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On Gronwall's Type Integral Inequalities with Singular Kernels

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Abstract. In this paper, the generalizations of Gronwall's type integral inequalities with singular kernels are established. In applications, theorems on stability estimates for the solutions of the nonliner integral equation and the integral-differential equation of the parabolic type are presented. Moreover, these inequalities can be used in the theory of fractional differential equations.

1. Introduction

Integral inequalities play an important role in the theory of differential equations. They are useful to investigate properties of solutions of differential equations, such as existence, uniqueness, and stability, see for instance [1–4]. Gronwall's integral inequality [5] is one of the most widely applied results in the theory of integral inequalities. Due to various motivations, several generalizations and applications of Gronwall's type integral inequality and their discrete analogues have been obtained and used extensively, see for instance [1, 6–9]. In [1], the following generalization of Gronwall's integral inequality with two dependent limits is obtained.

Theorem 1.1. *Assume that $v(t) \geq 0$ is a continuous function on $[-1, 1]$ and the inequalities*

$$v(t) \leq C + L\, sgn(t) \int_{-t}^{t} v(s)ds, \quad -1 \leq t \leq 1$$

hold, where $C = const \geq 0, L = const \geq 0$. Then for $v(t)$ the following inequalities

$$v(t) \leq C \exp\left(2L|t|\right), \quad -1 \leq t \leq 1$$

are satisfied.

In the present paper, two generalizations of Gronwall's type integral inequalities with singular kernels are presented. In applications, theorems on stability estimates for the solutions of the nonliner integral equation and the integral-differential equation of the parabolic type are presented. We note that these results may have an application in the theory of fractional differential equations [10].

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2. Integral Inequalities with Singular Kernels

We consider the generalizations of Gronwall's type integral inequalities with the singular kernel and two dependent limits.

Theorem 2.1. *Assume that $v(t) \geq 0$ is a continuous function on $[-1, 1]$, $a(t) \geq 0$ is an integrable function on $[-1, 1]$ and the inequalities*

$$v(t) \leq a(t) + L\, sgn(t) \int_{-t}^{t} (|t| - |s|)^{\beta-1} v(s) ds, \quad -1 \leq t \leq 1 \tag{1}$$

hold, where $L \geq 0$, $\beta \geq 0$. Then for $v(t)$ the inequalities

$$v(t) \leq a(t) + \sum_{n=1}^{\infty} 2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} sgn(t) \int_{-t}^{t} (|t| - |s|)^{n\beta-1} a(s) ds, \quad -1 \leq t \leq 1 \tag{2}$$

are satisfied, where $\Gamma(\beta)$ is the Gamma function.

Proof. We denote

$$Bv(t) = L\, sgn(t) \int_{-t}^{t} (|t| - |s|)^{\beta-1} v(s) ds, \quad -1 \leq t \leq 1. \tag{3}$$

Using (1), for nonnegative functions a and v we get

$$v(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n v(t), \quad -1 \leq t \leq 1, \tag{4}$$

where $n \in N$. We will now show that $B^n v(t) \to 0$ as $n \to \infty$, which will prove (2). For this we first prove that

$$B^n v(t) = 2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} sgn(t) \int_{-t}^{t} (|t| - |s|)^{n\beta-1} v(s) ds, \quad -1 \leq t \leq 1 \tag{5}$$

holds for any $n \in N$. Note that (5) follows directly from (3) when $n = 1$. Assume that (5) holds for some $n \in N$. Then for $0 \leq t \leq 1$ we have

$$B^{n+1} v(t) = L \int_{-t}^{t} (t - |s|)^{\beta-1} B^n v(s) ds = L\, 2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \int_{-t}^{t} (t - |s|)^{\beta-1} sgn(s) \int_{-s}^{s} (|s| - |\tau|)^{n\beta-1} v(\tau) d\tau ds$$

$$= L\, 2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_{0}^{t} (t - s)^{\beta-1} \int_{-s}^{s} (s - |\tau|)^{n\beta-1} v(\tau) d\tau ds + \int_{-t}^{0} (t + s)^{\beta-1} \int_{s}^{-s} (-s - |\tau|)^{n\beta-1} v(\tau) d\tau ds \right)$$

