[(a + bZ_i + cZ_i^2)(a^* + b^*Z_i + c^*Z_i^2)|X_i|^2] \\ &= (|a+c|^2 + |b|^2 + 2|c|^2)\mathbb{E}[|X_i|^2]. \end{aligned} \tag{24}$$

This yields to

$$|a+c|^2 + |b|^2 + 2|c|^2 < 1. \tag{25}$$

Therefore, we acquire

$$\frac{|y|(|y|+2) + 2((\theta-1)x-1)((\theta^*-1)x^*-1)}{2(\theta x-1)((x\theta)^*-1)} < 1, \tag{26}$$

which confirms the theoretical bound (17). This ends the proof. □

It should be remarked that if (14) is mean-square stable, then it is automatically asymptotic stable. In addition, our analysis shows that the stability region tends to be improved by increasing the value of θ even outside the range of $[0,1]$, i.e., even choosing $\theta = 2$ results in a method with better stability behavior in terms of x and y.

To clearly show the regions of stability, we illustrate these areas in terms of x-axis and y-axis in Figures 4-5 in contrast to the STM with $\theta = 0$ in Figure 4 (left). It is obvious that the fully implicit method (15) has a larger domain of MS-stability. Note that the region of stability in Figures 4-5 are colorized in light green.

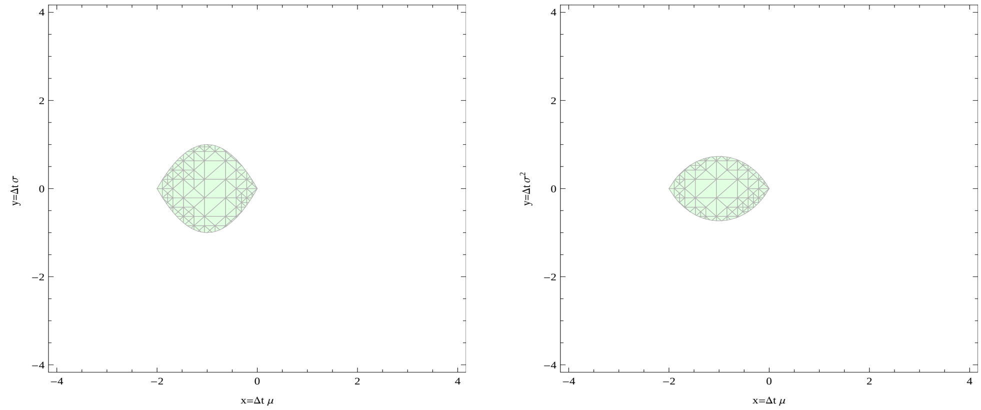

Figure 4: Stability region of STM for $\theta = 0$ (left) and (15) for $\theta = 0$ (right).

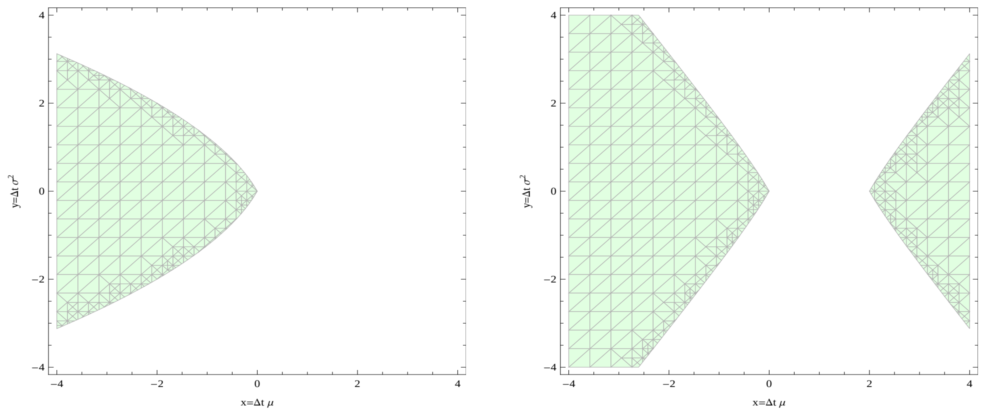

Figure 5: Stability region of (15) for the semi-implicit case of $\theta = 1/2$ (left) and the fully implicit case of $\theta = 1$ (right).

