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# On an Ill-posed Problem for a Biharmonic Equation

**Tynysbek Kal'menov**[a], **Ulzada Iskakova**[a]

[a] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*

**Abstract.** A local boundary value problem for the biharmonic equation in a rectangular domain is considered. Boundary conditions are given on all boundary of the domain. We show that the considered problem is self-adjoint. Herewith the problem is ill-posed. We show that the stability of solution to the problem is disturbed. Necessary and sufficient conditions of existence of the problem solution are found.

## 1. Introduction

The most known example of an ill-posed boundary value problem is the Cauchy problem for the Laplace equation. In $\Omega = \{(x,t) : 0 < x < \pi, 0 < t < T\}$ a problem for the equation

$$\triangle u \equiv u_{tt}(x,t) + u_{xx}(x,t) = 0, \quad (x,t) \in \Omega, \tag{1}$$

with the boundary conditions

$$u|_{x=0} = 0, \quad u|_{x=\pi} = 0, \quad 0 \le t \le T, \tag{2}$$

and the initial conditions

$$u|_{t=0} = \varphi_1(x), \quad \frac{\partial u}{\partial t}|_{t=0} = \varphi_2(x), \quad 0 \le x \le \pi \tag{3}$$

is considered.

The classic example of Hadamard showing [1] the instability of the solution

$$u_k(x,t) = \frac{\sin(kx)\sinh(kt)}{k^2}$$

to the Cauchy problem for the Laplace equation (1) with the boundary conditions (2) and the initial conditions

$$u\Big|_{t=0} = 0, \ \frac{\partial u}{\partial t}\Big|_{t=0} = \frac{1}{k}\sin(kx)$$

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*Email addresses:* kalmenov@math.kz ( Tynysbek Kal'menov ), ulzada@list.ru ( Ulzada Iskakova )

with respect to small changes of initial data is well-known.

The prominent Soviet mathematicians, academicians A.N. Tikhonov and M.M. Lavrent'ev, their disciples and followers proved that the Cauchy problem is conditionally well-posed for the Laplace equation and ill-posed for other problems [2, 3]. And also they suggested the regularization of these ill-posed problems.

In [4, 5] they obtained a necessary and sufficient condition of well-posedness in the space $L_2(\Omega)$ for problem (1) - (3) by the method of expansion with the help of the eigenfunctions of the mixed Cauchy problem for the Laplace equation with deviating argument. In [6] a nonlocal boundary value problem for the biharmonic equation in the disk is considered.

The main monograph for ill-posed boundary value problems for the biharmonic equation may be considered [7]. In it three essentially ill-posed internal boundary value problems for the biharmonic equation and the Cauchy problem for the abstract biharmonic equation was studied. In addition, some variants of these problems and the Cauchy problem, as well as the m-dimensional case, are considered. Ill-posed boundary value problems for the biharmonic equation were investigated extensively in the recent years (see, for example, [8 - 14]).

One of the reasons of the ill-posedness of boundary value problems for elliptic equations is considered to be a case when a part of the domain boundary is exempt from boundary conditions. A part of the boundary $t = T$, $0 \leq x \leq \pi$ is exempt from boundary conditions in the Cauchy problem considered above.

In the present paper we consider a local problem for an elliptic equation of the fourth order, ill-posedness of which is analogous to the ill-posedness of the Cauchy problem for the Laplace equation. Herewith boundary conditions are given on all boundary of the domain.

## 2. Statement of the Problem

**Problem 2.1.** *Find a solution to the biharmonic equation*

$$\triangle^2 u \equiv u_{tttt}(x,t) + 2u_{ttxx}(x,t) + u_{xxxx}(x,t) = 0, \quad (x,t) \in \Omega, \tag{4}$$

*satisfying boundary conditions in the spatial variable x:*

$$u|_{x=0} = 0, \ \triangle u|_{x=0} = 0; \quad u|_{x=\pi} = 0, \ \triangle u|_{x=\pi} = 0; \tag{5}$$

*and boundary conditions in the variable t:*

$$u|_{t=0} = \varphi_1(x), \ \frac{\partial u}{\partial t}|_{t=0} = \varphi_2(x), \quad 0 \leq x \leq \pi; \tag{6}$$

$$\triangle u|_{t=T} = \psi_1(x), \quad \frac{\partial \triangle u}{\partial t}|_{t=T} = \psi_2(x), \ \ 0 \leq x \leq \pi. \tag{7}$$

**Definition 2.2.** *The function $u \in C^4(\Omega) \bigcap C^3(\overline{\Omega})$ satisfying equation* (4) *and boundary conditions* (5) - (7) *is called a classic solution to the problem 2.1.*

## 3. Instability of Solution

Similarly to the Hadamard example one can construct an instability example of classic solution to the problem 2.1. Really, by direct calculation it is easy to obtain that the function

$$u_k(x,t) = \sin(kx)\left\{\frac{\cosh(kT)\sinh(kt)}{k^4} - t\frac{\cosh\left(k\left(T-t\right)\right)}{k^3}\right\}$$

is the solution to the problem 2.1 for the biharmonic equation (4) with the boundary conditions (5) and with conditions

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq \pi,$$

$$\triangle u|_{t=T} = 0, \quad \frac{\partial \triangle u}{\partial t}|_{t=T} = -\frac{2}{k}\sin kx, \quad 0 \le x \le \pi.$$

It is easy to see that the boundary data tend to zero for $k \to \infty$, but the solution $u_k(x,t)$ does not tend to zero in any norm. Consequently, the solution to the problem is instable. Therefore the considered problem 2.1 is ill-posed.