$$= L\, 2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_{0}^{t} v(\tau) \int_{\tau}^{t} (t - s)^{\beta-1} (s - \tau)^{n\beta-1} ds d\tau + \int_{-t}^{0} v(\tau) \int_{-\tau}^{t} (t - s)^{\beta-1} (s + \tau)^{n\beta-1} ds d\tau \right.$$

$$\left. + \int_{0}^{t} v(\tau) \int_{-t}^{-\tau} (t + s)^{\beta-1} (-s - \tau)^{n\beta-1} ds d\tau + \int_{-t}^{0} v(\tau) \int_{-t}^{\tau} (t + s)^{\beta-1} (-s + \tau)^{n\beta-1} ds d\tau \right)$$

$$= L\,2^n \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_0^t v(\tau) \int_0^{t-\tau} z^{\beta-1}(t-\tau-z)^{n\beta-1}dzd\tau + \int_{-t}^0 v(\tau) \int_0^{t+\tau} z^{\beta-1}(t+\tau-z)^{n\beta-1}dzd\tau \right)$$

$$= L\,2^n \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_0^t v(\tau)(t-\tau)^{(n+1)\beta-1}d\tau + \int_{-t}^0 v(\tau)(t+\tau)^{(n+1)\beta-1}d\tau \right) \int_0^1 \rho^{\beta-1}(1-\rho)^{n\beta-1}d\rho$$

$$= L\,2^n \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \frac{\Gamma(\beta)\Gamma(n\beta)}{\Gamma(n\beta+\beta)} \int_{-t}^t (t-|\tau|)^{(n+1)\beta-1}v(\tau)d\tau.$$

So, we have

$$B^{n+1}v(t) = 2^n \frac{(L\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} \int_{-t}^t (t-|s|)^{(n+1)\beta-1}v(s)ds, \quad 0 \le t \le 1. \tag{6}$$

In the similar way, for $-1 \le t < 0$ we have

$$B^{n+1}v(t) = L \int_t^{-t} (-t-|s|)^{\beta-1}B^n v(s)ds = L\,2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \int_t^{-t} (-t-|s|)^{\beta-1}sgn(s) \int_{-s}^s (|s|-|\tau|)^{n\beta-1}v(\tau)d\tau ds$$

$$= L\,2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_0^{-t} (-t-s)^{\beta-1} \int_{-s}^s (s-|\tau|)^{n\beta-1}v(\tau)d\tau ds + \int_t^0 (-t+s)^{\beta-1} \int_s^{-s} (-s-|\tau|)^{n\beta-1}v(\tau)d\tau ds \right)$$

$$= L\,2^{n-1} \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_0^{-t} v(\tau) \int_\tau^{-t} (-t-s)^{\beta-1}(s-\tau)^{n\beta-1}dsd\tau + \int_t^0 v(\tau) \int_{-\tau}^{-t} (-t-s)^{\beta-1}(s+\tau)^{n\beta-1}dsd\tau \right.$$

$$\left. + \int_0^{-t} v(\tau) \int_t^{-\tau} (-t+s)^{\beta-1}(-s-\tau)^{n\beta-1}dsd\tau + \int_t^0 v(\tau) \int_t^{\tau} (-t+s)^{\beta-1}(-s+\tau)^{n\beta-1}dsd\tau \right)$$

$$= L\,2^n \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_0^{-t} v(\tau) \int_0^{-t-\tau} z^{\beta-1}(-t-\tau-z)^{n\beta-1}dzd\tau + \int_t^0 v(\tau) \int_0^{-t+\tau} z^{\beta-1}(-t+\tau-z)^{n\beta-1}dzd\tau \right)$$

$$= L\,2^n \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\int_0^{-t} v(\tau)(-t-\tau)^{(n+1)\beta-1}d\tau + \int_t^0 v(\tau)(-t+\tau)^{(n+1)\beta-1}d\tau \right) \int_0^1 \rho^{\beta-1}(1-\rho)^{n\beta-1}d\rho$$

$$= L\,2^n \frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)} \frac{\Gamma(\beta)\Gamma(n\beta)}{\Gamma(n\beta+\beta)} \int_t^{-t} (-t-|\tau|)^{(n+1)\beta-1}v(\tau)d\tau.$$