Now we are able to propose our new three-step scheme for solving SDEs according to (15) in what follows ($\theta \neq 0$ and better to be chosen as $\theta = 1$):

$$
\begin{aligned}
\widehat{X}_i &= X_i + f(t_i, X_i)\Delta t_i + g(t_i, X_i)\Delta W_i, \\
\overline{X}_i &= X_i + (1-\theta) f(t_i, X_i)\Delta t_i + \theta f(t_i, \widehat{X}_i)\Delta t_i + g(t_i, X_i)\Delta W_i, \\
X_{i+1} &= X_i + (1-\theta) f(t_i, X_i)\Delta t_i + \theta f(t_i, \overline{X}_i)\Delta t_i + g(t_i, X_i)\Delta W_i \\
&\quad + \frac{1}{2} g(t_i, X_i)\frac{\partial g}{\partial x}(t_i, X_i)\left(\Delta W_i^2 - \Delta t_i\right).
\end{aligned}
\tag{27}
$$

To provide higher strong order schemes, we propose a theta-version of the high order scheme of Kloeden et al. [6] for scalar SDEs with $\frac{3}{2}$ strong order as follows:

$$\begin{aligned} X_{i+1} \;=\; & X_i + (1-\theta) f(t_i, X_i)\Delta t_i + \theta f(t_i, X_{i+1})\Delta t_i + g(t_i, X_i)\Delta W_i \\ & + \frac{1}{2} g(t_i, X_i)\frac{\partial g}{\partial x}(t_i, X_i)\left(\Delta W_i^2 - \Delta t_i\right) \\ & + \frac{\partial f}{\partial x}(t_i, X_i) g(t_i, X_i)\Delta Z_i + \frac{1}{2}\left(f(t_i, X_i)\frac{\partial f}{\partial x}(t_i, X_i) + \frac{1}{2} g(t_i, X_i)^2 \frac{\partial^2 f}{\partial x^2}(t_i, X_i)\right)\Delta t_i^2 \\ & + \left(f(t_i, X_i)\frac{\partial g}{\partial x}(t_i, X_i) + \frac{1}{2} g(t_i, X_i)^2 \frac{\partial^2 g}{\partial x^2}(t_i, X_i)\right)(\Delta W_i \Delta t_i - \Delta Z_i) \\ & + \frac{1}{2} g(t_i, X_i)\left(g(t_i, X_i)\frac{\partial^2 g}{\partial x^2}(t_i, X_i) + \left(\frac{\partial g}{\partial x}(t_i, X_i)\right)^2\right)\left(\frac{1}{3}\Delta W_i^2 - \Delta t_i\right)\Delta W_i, \end{aligned} \tag{28}$$

wherein the additional random variable ΔZ_i is normally distributed with mean 0, variance $\mathbb{E}(\Delta Z_i^2) = \frac{1}{3}\Delta t_i^3$ and correlated with ΔW_i with covariance $\mathbb{E}(\Delta Z_i \Delta W_i) = \frac{1}{2}\Delta t_i^2$. Note that ΔZ_i can be generated as $\Delta Z_i = \frac{1}{2}\Delta t_i\left(\Delta W_i + \Delta V_i/\sqrt{3}\right)$, wherein ΔV_i is chosen independently from $\sqrt{\Delta t_i}N(0,1)$. This implicit scheme is named as KSTM.

We provide our lemma for the mean square stability of (28) in what follows.

Lemma 3.2. *Consider similar assumptions as in Lemma 3.1. Choosing $\theta \in [0,1]$ in the derived SDE solver (28), the method is MS-stable when*

$$18\left|\frac{y}{6-6x\theta}\right|^2 + 6\left|\frac{y^{3/2}}{6-6x\theta}\right|^2 + \left|\frac{2+2x+x^2-2x\theta}{2-2x\theta}\right|^2 + \left|\frac{(1+x)\sqrt{y}}{-1+x\theta}\right|^2 < 1, \tag{29}$$

where $x = \Delta t\mu$, and $y = \Delta t\sigma^2$.

