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On the NBVP for Semilinear Hyperbolic Equations

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Abstract. This paper is concerned with establishing the solvability of the nonlocal boundary value problem for the semilinear hyperbolic equation in a Hilbert space. For the approximate solution of this problem, the first order of accuracy difference scheme is presented. Under some assumptions, the convergence estimate for the solution of this difference scheme is obtained. Moreover, these results are supported by a numerical example.

1. Introduction

Second order linear and semilinear hyperbolic equations are of keen interest in fluid dynamics, acoustics, mathematical physics, electromagnetic, etc. The article [1] investigates the existence and uniqueness of weak solutions for the semilinear degenerate hyperbolic Goursat problem. The authors of [2] study the blow-up property of weak solutions to an initial and boundary value problem for models many real physical problems such as viscoelastic fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity, etc. In paper [3], in the conic domain, the existence or nonexistence of global solutions of a multidimensional version of the first Darboux problem for wave equations with power nonlinearity is investigated. For the approximate solution of such types of partial differential equations efforts are being made to develop efficient and high accuracy finite difference methods (see [4]-[10] and the references therein). The authors of [11] investigate a nonlocal boundary value problem for semilinear hyperbolic-parabolic equations in a Hilbert space. The first and second order accuracy difference schemes approximately solving this problem are studied. The convergence estimates for the solution of these difference schemes are obtained. In paper [12], the unique solvability of local and nonlocal boundary value problems for the semilinear Schrödinger equation in a Hilbert space is investigated. The convergence estimates for the solution of difference schemes are established. The authors of [13] study the initial value problem for the semilinear integral-differential equation of the hyperbolic type. The convergence estimates for the solutions of the first and second order of accuracy difference schemes are obtained.

In the present paper, we consider the nonlocal boundary value problem for semilinear hyperbolic equation

$$\begin{aligned} \frac{d^2u(t)}{dt^2} + Au(t) &= f(t, u(t)), \quad 0 \le t \le T, \\ u'(0) &= \rho u'(T) + \psi, \\ u(0) &= \sigma u(T) + \varphi \end{aligned} \tag{1}$$

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in a Hilbert space H with the self-adjoint positive definite linear operator $A \geq \delta I$ ($\delta > 0$), under the assumption

$$\left|\sigma\rho + 1\right| > |\sigma| + \left|\rho\right| \tag{2}$$

where $f(t, u(t))$ is the given function, ρ and σ are constants, $\varphi \in D(A)$ and $\psi \in D(A^{1/2})$ are given initial value functions.

A function $u(t)$ is the solution of problem (1) if the following conditions are satisfied:

i) $u(t)$ is twice continuously differentiable on the interval $(0, T)$, continuously differentiable on the segment $[0, T]$ and satisfies the equation and nonlocal boundary conditions of problem (1).

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, T]$, and the function $Au(t)$ is continuous on the segment $[0, T]$.

In this work, we prove that problem (1) has a unique solution in $C([0, T], H)$. For the approximate solution of (1), the first order of accuracy difference scheme is presented. Under some assumptions, the convergence estimate for the solution of this difference scheme is obtained. To validate the main results, this difference scheme is applied to the one dimensional semilinear hyperbolic problem.

2. Existence and Uniqueness

In this section, the uniqueness of the solution of semilinear hyperbolic problem (1) is considered. First of all, let us prove that the problem (1) has a unique solution in $C([0, T], H)$. Here, $C([0, T], H)$ is the space of all continuous function, $v(t)$ defined on the interval $[0, T]$ in Hilbert space H. We use the norm

$$\|v\|_{C([0,T],H)} = \max_{0\leq t\leq T} \|v(t)\|_H . \tag{3}$$

Strongly continuous cosine and sine operator functions $\{c(t), s(t), t \geq 0\}$, see [15, 16], are defined by formulas

$$c\,(t) = \frac{e^{-itA^{1/2}} + e^{itA^{1/2}}}{2}, \quad s\,(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

Lemma 2.1. *The following estimates hold:*

$$\|c(t)\|_{H\to H} \leq 1, \quad \left\|A^{1/2}s(t)\right\|_{H\to H} \leq 1, \quad t \geq 0. \tag{4}$$

Lemma 2.2. *Assume that the assumption (2) is satisfied. Then, operator*

$$P = [1 + \sigma\rho]\, I - (\rho + \sigma)\, c(T)$$

has an inverse and

$$\|P^{-1}\|_{H\to H} \leq \frac{1}{\left|\sigma\rho + 1\right| - \left|\sigma + \rho\right|} \tag{5}$$

is satisfied.