So, we have

$$B^{n+1}v(t) = 2^n \frac{(L\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} \int_t^{-t} (-t-|s|)^{(n+1)\beta-1}v(s)ds, \quad -1 \le t < 0. \tag{7}$$

Combining (6) and (7), we prove by induction that (5) holds for any $n \in N$. Since $B^n v(t) \geq 0$ for any $n \in N$ and $t \in [-1, 1]$, we have

$$\frac{B^{n+1}v(t)}{B^n v(t)} = \frac{2L\Gamma(\beta)\Gamma(n\beta)}{\Gamma((n+1)\beta)} \frac{\int\limits_{-|t|}^{|t|} (|t| - |s|)^{(n+1)\beta-1} v(s)ds}{\int\limits_{-|t|}^{|t|} (|t| - |s|)^{n\beta-1} v(s)ds} \leq \frac{2L\Gamma(\beta)\Gamma(n\beta)}{\Gamma((n+1)\beta)} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $\lim\limits_{n\to\infty} B^n v(t) = 0$. Then, by letting $n \to \infty$ in (4) and using (5), we obtain the inequalities (2). Theorem 2.1 is proved.

Note that by putting $a(t) \equiv const$ in the Theorem 2.1, we obtain the following result.

Corollary 2.2. *Assume that $v(t) \geq 0$ is a continuous function on $[-1, 1]$ and the inequalities*

$$v(t) \leq C + L\, sgn(t) \int\limits_{-t}^{t} (|t| - |s|)^{\beta-1} v(s)ds, \quad -1 \leq t \leq 1$$

hold, where $C \geq 0$, $L \geq 0$, $\beta \geq 0$. Then for $v(t)$ the inequalities

$$v(t) \leq C \sum_{n=0}^{\infty} \frac{(2L|t|\Gamma(\beta))^n}{\Gamma(n\beta + 1)}, \quad -1 \leq t \leq 1$$

are satisfied.

Note that by putting $\beta = 1$ in the Corollary 2.2, we obtain the Theorem 1.1.

Theorem 2.3. *Assume that $v(t) \geq 0$ is a continuous function on $[-1, 1]$, $a(t) \geq 0$ is an integrable function on $[-1, 1]$ and the inequalities*

$$v(t) \leq a(t) + L\, sgn(t) \int\limits_{-t}^{t} |s|^{\alpha-1} v(s)ds, \quad -1 \leq t \leq 1 \tag{8}$$

hold, where $L \geq 0$, $1 \geq \alpha > 0$. Then for $v(t)$ the following inequalities hold

$$v(t) \leq a(t) + \sum_{n=1}^{\infty} \frac{2^{n-1}L^n}{\alpha^{n-1}(n-1)!} sgn(t) \int\limits_{-t}^{t} |s|^{\alpha-1} \left(|t|^\alpha - |s|^\alpha\right)^{n-1} a(s)ds, \quad -1 \leq t \leq 1. \tag{9}$$

Proof. We denote

$$Bv(t) = L\, sgn(t) \int\limits_{-t}^{t} |s|^{\alpha-1} v(s)ds, \quad -1 \leq t \leq 1. \tag{10}$$