Proof. Just like Lemma 3.1, applying (28) on (14) results in

$$X_{i+1} = -\frac{\left(6 + 3\Delta\mu(2 - 2\theta + \Delta\mu) + 6Z_i\sqrt{\Delta}(1+\Delta\mu)\sigma + 3\left(-1 + Z_i^2\right)\Delta\sigma^2 + Z_i\left(-3 + Z_i^2\right)\Delta^{3/2}\sigma^3\right)}{-6 + 6\Delta\theta\mu} X_i. \tag{30}$$

Now, the Itô stability function for KSTM can be written as

$$P(Z_i) = -\frac{\left(6 + 6\Delta\mu - 6\Delta\theta\mu + 3\Delta^2\mu^2 + 6Z_i\sqrt{\Delta}\sigma + 6Z_i\Delta^{3/2}\mu\sigma - 3\Delta\sigma^2 + 3Z_i^2\Delta\sigma^2 - 3Z_i\Delta^{3/2}\sigma^3 + Z_i^3\Delta^{3/2}\sigma^3\right)}{6(-1 + \Delta\theta\mu)}. \tag{31}$$

Now we choose

$$a = \frac{6 + 6x - 6\,\theta x + 3x^2 - 3y}{6 - 6\theta x}, \tag{32}$$

$$b = \frac{6\,y^{1/2} + 6\,xy^{1/2} - 3\,y^{3/2}}{6 - 6\,\theta x}, \tag{33}$$

$$c = \frac{3\,y}{6 - 6\,\theta x}, \tag{34}$$

and

$$d = \frac{y^{3/2}}{6 - 6\,\theta x}, \tag{35}$$

where $x = \Delta t\mu$, and $y = \Delta t\sigma^2$. Similarly, we should study the regions at which $|P(Z_i)| < 1$. Accordingly for the MS-stability, we could write

$$\begin{aligned} \mathbb{E}[|X_{i+1}|^2] &= \mathbb{E}[(a + bZ_i + cZ_i^2 + dZ_i^3)(a^* + b^*Z_i + c^*Z_i^2 + d^*Z_i^3)|X_i|^2] \\ &= (|a + c|^2 + |b + 3d|^2 + 2|c|^2 + 6|d|^2)\mathbb{E}[|X_i|^2]. \end{aligned} \tag{36}$$

This yields to

$$|a + c|^2 + |b + 3d|^2 + 2|c|^2 + 6|d|^2 < 1. \tag{37}$$

Therefore, we acquire the theoretical bound (29). This ends the proof. □

We have drawn the MS-stability regions of the proposed implicit solver (28) in Figures 6-7. It is seen that although the strong order is 3/2, the stability region will not improve significantly by choosing even $\theta = 3/2$ which is outside the interval [0,1] for KSTM.

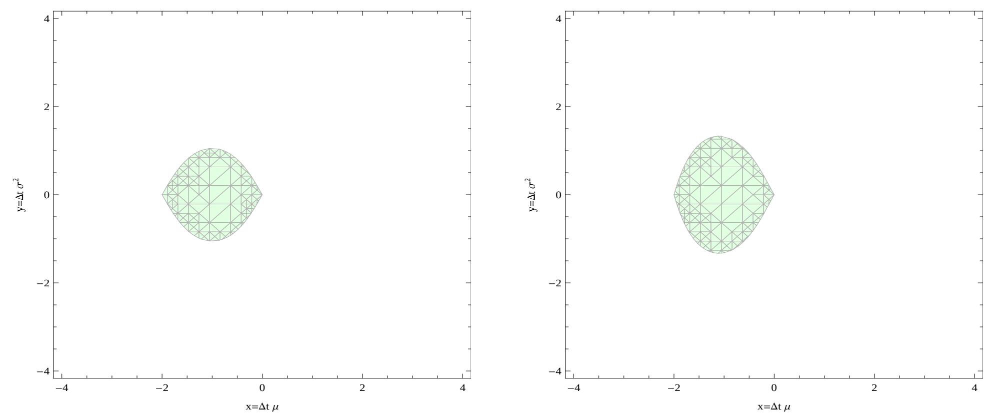

Figure 6: Stability region of KSTM for $\theta = 0$ (left) and $\theta = 1/2$ (right).