Proof. Applying the estimates (4) and triangle inequality, we obtain

$$\|P\|_{H\to H} = \left\|[1 + \sigma\rho]\, I - (\sigma + \rho)\, c(T)\right\|_{H\to H} \geq \left|1 + \sigma\rho\right| - \left|\sigma + \rho\right| > 0.$$

Estimate (5) follows from this estimation. □

Theorem 2.3. *Assume that f is continuous function in $[0, T] \times H$ and there exists $K > 0$ such that f satisfies the Lipschitz condition:*

$$\left\|A^{-\frac{1}{2}}[f(t, v) - f(t, u)]\right\|_H \leq K\, \|v - u\|_H\, , \quad t \in [0, T] \tag{6}$$

for all $u, v \in H$. Let the assumption (2) and

$$KT\left(1+\frac{4|\sigma||\rho|+|\sigma|+|\rho|}{|\sigma\rho+1|-|\sigma+\rho|}\right)<1 \tag{7}$$

hold. Then, the nonlocal boundary value problem (1) has a unique solution in $C([0,T],H)$.

Proof. For the solution of (1), we have (see [17])

$$\begin{aligned} u(t) &= c(t)P^{-1}\left\{[I-\rho c(T)]\left[\sigma\int_0^T s(T-\lambda)f(\lambda,u(\lambda))d\lambda+\varphi\right]\right. \\ &\left.+\sigma s(T)\left[\rho\int_0^T c(T-\lambda)f(\lambda,u(\lambda)))d\lambda+\psi\right]\right\} \\ &+s(t)P^{-1}\left\{[I-\sigma c(T)]\left[\rho\int_0^T c(T-\lambda)f(\lambda,u(\lambda)))d\lambda+\psi\right]\right. \\ &\left.-\rho As(T)\left[\sigma\int_0^T s(T-\lambda)f(\lambda,u(\lambda)))d\lambda+\varphi\right]\right\}+\int_0^t s(t-\lambda)f(\lambda,u(\lambda))d\lambda. \end{aligned}$$

We denote the right-hand side by operator $Fu(t)$, which maps $C([0,T],H)$ into $C([0,T],H)$. We can prove that F is a contraction operator on $C([0,T],H)$. By using (4) and Lipschitz condition (6), we get

$$\begin{aligned} \|Fv(t)-Fu(t)\|_H &\le \left\|P^{-1}\right\|_{H\to H}\left\{\left(1+|\rho|\right)|\sigma|\int_0^T\left\|A^{-1/2}\left[f(\lambda,v(\lambda))-f(\lambda,u(\lambda))\right]\right\|_H d\lambda\right. \\ &+|\sigma||\rho|\int_0^T\left\|A^{-1/2}\left[f(\lambda,v(\lambda))-f(\lambda,u(\lambda))\right]\right\|_H d\lambda \\ &+(1+|\sigma|)|\rho|\int_0^T\left\|A^{-1/2}\left[f(\lambda,v(\lambda))-f(\lambda,u(\lambda))\right]\right\|_H d\lambda \\ &\left.+|\sigma||\rho|\int_0^T\left\|A^{-1/2}\left[f(\lambda,v(\lambda))-f(\lambda,u(\lambda))\right]\right\|_H d\lambda\right\} \\ &+\int_0^T\left\|A^{-1/2}\left[f(\lambda,v(\lambda))-f(\lambda,u(\lambda))\right]\right\|_H d\lambda \\ &\le\left[1+\left\|P^{-1}\right\|_{H\to H}\left(|\sigma|+|\rho|+4|\sigma||\rho|\right)\right]\int_0^T\left\|A^{-1/2}\left[f(\lambda,v(\lambda))-f(\lambda,u(\lambda))\right]\right\|_H d\lambda \end{aligned}$$

for any $t \in [0,T]$. So,

$$\|Fu-Fv\|_{C([0,T],H)} \le \alpha\|u-v\|_{C([0,T],H),} \quad \text{where} \quad \alpha=KT\left(1+\frac{4|\sigma||\rho|+|\sigma|+|\rho|}{|\sigma\rho+1|-|\sigma+\rho|}\right).$$