Using (8), for nonnegative functions a and v we get

$$v(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n v(t), \quad -1 \leq t \leq 1, \tag{11}$$

where $n \in N$. Let us prove that

$$B^n v(t) = \frac{2^{n-1} L^n}{\alpha^{n-1}(n-1)!} sgn(t) \int_{-t}^{t} |s|^{\alpha-1} \left(|t|^\alpha - |s|^\alpha\right)^{n-1} v(s) ds, \quad -1 \le t \le 1 \tag{12}$$

holds for any $n \in N$. Note that (12) follows directly from (10) when $n = 1$. Assume that (12) holds for some $n \in N$. Then for $0 \le t \le 1$ we have

$$B^{n+1} v(t) = L \int_{-t}^{t} |s|^{\alpha-1} B^n v(s) ds = \frac{2^{n-1} L^{n+1}}{\alpha^{n-1}(n-1)!} \int_{-t}^{t} |s|^{\alpha-1} sgn(s) \int_{-s}^{s} |\tau|^{\alpha-1} \left(|s|^\alpha - |\tau|^\alpha\right)^{n-1} v(\tau) d\tau ds$$

$$= \frac{2^{n-1} L^{n+1}}{\alpha^{n-1}(n-1)!} \left(\int_{0}^{t} s^{\alpha-1} \int_{-s}^{s} |\tau|^{\alpha-1} \left(s^\alpha - |\tau|^\alpha\right)^{n-1} v(\tau) d\tau ds + \int_{-t}^{0} (-s)^{\alpha-1} \int_{s}^{-s} |\tau|^{\alpha-1} \left((-s)^\alpha - |\tau|^\alpha\right)^{n-1} v(\tau) d\tau ds \right)$$

$$= \frac{2^{n-1} L^{n+1}}{\alpha^{n-1}(n-1)!} \left(\int_{0}^{t} \tau^{\alpha-1} v(\tau) \int_{\tau}^{t} s^{\alpha-1} \left(s^\alpha - \tau^\alpha\right)^{n-1} ds d\tau + \int_{-t}^{0} (-\tau)^{\alpha-1} v(\tau) \int_{-\tau}^{t} s^{\alpha-1} \left(s^\alpha - (-\tau)^\alpha\right)^{n-1} ds d\tau \right.$$

$$\left. + \int_{0}^{t} \tau^{\alpha-1} v(\tau) \int_{-t}^{-\tau} (-s)^{\alpha-1} \left((-s)^\alpha - \tau^\alpha\right)^{n-1} ds d\tau + \int_{-t}^{0} (-\tau)^{\alpha-1} v(\tau) \int_{-t}^{\tau} (-s)^{\alpha-1} \left((-s)^\alpha - (-\tau)^\alpha\right)^{n-1} ds d\tau \right)$$

$$= \frac{2^n L^{n+1}}{\alpha^n n!} \left(\int_{0}^{t} \tau^{\alpha-1} \left(t^\alpha - \tau^\alpha\right)^n v(\tau) d\tau + \int_{-t}^{0} (-\tau)^{\alpha-1} \left(t^\alpha - (-\tau)^\alpha\right)^n v(\tau) d\tau \right).$$

So, we have

$$B^{n+1} v(t) = \frac{2^n L^{n+1}}{\alpha^n n!} \int_{-t}^{t} |s|^{\alpha-1} \left(t^\alpha - |s|^\alpha\right)^n v(s) ds, \quad 0 \le t \le 1. \tag{13}$$

In the similar way, for $-1 \le t < 0$ we have

$$B^{n+1} v(t) = L \int_{t}^{-t} |s|^{\alpha-1} B^n v(s) ds = \frac{2^{n-1} L^{n+1}}{\alpha^{n-1}(n-1)!} \int_{t}^{-t} |s|^{\alpha-1} sgn(s) \int_{-s}^{s} |\tau|^{\alpha-1} \left(|s|^\alpha - |\tau|^\alpha\right)^{n-1} v(\tau) d\tau ds$$

$$= \frac{2^{n-1} L^{n+1}}{\alpha^{n-1}(n-1)!} \left(\int_{0}^{-t} s^{\alpha-1} \int_{-s}^{s} |\tau|^{\alpha-1} \left(s^\alpha - |\tau|^\alpha\right)^{n-1} v(\tau) d\tau ds + \int_{t}^{0} (-s)^{\alpha-1} \int_{s}^{-s} |\tau|^{\alpha-1} \left((-s)^\alpha - |\tau|^\alpha\right)^{n-1} v(\tau) d\tau ds \right)$$