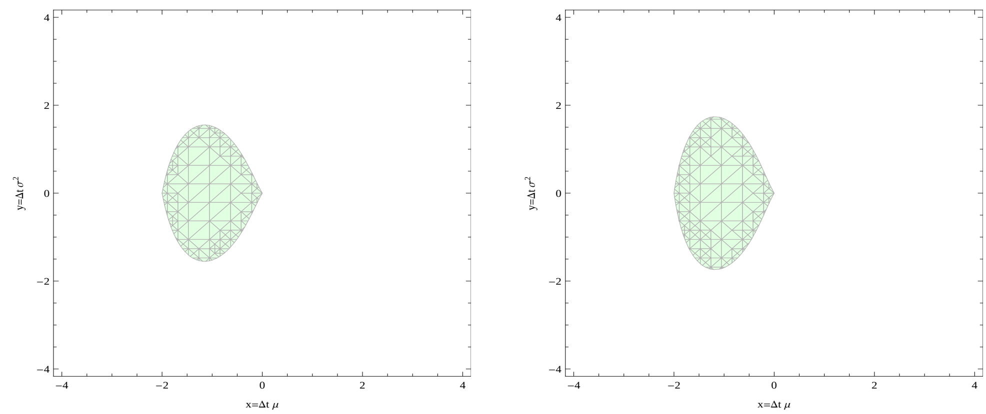

Figure 7: Stability region of KSTM for $\theta = 1$ (left) and $\theta = 3/2$ (right).

We can observe the computational order of convergence for (15) and (28) numerically in the next section using the dependance of the global truncation error on the step size. Toward this goal, we will use the fact that a function $f(\Delta) = A\Delta^\gamma$ becomes linear in logarithmic coordinates. That is,

$$\log_a f(\Delta) = \log_a A + \gamma \log_a \Delta, \tag{38}$$

for logarithms to the base $a \neq 1$. In this work, and so as to check the convergence order, we take steps of the form $\Delta = a^{-k}$ for $k = 1, 2, \ldots$, and $a > 1$.

This will be the goal of the next section to inquire into the computational order and efficiency of the presented variation of the modified STMs.

4. Experimental Results

Here, we show the numerical behavior of the methods extracted from (15) along with (27) for various choices of θ on some standard and hard nonlinear SDEs. By STM1, STM2, MSTM1, MSTM2, MSTM3, we refer to (3) with $\theta = 0$, (3) with $\theta = 1$, (15) with $\theta = 0$, (15) with $\theta = 1$ and (27), respectively. We furthermore do not include (28) in the numerical comparisons since by choosing values of θ in $(0, 1]$ the stability region would not improve in an important manner.

In the tests made, double precision arithmetics has been used and 2^{12} increments are used for reproducing the Wiener processes. We employ the computer algebra system Mathematica 8 in our calculations.

To compare the methods from the strong point of view, we report the mean values of the (absolute) error $e(T) = |y(T) - Y(T)|$ at the end point T over $k = 500$ realizations.

Example 4.1. *As the first example, we take into account (14) with* $X(t_0) = 1$, *while the exact solution* $X_t = \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)$, $0 \leq t \leq 1$, $\mu = 4$ *and* $\sigma = 3$.

In this experiment, the EM is stable only if $\Delta t < \frac{17}{16}$. The results of comparisons are brought forward in Table 1. The fits for the two-logarithm of the errors by decreasing the step sizes are given in Table 2 to show the slope of 1 for the MSTM1-MSTM3 in terms of strong convergence.

	STM1	STM2	MSTM1	MSTM2	MSTM3
Δt\Error					
2^{-6}	2.894×10^{-2}	8.285×10^{-1}	7.025×10^{-2}	3.911×10^{-2}	6.561×10^{-2}
2^{-7}	6.734×10^{-2}	1.020×10^{-1}	8.553×10^{-2}	4.953×10^{-2}	4.962×10^{-2}
2^{-8}	6.935×10^{-2}	8.019×10^{-2}	2.561×10^{-2}	2.561×10^{-2}	2.369×10^{-2}
2^{-9}	4.636×10^{-2}	5.253×10^{-2}	1.834×10^{-2}	9.774×10^{-3}	9.247×10^{-3}
2^{-10}	2.227×10^{-2}	2.574×10^{-2}	4.626×10^{-3}	6.484×10^{-4}	5.480×10^{-4}
2^{-11}	5.847×10^{-3}	3.867×10^{-3}	2.702×10^{-3}	7.794×10^{-4}	3.886×10^{-4}
2^{-12}	2.524×10^{-3}	1.551×10^{-3}	1.282×10^{-3}	3.212×10^{-4}	1.231×10^{-4}

Table 1: Comparisons of mean results for Example 4.1.