From (7) it follows that F is a contraction operator on $C([0,T],H)$. Then by the fixed point theorem, the nonlocal boundary value problem (1) has a unique solution in $C([0,T],H)$. □

3. The First Order of Accuracy Difference Scheme

For the approximate solution of the nonlocal boundary value problem (1), we construct the first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_{k+1}=f\left(t_k,u_k\right), \ \ t_k=k\tau,\ 1\le k\le N-1,\ \ N\tau=T, \\ u_0=\sigma u_N+\varphi,\ \ \left(I+\tau^2A\right)\tau^{-1}\left(u_1-u_0\right)=\rho\tau^{-1}(u_N-u_{N-1})+\psi. \end{cases} \tag{8}$$

The following theorem is about the uniqueness of the solution $u^\tau=\{u_k\}_{k=1}^N$ of difference scheme (8) in the space of grid functions $C\left([0,T]_\tau,H\right)$ with the norm

$$\|u^\tau\|_{C([0,T]_\tau,H)}=\max_{1\le k\le N}\|u_k\|_H\,. \tag{9}$$

Lemma 3.1. *Let R, $\tilde{R}$, and P_τ be defined as*

$$R=\left(I+i\tau A^{1/2}\right)^{-1},\ \tilde{R}=\left(I-i\tau A^{1/2}\right)^{-1},$$

$$P_\tau=I-\frac{\sigma}{2}\left(R^{N-1}+\tilde{R}^{N-1}\right)-\frac{\rho}{2}\left(R^N+\tilde{R}^N\right)+\sigma\rho R^N\tilde{R}^N.$$

The estimates hold:

$$\left\|P_\tau^{-1}\right\|_{H\to H}\le\frac{1}{1-|\sigma|\left|\rho\right|-|\sigma|-\left|\rho\right|},\ \ \|R\|_{H\to H}\le1,\ \left\|\tilde{R}\right\|_{H\to H}\le1,\ \left\|\tau A^{1/2}R\right\|_{H\to H}\le1,\ \left\|\tau A^{1/2}\tilde{R}\right\|_{H\to H}\le1. \tag{10}$$

Theorem 3.2. *Assume that f satisfies (6) in space $C\left([0,T]_\tau,H\right)$ and assumption (7) holds. Then the difference scheme (8) has a unique solution in $C\left([0,T]_\tau,H\right)$.*

Proof. The proof of this theorem is similar to the Theorem 2.3 and based on the operator F which is defined on the space $C\left([0,T]_\tau,H\right)$ by the help of [17]

$$\begin{aligned} Fu_0 \ &= \ P_\tau^{-1}\left\{\left[I-\rho\left(R-\tilde{R}\right)^{-1}\frac{\tau}{i}A^{1/2}\left(R^N+\tilde{R}^N\right)\right]R\tilde{R}\left[-\sigma\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left(R^{N-s}-\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\varphi\right]\right. \\ &\quad \left.+\ \sigma\left(R-\tilde{R}\right)^{-1}\tau\left(R^N-\tilde{R}^N\right)R\tilde{R}\left[-\rho\sum_{s=1}^{N-2}\frac{\tau}{4}\left(R^{N-s}+\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\rho\tau R\tilde{R}f\left(t_{N-1},u_{N-1}\right)+\psi\right]\right\}, \end{aligned}$$