$$= \frac{2^{n-1} L^{n+1}}{\alpha^{n-1}(n-1)!} \left(\int_{0}^{-t} \tau^{\alpha-1} v(\tau) \int_{\tau}^{-t} s^{\alpha-1} \left(s^\alpha - \tau^\alpha\right)^{n-1} ds d\tau + \int_{t}^{0} (-\tau)^{\alpha-1} v(\tau) \int_{-\tau}^{-t} s^{\alpha-1} \left(s^\alpha - (-\tau)^\alpha\right)^{n-1} ds d\tau \right.$$

$$\left. + \int_{0}^{-t} \tau^{\alpha-1} v(\tau) \int_{t}^{-\tau} (-s)^{\alpha-1} \left((-s)^\alpha - \tau^\alpha\right)^{n-1} ds d\tau + \int_{t}^{0} (-\tau)^{\alpha-1} v(\tau) \int_{t}^{\tau} (-s)^{\alpha-1} \left((-s)^\alpha - (-\tau)^\alpha\right)^{n-1} ds d\tau \right)$$

$$= \frac{2^n L^{n+1}}{\alpha^n n!} \left(\int_{0}^{-t} \tau^{\alpha-1} \left((-t)^\alpha - \tau^\alpha\right)^n v(\tau) d\tau + \int_{t}^{0} (-\tau)^{\alpha-1} \left((-t)^\alpha - (-\tau)^\alpha\right)^n v(\tau) d\tau \right).$$

So, we have

$$B^{n+1}v(t) = \frac{2^n L^{n+1}}{\alpha^n n!} \int_t^{-t} |s|^{\alpha-1} \left((-t)^\alpha - |s|^\alpha\right)^n v(s)ds, \quad -1 \le t < 0. \tag{14}$$

Combining (13) and (14), by induction we prove that (12) holds for any $n \in N$. Since $B^n v(t) \ge 0$ for any $n \in N$ and

$$\frac{B^{n+1}v(t)}{B^n v(t)} = \frac{2L}{n\alpha} \frac{\int_{-|t|}^{|t|} |s|^{\alpha-1} \left(|t|^\alpha - |s|^\alpha\right)^n v(s)ds}{\int_{-|t|}^{|t|} |s|^{\alpha-1} \left(|t|^\alpha - |s|^\alpha\right)^{n-1} v(s)ds} \le \frac{2L}{n\alpha} \to 0 \quad \text{as} \quad n \to \infty,$$

we get $\lim_{n\to\infty} B^n v(t) = 0$. Then, by letting $n \to \infty$ in (11) and using (12), we obtain the inequalities (9). Theorem 2.3 is proved.

Note that by putting $a(t) \equiv const$ in the Theorem 2.3, we obtain the following result.

Corollary 2.4. *Assume that $v(t) \ge 0$ is a continuous function on $[-1, 1]$ and the inequalities*

$$v(t) \le C + L\, sgn(t) \int_{-t}^{t} |s|^{\alpha-1} v(s)ds, \quad -1 \le t \le 1$$

hold, where $C \ge 0$, $L \ge 0$, $1 \ge \alpha > 0$. Then for $v(t)$ the following inequalities hold

$$v(t) \le C \exp\left(\frac{2L|t|^\alpha}{\alpha}\right), \quad -1 \le t \le 1.$$

Note that by putting $\alpha = 1$ in the Corollary 2.4, we obtain the Theorem 1.1.

3. Applications

First, we consider the nonliner integral equation

$$x(t) = g(t) + \int_{-t}^{t} f\left(t, s; x(s)\right) ds, \quad -1 \le t \le 1. \tag{15}$$

Assume that the function $g(t)$ is continuous on $[-1, 1]$. Suppose the kernel f of the equation (15) is continuous on $[-1, 1] \times [-1, 1] \times (-\infty, \infty)$ and f satisfies on $[-1, 1] \times [-1, 1] \times (-\infty, \infty)$ the following condition

$$\left| f\left(t, s; x(s)\right) \right| \le L(|t| - |s|)^{\beta-1} |x(s)|, \quad 0 < |s| < |t| \le 1 \tag{16}$$

for $L \ge 0$ and $\beta \ge 0$.