STM1	STM2	MSTM1	MSTM2	MSTM3
$-3.03 + 0.43t$	$6.57 + 0.80t$	$3.60 + 1.09t$	$0.91 + 1.00t$	1.70+ 1.08t

Table 2: The linear fit for 2-logarithm of (mean) errors in Example 4.1.

Example 4.2. *Consider the nonlinear SDE*

$$\begin{aligned} dX_t &= \tfrac{1}{3} X_t^{\frac{1}{3}} dt + \tfrac{1}{3} X_t^{\frac{2}{3}} dW_t, \\ X_0 &= 1, \end{aligned} \tag{39}$$

with the exact solution $X_t = \left(X_0^{1/3} + \frac{1}{3} W_t\right)^3$, *wherein* $0 \leq t \leq 1$.

	STM1	STM2	MSTM1	MSTM2	MSTM3
Δt\Error					
2^{-7}	2.086×10^{-2}	1.999×10^{-2}	3.172×10^{-4}	5.787×10^{-4}	2.669×10^{-3}
2^{-8}	1.659×10^{-2}	1.665×10^{-2}	1.314×10^{-4}	1.886×10^{-4}	1.360×10^{-3}
2^{-9}	2.737×10^{-2}	2.755×10^{-2}	1.000×10^{-4}	7.441×10^{-5}	6.055×10^{-4}
2^{-10}	4.306×10^{-3}	4.407×10^{-3}	1.724×10^{-4}	7.294×10^{-5}	7.444×10^{-5}
2^{-11}	1.653×10^{-3}	1.653×10^{-3}	2.430×10^{-5}	2.661×10^{-5}	5.240×10^{-5}
2^{-12}	1.839×10^{-3}	1.864×10^{-3}	1.377×10^{-5}	1.100×10^{-5}	2.513×10^{-5}

Table 3: Comparisons of mean results for Example 4.2.

STM1	STM2	MSTM1	MSTM2	MSTM3
$-1.10 + 0.57t$	$-1.21 + 0.56t$	$-5.71 + 0.83t$	$-3.64 + 1.05t$	1.78+ 1.02t

Table 4: The linear fit for 2-logarithm of (mean) errors in Example 4.2.

The results for this experiment are provided in Table 3. Furthermore, we have found the best linear fit for the two-logarithm of the errors by decreasing the step sizes again. These results are shown in Table 4. The slope of the lines clearly show $\gamma = 1$ for MSTM2 and MSTM3.

The numerical experiments were based on enough sample paths ($k = 500$) of the driving Brownian motion and are therefore of relevance with respect to investigating mean squared error rates.

However, to further show the consistent behavior of MSTM2 and MSTM3 in contrasts to the other schemes, we provide Figure 8 at which comparisons of the exact solution of numerical solution only for one sample path are given when $\Delta t = 1/4$. Results show a good agreement between the theoretical discussion and the numerical behaviors for MSTM3.

From the numerical comparisons given in Tables 1-4, it is clear that the accuracy in computing the approximated solution at the time T increases, showing stable nature of our proposed variations. Also, the presented implicit methods show consistent convergence behavior. The proposed method MSTM3 widely possess the numerical rate of convergence $\gamma = 1$.