$$\begin{aligned} Fu_1 \ &= \ P_\tau^{-1}\left\{\left[I-\rho\left(R-\tilde{R}\right)^{-1}\frac{\tau}{i}A^{1/2}\left(R^N+\tilde{R}^N\right)\right]R\tilde{R}\left[-\sigma\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left(R^{N-s}-\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\varphi\right]\right. \\ &\quad \left.+\ \sigma\left(R-\tilde{R}\right)^{-1}\tau\left(R^N-\tilde{R}^N\right)R\tilde{R}\left[-\rho\sum_{s=1}^{N-2}\frac{\tau}{4}\left(R^{N-s}+\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\rho\tau R\tilde{R}f\left(t_{N-1},u_{N-1}\right)+\psi\right]\right\} \\ &\quad +\tau P_\tau^{-1}R\tilde{R}\left\{\left[I-\sigma\left[\frac{1}{2}\left(R^{N-1}+\tilde{R}^{N-1}\right)\right]\right]\left[-\rho\sum_{s=1}^{N-2}\frac{\tau}{4}\left(R^{N-s}+\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\rho\tau R\tilde{R}\right.\right. \\ &\quad \left.\left.\times\ f\left(t_{N-1},u_{N-1}\right)+\psi\right]+\rho\frac{A^{1/2}}{2i}\left(R^{N-1}-\tilde{R}^{N-1}\right)\left[-\sigma\sum_{s=1}^{N-1}\frac{\tau A^{-1/2}}{2i}\left(R^{N-s}-\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\varphi\right]\right\}, \end{aligned}$$

$$
\begin{aligned}
Fu_k &= \frac{1}{2}\left(R^{k-1}+\tilde{R}^{k-1}\right)P_\tau^{-1}\left\{\left[I-\rho\left(R-\tilde{R}\right)^{-1}\frac{\tau}{i}A^{1/2}\left(R^N+\tilde{R}^N\right)\right]R\tilde{R}\right.\\
&\times\left[-\sigma\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left(R^{N-s}-\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\varphi\right]+\sigma\left(R-\tilde{R}\right)^{-1}\tau\left(R^N-\tilde{R}^N\right)R\tilde{R}\\
&\times\left.\left(-\frac{\rho}{4}\sum_{s=1}^{N-2}\tau\left(R^{N-s}+\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\rho\tau R\tilde{R}f\left(t_{N-1},u_{N-1}\right)+\psi\right)\right\}+\tau\left(R-\tilde{R}\right)^{-1}R\tilde{R}\\
&\times\left(R^k-\tilde{R}^k\right)P_\tau^{-1}\left\{\left[I-\sigma\left(\frac{1}{2}\left(R^{N-1}+\tilde{R}^{N-1}\right)\right)\right]\left[-\rho\sum_{s=1}^{N-2}\frac{\tau}{4}\left(R^{N-s}+\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)\right.\right.\\
&\left.+\rho\tau R\tilde{R}f\left(t_{N-1},u_{N-1}\right)+\psi\right]+\rho\frac{1}{2i}A^{1/2}\left(R^{N-1}-\tilde{R}^{N-1}\right)\left[-\sigma\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\right.\\
&\times\left.\left.\left(R^{N-s}-\tilde{R}^{N-s}\right)f\left(t_s,u_s\right)+\varphi\right]\right\}-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left(R^{k-s}-\tilde{R}^{k-s}\right)f\left(t_s,u_s\right),\quad 2\le k\le N.
\end{aligned}
$$

By using triangle inequality, (6) and (10) we get

$$
\begin{aligned}
\|Fv_0-Fu_0\|_H &\le \left\|P_\tau^{-1}\right\|_{H\to H}\left\{\left(1+2\left|\rho\right|\right)|\sigma|\,\tau\sum_{s=1}^{N-1}\left\|A^{-1/2}\left[f\left(t_s,v_s\right)-f\left(t_s,u_s\right)\right]\right\|_H\right\}\\
&\le KT\left\|P_\tau^{-1}\right\|_{H\to H}\left(1+2\left|\rho\right|\right)|\sigma|\,\|v_k-u_k\|_H\,,
\end{aligned}
$$