Theorem 3.1. *Suppose that the assumption (16) holds. Then, for the solution of equation (15) the following stability estimate*

$$|x(t)| \le \max_{-1 \le t \le 1} \left| g(t) \right| \sum_{n=0}^{\infty} \frac{\left(2L|t|\Gamma(\beta)\right)^n}{\Gamma(n\beta + 1)}$$

is satisfied for any $t \in [-1, 1]$.

The proof of the Theorem 3.1 is based on the Theorem 2.1.

Second, we consider the integral-differential equation (see [11])

$$\frac{du(t)}{dt} + sgn(t)Au(t) = \int_{-t}^{t} B(s)u(s)ds + f(t), \quad -1 \le t \le 1 \tag{17}$$

in an arbitrary Banach space E with unbounded linear operators A and $B(t)$ in E with dense domain $D(A) \subset D(B(t))$ and

$$\|B(t)A^{-1}\|_{E\to E} \le \frac{M_1}{|t|^{1-\alpha}}, \quad 1 \ge \alpha > 0, \quad t \in [-1,0) \cup (0,1]. \tag{18}$$

A function $u(t)$ is called a solution of the equation (17) if the following conditions are satisfied:

i) $u(t)$ is continuously differentiable on $[-1,1]$. The derivatives at the endpoints are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the functions $Au(t)$ and $B(t)u(t)$ are continuous on $[-1,1]$.

iii) $u(t)$ satisfies the equation (17).

A solution of the equation (17) defined in this manner will from now on be referred to as a solution of the equation (17) in the space $C(E) = C([-1,1],E)$ of all continuous functions $\varphi(t)$ defined on $[-1,1]$ with values in E equipped with the norm

$$\|\varphi\|_{C(E)} = \max_{-1\le t\le 1} \|\varphi(t)\|_E.$$

We consider (17) under the assumption that the operator $-A$ generates an analytic semigroup $\exp\{-tA\}$ $(t \ge 0)$, i.e. the following estimates hold:

$$\|e^{-tA}\|_{E\to E} \le M, \quad \|tAe^{-tA}\|_{E\to E} \le M, \quad 0 \le t \le 1. \tag{19}$$

Theorem 3.2. *Suppose that assumptions (18) and (19) for the operators A and $B(t)$ hold. Assume that $f(t)$ is a continuously differentiable on $[-1,1]$ function. Then there is a unique solution of the equation (17) and the following stability inequalities*

$$\left\|\frac{du(t)}{dt}\right\|_E, \ \|Au(t)\|_E \le M^* \left[\|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right]$$

hold for any $t \in [-1,1]$, where M^* does not depend on $f(t)$ and t.

The proof of the Theorem 3.2 is based on the following formula [11]

$$u(t) = sgn(t)A^{-1}f(t) - sgn(t)e^{-|t|A}A^{-1}f(0) - sgn(t)\int_0^t e^{-(|t|-|s|)A}A^{-1}f'(s)ds$$
$$+ sgn(t)\int_{-t}^{t}\left[I - e^{-(|t|-|s|)A}\right]A^{-1}B(s)u(s)ds, \quad -1 \le t \le 1$$

and the Theorem 2.3.

We note that the inequality

$$\max_{-1\le t\le 1}\left\|\frac{du(t)}{dt}\right\|_E+\max_{-1\le t\le 1}\|Au(t)\|_E\le M^*\max_{-1\le t\le 1}\|f(t)\|_E$$

does not hold in general in the arbitrary Banach space E and for the general strong positive operator A (see [12]). Nevertheless, we can establish the following theorem.