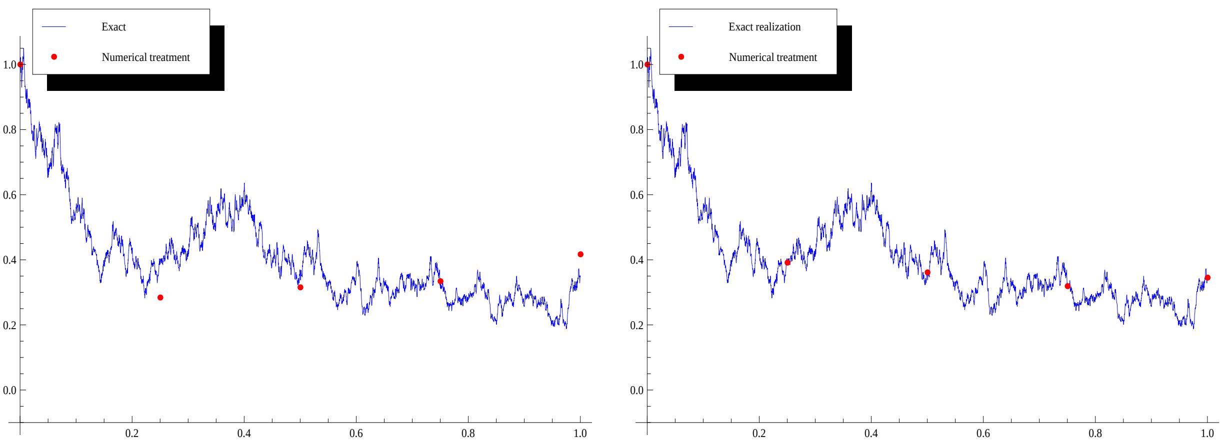

Figure 8: Comparison of one sample solution with the numerical treatments obtained from STM1 (left) and the MSTM2 (right).

5. Conclusions

Several authors (see e.g. [7] and the references therein) proposed various numerical schemes for the SDE (1), which recursively compute sample paths (trajectories) of solution X_t at the grid points. In this paper, we have tried to motivate our goal by finding a solver at which the convergence is high in the strong sense while it is implicit. This was done in the Section 3 when we discussed some modified versions of the STM.

Summarizing, we can conclude that the novel computational scheme (by varying $0 < \theta \leq 1$) has a charming performance for solving Itô-type SDEs. In the applied examples, the new computational methods showed good stability and precision in the results.

There are many open questions regarding (asymptotic) stability of the scheme (27) and (28). For example, are the generalizations of our results to nonlinear SDEs possible? Investigating these queries could be considered for future studies.

Acknowledgement

The authors thank to anonymous referees for their valuable comments and for the suggestions to improve the readability of the paper.

References

[1] D.J. Higham, A-stability and stochastic mean-square stability, BIT, 40 (2000), 404-409.
[2] D.J. Higham, Mean-square and asymptotic stability of the stochastic theta method, SIAM J. Numer. Anal., 38 (2000), 753-769.
[3] S.M. Iacus, Simulation and Inference for Stochastic Differential Equations with *R* Examples, Springer, New York, USA, 2008.
[4] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, Berlin, 1999.
[5] P.E. Kloeden, E. Platen, H. Schurz, Numerical Solution of SDE through Computer Experiments, Springer-Verlag, Germany, 2003.
[6] P.E. Kloeden, E. Platen, H. Schurz, M. Sørensen, On effects of discretization on estimators of drift parameters for diffusion processes, J. Appl. Prob. 33 (1996), 1061-1076.
[7] Y. Komori, Y. Saito, and T. Mitsui, Some issues in discrete approximate solution for stochastic differential equations, Comput. Math. Appl., 28 (1994), 269-278.
[8] X. Mao, Exponential stability of equidistant Euler-Marauyama approximations of stochastic differential delay equations, J. Comput. Appl. Math. 200 (2007), 297-316.
[9] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.
[10] M. Milošević, On the approximations of solutions to stochastic differential delay equations with Poisson random measure via Taylor series, Filomat, 27 (2013), 201-214.
[11] G. Milstein, Numerical Integration of Stochastic Differential Equations, Dordrecht, Kluwer, 1995.
[12] Y. Saito, T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal., 33 (1996), 2254-2267.
[13] W. Wang, L. Wen, S. Li, Nonlinear stability of θ-methods for neutral differential equations in Banach space, Appl. Math. Comput. 198 (2008), 742-753.
[14] Wolfram Mathematica, Tutorial Collection: Advanced Numerical Differential Equation Solving in Mathematica, ISBN: 978-1-57955-058-5, 2008.
[15] X. Zong, F. Wu, Choice of θ and mean-square exponential stability in the stochastic theta method of stochastic differential equations, J. Comput. Appl. Math., 255 (2014), 837-847.