$$
\begin{aligned}
\|Fv_1-Fu_1\|_H &\le \left\|P_\tau^{-1}\right\|_{H\to H}\left(|\sigma|+\left|\rho\right|+4|\sigma|\left|\rho\right|\right)\tau\sum_{s=1}^{N-1}\left\|A^{-1/2}\left[f\left(t_s,v_s\right)-f\left(t_s,u_s\right)\right]\right\|_H\\
&\le KT\left\|P_\tau^{-1}\right\|_{H\to H}\left(|\sigma|+\left|\rho\right|+4|\sigma|\left|\rho\right|\right)\|v_k-u_k\|_H\,,
\end{aligned}
$$

$$
\begin{aligned}
\|Fv_k-Fu_k\|_H &\le \left[1+\left\|P_\tau^{-1}\right\|_{H\to H}\left(|\sigma|+\left|\rho\right|+4|\sigma|\left|\rho\right|\right)\right]\tau\sum_{s=1}^{N-1}\left\|A^{-1/2}\left[f\left(t_s,v_s\right)-f\left(t_s,u_s\right)\right]\right\|_H\\
&\le KT\left(1+\left\|P_\tau^{-1}\right\|_{H\to H}\left(|\sigma|+\left|\rho\right|+4|\sigma|\left|\rho\right|\right)\right)\|v_k-u_k\|_H\,.
\end{aligned}
$$

□

Theorem 3.3. *Let u_k represent the solution of difference scheme (8) at t_k and $u\left(t_k\right)$ be the exact solution of (1). If $u'''(t)$ and $Au'(t)$ are continuous functions and f satisfies the Lipschitz condition (6), the convergence estimate*

$$\|u_k-u\left(t_k\right)\|_H\le L\tau$$

is satisfied, where L is independent of τ.

Proof. Using problem (1) and first order difference scheme (8), we get

$$
\begin{cases}
\tau^{-2}\left(z_{k+1}-2z_k+z_{k-1}\right)+Az_{k+1}=a_{k+1},\ 1\le k\le N-1,\\
z_0=\sigma z_N+a_0,\quad \left(I+\tau^2A\right)\tau^{-1}\left(z_1-z_0\right)-\rho\tau^{-1}(z_N-z_{N-1})=a_1,
\end{cases}
\tag{11}
$$

where $z_k=u\left(t_k\right)-u_k$,

$$a_0=u_0-u(t_0)-\sigma(u_N-u(t_N))=0,$$

$$a_1 = u'(0) - \rho u'(T) - \left[\left(I + \tau^2 A\right)\tau^{-1}\left(u(\tau) - u(0)\right) - \rho\tau^{-1}\left(u(T) - u(T-\tau)\right)\right] = O(\tau),$$

and

$$a_{k+1} = f(t_k, u_k) - f(t_k, u(t_k)) + \frac{d^2u(t_k)}{dt^2} - \tau^{-2}\left(u(t_{k+1}) - 2u(t_k) + u(t_{k-1})\right)$$

$$+Au(t_k) - Au(t_{k+1}), 1 \le k \le N-1.$$

For the solution of (11), we have

$$z_0 = P_\tau^{-1}\left\{\left[I - \rho\left(R - \tilde{R}\right)^{-1}\left(R^N + \tilde{R}^N\right)\frac{\tau}{i}A^{1/2}R\tilde{R}\right]\left[-\sigma\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s} - \tilde{R}^{N-s}\right]a_{s+1}\right]\right.$$

$$\left.+\tau\sigma\left(R - \tilde{R}\right)^{-1}\left(R^N - \tilde{R}^N\right)R\tilde{R}\left[-\rho\sum_{s=1}^{N-2}\frac{\tau}{4}\left[R^{N-s} - \tilde{R}^{N-s}\right]a_{s+1} + \rho\tau R\tilde{R}a_N + a_1\right]\right\},$$

$$z_1 = \tau P_\tau^{-1}R\tilde{R}\left\{\left[I - \frac{\sigma}{2}\left(R^{N-1} + \tilde{R}^{N-1}\right)\right]\left[-\rho\sum_{s=1}^{N-2}\frac{\tau}{4}\left[R^{N-s} - \tilde{R}^{N-s}\right]a_{s+1} + \rho\tau R\tilde{R}a_N + a_1\right]\right.$$