Theorem 3.3. *Suppose that the estimates (19) for the operator A hold and*

$$\|B(t)A^{-1}\|_{E_\lambda\to E_\lambda}\le\frac{M_1}{|t|^{1-\alpha}},\quad 1\ge\alpha>0,\quad t\in[-1,0)\cup(0,1].$$

Assume that $f(t)$ *is a continuous function on* $[-1,1]$. *Then there is a unique solution of the equation (17) and stability inequalities*

$$\left\|\frac{du(t)}{dt}\right\|_{E_\lambda},\ \|Au(t)\|_{E_\lambda}\le M^*(\lambda)\max_{-1\le t\le 1}\|f(t)\|_{E_\lambda}$$

hold for any $t\in[-1,1]$, *where* $M^*(\lambda)$ *does not depend on* $f(t)$ *and* t. *Here the fractional spaces* $E_\lambda=E_\lambda(E,A)$ $(0<\lambda<1)$, *consisting of all* $v\in E$ *for which the following norms are finite:*

$$\|\ v\ \|_{E_\lambda}=\sup_{0<z} z^{1-\lambda}\ \|\ Aexp\{-zA\}v\ \|_E\ .$$

Finally, we note that this approach allows us to extend our discussion to the study of the initial-value problem

$$\begin{cases}\dfrac{d^2u(t)}{dt^2}+Au(t)=\displaystyle\int_{-t}^{t}B(\rho)u(\rho)d\rho+f(t), & -1\le t\le 1,\\ u(0)=u_0,\quad u'(0)=u'_0\end{cases}\tag{20}$$

for integral-differential equation in an arbitrary Banach space E with unbounded linear operators A and $B(t)$ in E with dense domain $D(A)\subset D(B(t))$ satisfying the assumption (18). The stability estimates for the solution of the equation (20) can be obtained in the similar way.

4. Conclusions

In this paper, two generalizations of Gronwall's type integral inequalities with singular kernels are presented. In applications, theorems on stability estimates for the solutions of the nonliner integral equation and the integral-differential equation of the parabolic type are presented. Moreover, applying the results of the paper [10], the fractional differential equation

$$D^\alpha_\pm v(t)=f\left(t,v(t)\right),\quad 0<\alpha<1,\quad -1\le t\le 1$$

can be investigated in the similar way, where

$$D^\alpha_\pm v(t)=\frac{1}{\Gamma(1-\alpha)}sgn(t)\int_{-t}^{t}(|t|-|s|)^{-\alpha}v'(s)ds.$$

The stability estimates for the solution of this fractional differential equation can be obtained in the similar way.

References

[1] M. Ashyraliyev, A Note on the Stability of the Integral-Differential Equation of the Hyperbolic Type in a Hilbert Space, Numerical Functional Analysis and Optimization 29 (2008) 750–769.
[2] S. G. Krein, Linear Differential Equations in a Banach Space, Nauka, Moscow, 1966.
[3] E. F. Beckenbach, R. Bellman, Inequalities, Springer, Berlin, 1961.
[4] S. Ashirov, Y. D. Mamedov, A Volterra-type integral equations, Ukr. Mat. Journal 40:4 (1988) 438–442.
[5] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Annals of Mathematics 20:4 (1919) 292–296.
[6] T. Nurimov, D. Filatov, Integral Inequalities, FAN, Tashkent, 1991.
[7] A. Corduneanu, A note on the Gronwall inequality in two independent variables, Journal of Integral Equations 4:3 (1982) 271–276.
[8] Z. Direk, M. Ashyraliyev, FDM for the integral-differential equation of the hyperbolic type, Advances in Difference Equations 2014:132 (2014) 1–8.
[9] A. Ashyralyev, E. Misirli, O. Mogol, A note on the integral inequalities with two dependent limits, Journal of Inequalities and Applications 2010 (2010) 1–18.
[10] A. Ashyralyev, A note on fractional derivatives and fractional powers of operators, Journal of Mathematical Analysis and Applications 357 (2009) 232–236.
[11] M. Ashyraliyev, A note on the stability of the integral-differential equation of the parabolic type in a Banach space, Abstract and Applied Analysis 2012 (2012) 1–18.
[12] A. Ashyralyev, P. E. Sobolevskii, Well-Posedness of Parabolic Difference Equations, Birkhauser, Basel, 1994.