$$\left.-\sigma\rho\frac{A^{1/2}}{2i}\left(R^{N-1} - \tilde{R}^{N-1}\right)\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s} - \tilde{R}^{N-s}\right]a_{s+1}\right\} + z_0,$$

and

$$z_k = \frac{1}{2}\left(R^{k-1} + \tilde{R}^{k-1}\right)z_0 + \left(R - \tilde{R}\right)^{-1}\left(R^k - \tilde{R}^k\right)(z_1 - z_0) - \sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s} - \tilde{R}^{k-s}\right]a_{s+1},\ 2 \le k \le N.$$

By using triangle inequality and (6), we obtain

$$\begin{aligned}
\left\|A^{-1/2}a_{k+1}\right\|_H &\le \left\|A^{-1/2}\left(f(t_k, u_k) - f(t_k, u(t_k))\right)\right\|_H + M(\delta)\left\|Au(t_k) - Au(t_{k+1})\right\|_H \\
&\quad + M(\delta)\left\|\frac{d^2u(t_k)}{dt^2} - \tau^{-2}\left(u(t_{k+1}) - 2u(t_k) + u(t_{k-1})\right)\right\|_H \\
&\le K\left\|u_k - u(t_k)\right\|_H + M\tau,\ \ 1 \le k \le N-1, \\
\left\|A^{-1/2}a_1\right\|_H &\le L_1\tau,
\end{aligned}$$

where L_1 does not depend on τ. Applying the estimates (10) and Lipschitz condition (6), we get

$$\|z_0\|_H \le \left\|P_\tau^{-1}\right\|_{H\to H}\left(1 + 2\left|\rho\right|\right)|\sigma|\,\tau\sum_{s=1}^{N-1}\left\|A^{-1/2}a_{s+1}\right\|_H + M_1\tau, \tag{12}$$

$$\|z_1\|_H \le \left\|P_\tau^{-1}\right\|_{H\to H}\left(|\sigma| + \left|\rho\right| + 4\left|\rho\right||\sigma|\right)\tau\sum_{s=1}^{N-1}\left\|A^{-1/2}a_{s+1}\right\|_H + M_2\tau, \tag{13}$$

and

$$\|z_k\|_H \le \left[1 + \left\|P_\tau^{-1}\right\|_{H\to H}\left(|\sigma| + \left|\rho\right| + 4\left|\rho\right||\sigma|\right)\right]\sum_{s=1}^{N-1}\left\|A^{-1/2}a_{s+1}\right\|_H\tau + M_3\tau,\ 2 \le k \le N. \tag{14}$$

Applying (14), we get

$$\|z_k\|_H \le K\beta \sum_{s=1}^{N} \|z_s\|_H \,\tau + M_4\tau, \quad 1 \le k \le N,$$

where $\beta = \alpha/KT$. It follows that

$$\max_{1\le k\le N} \|z_k\|_H \le \alpha \max_{1\le s\le N} \|z_s\|_H + M_4\tau.$$

Therefore, using (7) we have

$$\max_{1\le k\le N} \|z_k\|_H \le \frac{1}{1-\alpha} M_4\tau = M_5\tau, \quad \|z_0\|_H \le L_2\tau.$$

That shows us that the unique solution of difference scheme (8) converges to the unique solution of (1) in a Hilbert space H. □

To obtain the unique solution of the difference scheme (8), the recursive relation

$$\begin{cases} \tau^{-2}\left({}_{(m)}u_{k+1} - 2{}_{(m)}u_k + {}_{(m)}u_{k-1}\right) + A{}_{(m)}u_{k+1} = f\left(t_k, {}_{(m-1)}u_k\right), \ t_k = k\tau, \ 1 \le k \le N-1, \ N\tau = T, \\ {}_{(m)}u_0 = \sigma{}_{(m)}u_N + \varphi, \ (I + \tau^2 A)\tau^{-1}({}_{(m)}u_1 - {}_{(m)}u_0) = \rho\tau^{-1}({}_{(m)}u_N - {}_{(m)}u_{N-1}) + \psi \end{cases} \tag{15}$$

is used, where ${}_{(m)}u^\tau = \left\{{}_{(m)}u_k\right\}_{k=1}^{N}$, $m = 1, 2, 3, \ldots$ and ${}_{(0)}u^\tau$ is given. The sequence of the solutions converges to the unique solution u^τ of difference scheme (8).

4. Numerical Analysis

In this section, we apply the first order of accuracy difference scheme (8) to the semilinear hyperbolic problem.

Example 4.1. *Consider the problem*

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} - \dfrac{\partial^2 u(x,t)}{\partial x^2} - u_x(x,t)) + u(x,t) = f(x,t,u), \ \ 0 < t < 1, \ 0 < x < \pi, \\ f(x,t,u) = 0.05\sin((e^t - t - 1)\sin x) + (3e^t - 2t - 2)\sin x - (e^t - t - 1)\cos x - \dfrac{\sin u}{20}, \\ u(x,0) = \dfrac{1}{2}u(x,1) + \varphi(x), \ \ \varphi(x) = \dfrac{2-e}{2}\sin x, \ \ 0 \le x \le \pi, \\ u_t(x,0) = \dfrac{1}{2}u_t(x,1) + \psi(x), \ \psi(x) = \dfrac{1-e}{2}\sin x, \ 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, \ \ 0 \le t \le 1 \end{cases} \tag{16}$$

which has exact solution $u = (e^t - t - 1)\sin x$.

Note that, the inequality (7) holds with $K < \frac{1}{12}$, $\sigma = \rho = \frac{1}{2}$, the function $f(t,x,u)$ satisfies the Lipschitz condition (6). For the approximate solutions of nonlocal boundary value problem (16), we apply (15).

$$\begin{cases} \dfrac{{}_{(m)}u_n^{k+1} - 2{}_{(m)}u_n^k +{}_{(m)} u_n^{k-1}}{\tau^2} - \dfrac{{}_{(m)}u_{n+1}^{k+1} - 2{}_{(m)}u_n^{k+1} +{}_{(m)} u_{n-1}^{k+1}}{h^2} - \dfrac{{}_{(m)}u_{n+1}^{k+1} -{}_{(m)} u_{n-1}^{k+1}}{2h} +{}_{(m)} u_n^{k+1} \\ = 0.05\sin\left((e^{t_k} - t_k - 1)\sin x_n\right) + (3e^{t_k} - 2t_k - 2)\sin x_n - (e^{t_k} - t_k - 1)\cos x_n - 0.05\sin_{(m-1)} u_n^k, \\ N\tau = 1,\ x_n = nh,\ 1 \le n \le M-1,\ Mh = \pi,\ t_k = k\tau,\ 1 \le k \le N-1, \\ {}_{(m)}u_n^0 - \dfrac{1}{2}{}_{(m)}u_n^N = \dfrac{2-e}{2}\sin x_n,\ 0 \le n \le M, \\ {}_{(m)}u_n^1 -{}_{(m)} u_n^0 + \dfrac{1}{2}{}_{(m)}u_n^N - \dfrac{1}{2}{}_{(m)}u_n^{N-1} = \tau\dfrac{1-e}{2}\sin x_n,\ \ 0 \le n \le M, \\ {}_{(0)}u_n^k = \sin x,\ {}_{(m)}u_0^k ={}_{(m)} u_M^k = 0,\ 0 \le k \le N,\ \ 0 \le n \le M \end{cases} \tag{17}$$

where $m = 1, 2, 3, ...$ To solve the difference scheme (17) modified Gauss elimination method is used. The solution of difference scheme (17) is recorded for different values of M and N. The errors are computed as following:

$$ {}_{(m)}E_M^N = \max_{\substack{1\le n\le M-1 \\ 1\le k\le N-1}} \left|{}_{(m)}u_n^k - u(x_n, t_k)\right| $$

where ${}_{(m)}u_n^{k+1}$ denotes the numerical solution, $u(x_n, t_k)$ denotes the exact solution at (x_n, t_k). Results of simulations are shown in the following table. We observe that the error decreases as N and M increases. Note that, error is approximately cut in half when N and M are doubled.

Table 1: Error analysis for the first order of accuracy difference scheme

N/M	20/20	40/40	80/80	160/160
Error	0.0901	0.0423	0.0195	0.0091

To summarize, for the solution of the problem (1), the uniqueness of the solution is established. The convergence estimates are established for the solution of first order of accuracy difference scheme (8). Finally, this difference scheme is applied to the one dimensional semilinear hyperbolic problems. Furthermore, this technique can be applied to the higher order of accuracy difference schemes. Without proof extended abstract of this work was printed in [14].

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Competing interests

The authors declare that they have no competing interests.

References

[1] D. Lupo, K. Payne, N. I. Popivanov, On the degenerate hyperbolic Goursat problem for linear and nonlinear equations of Tricomi type, Nonlinear Analysis 108 (2014) 29-56.

[2] L. Sun, B. Guo, W. Gao, A lower bound for the blow-up time to a damped semilinear wave equation, Applied Mathematics Letters 37 (2014) 22-25.
[3] S. Kharibegashvili, Solvability of characteristic boundary value problem for nonlinear equations with iterated wave operator in the principal part, Electronic Journal of Differential Equations 2008(72) (2008) 1–12.
[4] X. Liu, X. Cui, J. Sun, FDM for multi-dimensional nonlinear coupled system of parabolic and hyperbolic equations, Journal of Computational and Applied Mathematics 186(2) (2006) 432–449.
[5] A. Ashyralyev, M.E. Koksal, On the numerical solution of hyperbolic PDEs with variable space operator, Numerical Methods for Partial Differential Equations 25(5) (2009) 1084–1096.
[6] A. Ashyralyev, O. Yildirim, On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations, Taiwanese Journal of Mathematics 14(1) (2010) 165–194.
[7] A. Ashyralyev, N. Aggez, On the solution of NBVP for multidimensional hyperbolic equations, The Scientific World Journal, 2014 (2014) 1–22.
[8] A. Ashyralyev, N. Aggez, Nonlocal boundary value hyperbolic problems involving integral conditions, Boundary Value Problems 2014 (2014) 1–10.
[9] W. D. Li, Z. Z. Sun, L. Zhao, An analysis for a high-order difference scheme for numerical solution to utt = A(x,t)uxx + F(x,t,u,ut,ux), Numerical Methods for Partial Differential Equations 23(2) 2007 484–498.
[10] R. K. Mohanty, V. Gopal, High accuracy arithmetic average type discretization for the solution of two-space dimensional nonlinear wave equations, International Journal of Modeling Simulation and Scientific Computing 3 (2012) 1–18.
[11] A. Ashyralyev, A. H. Yurtsever, On a nonlocal boundary value problem for semilinear hyperbolic-parabolic equations, Nonlinear Analysis Theory Methods and Applications 47 (2001) 358–359.
[12] A. Ashyralyev, A. Sirma, A note on the numerical solution of the semilinear Schrödinger equation, Nonlinear Analysis Theory Methods and Applications 71 (2009) 2507–2516.
[13] Z. Direk, M. Ashyraliyev, Finite difference method for the semilinear integral-differential equation of the hyperbolic type, AIP Conference Proceedings 1676 (2015), 020048-1–020048-4.
[14] N. Aggez, G. Yucel, NBVP for semilinear hyperbolic equations, AIP Conference Proceedings 1676 (020057) (2015) 1-4.
[15] H. O. Fattorini, Second Order Linear Differential Equations in Banach Space, Notas de Matematica, New York, 1985.
[16] S. Piskarev, Y. Shaw, On certain operator families related to cosine operator function, Taiwanese Journal of Mathematics 1(4) (1997) 527–546.
[17] A. Ashyralyev, N. Aggez, A note on the difference schemes of the nonlocal boundary value problems for hyperbolic equations, Numerical Functional Analysis and Optimization, 25(5-6) (2004) 1–24.