## 4. Symmetry and Positivity of an Operator of the Problem

Consider the problem with homogeneous boundary conditions (6) - (7):

$$u|_{t=0} = 0, \; \frac{\partial u}{\partial t}|_{t=0} = 0, \; 0 \le x \le \pi, \tag{6'}$$

$$\triangle u|_{t=T} = 0, \; \frac{\partial \triangle u}{\partial t}|_{t=T} = 0, \; 0 \le x \le \pi. \tag{7'}$$

Let $L$ be an operator in $L_2(\Omega)$ being a closure of the operator given by the differential expression

$$Lu \equiv u_{tttt}(x,T) + 2u_{ttxx}(x,t) + u_{xxxx}(x,t), \quad (x,t) \in \Omega$$

on a linear manifold of functions $u \in C^4(\Omega) \bigcap C^3(\overline{\Omega})$ satisfying the boundary conditions (5), (6′), (7′).

We show that the operator $L$ is symmetric. Let $u, v \in D(L)$ be two arbitrary elements from the definition domain of the operator $L$. For these elements there exist corresponding sequences of smooth functions $u_n, v_n \in C^4(\Omega) \bigcap C^3(\overline{\Omega})$ satisfying the boundary conditions (5), (6′), (7′) such that

$$\lim_{n\to\infty} u_n = u, \lim_{n\to\infty} Lu_n = Lu; \; \lim_{n\to\infty} v_n = v, \lim_{n\to\infty} Lv_n = Lv$$

in $L_2(\Omega)$.

Then by direct calculation we obtain for all $u, v \in D(L)$

$$(Lu, v) - (u, Lv) = \lim_{n\to\infty}\left\{(Lu_n, v_n) - (u_n, Lv_n)\right\} = 0.$$

Consequently the operator $L$ is symmetric. In this sense the boundary value problem (4) - (7) is self-adjoint.

Similarly, for all $u \in D(L)$ we obtain $(Lu, u) = \|\triangle u\|^2 \ge 0$. Consequently the operator $L$ is positive.

## 5. Construction of a Formal Solution of Problem (4) - (7)

By $\omega_k(x) = \sqrt{2/\pi}\sin(kx)$, $k = 1, 2, ...$, let denote an orthonormal basis in $L_2(0,\pi)$. The solution to problem (4) - (7) can be represented in the form of an expansion into the orthogonal series

$$u(x,t) = \sum_{k=1}^{\infty} \omega_k(x) v_k(t). \tag{8}$$

By considering that series (8) converges and allows a term by term differentiation (the required number of times), we construct a formal solution to the problem. Satisfying (8) to equation (4) and to the boundary conditions (6), (7), for $v_k(t)$ we obtain the problems

$$v_k^{(IV)}(t) - 2k^2 v_k''(t) + k^4 v_k(t) = 0, \quad 0 < t < T, \tag{9}$$

$$v_k(0) = \varphi_{1k}, \quad v_k'(0) = \varphi_{2k}, \tag{10}$$

$$v_k''(T) - k^2 v_k(T) = \psi_{1k}, \quad v_k'''(T) - k^2 v_k'(T) = \psi_{2k}. \tag{11}$$

Here $\varphi_{ik}$ and $\psi_{ik}$ are Fourier coefficients of the expansion according to the orthonormal basis $\{\omega_k(x)\}_{k=1}^{\infty}$ of the functions $\varphi_i(x)$ and $\psi_i(x)$ respectively:

$$\varphi_i(x) = \sum_{k=1}^{\infty} \varphi_{ik}\omega_k(x), \quad \psi_i(x) = \sum_{k=1}^{\infty} \psi_{ik}\omega_k(x), \quad , i = 1, 2.$$

Equation (9) has a general solution

$$v(t) = (C_1 t + C_2)e^{kt} + (C_3 t + C_4)e^{-kt}.$$

This solution satisfies the boundary conditions (10), (11). Then we get the system of linear equations

$$\left\{ \begin{array}{rrrrl} & C_2 & & +C_4 & = \varphi_{1k}, \\ C_1 & +kC_2 & +C_3 & -kC_4 & = \varphi_{2k}, \\ 2ke^{kT}C_1 & & -2ke^{-kT}C_3 & & = \psi_{1k}, \\ 2k^2e^{kT}C_1 & & +2k^2e^{-kT}C_3 & & = \psi_{2k}. \end{array} \right. \tag{12}$$

A determinant of this system equals to

$$\Delta = 16k^4.$$

Since $\Delta \geq 16$, then system (12) has a unique solution. By the direct calculation we get

$$C_1 = \frac{1}{4k^2}e^{-kT}\Big\{k\psi_1 + \psi_2\Big\},$$

$$C_2 = \frac{1}{2k}\Big\{k\varphi_1 + \varphi_2\Big\} + \frac{1}{8k^2}\Big\{e^{kT} - e^{-kT}\Big\}\psi_1 - \frac{1}{8k^3}\Big\{e^{kT} + e^{-kT}\Big\}\psi_2,$$

$$C_3 = -\frac{1}{4k^2}e^{kT}\Big\{k\psi_1 - \psi_2\Big\},$$

$$C_4 = \frac{1}{2k}\Big\{k\varphi_1 - \varphi_2\Big\} - \frac{1}{8k^2}\Big\{e^{kT} - e^{-kT}\Big\}\psi_1 + \frac{1}{8k^3}\Big\{e^{kT} + e^{-kT}\Big\}\psi_2.$$

Consequently the solution to problem (9) - (11) has the form

$$v_k(t) = -\frac{t}{2k}\sinh\Big(k(T-t)\Big)\psi_1 + \frac{t}{2k^2}\cosh\Big(k(T-t)\Big)\psi_2$$

$$+\Big\{\varphi_1 + \frac{1}{k}\varphi_2 + \frac{1}{2k^2}\sinh(kT)\psi_1 - \frac{1}{2k^3}\cosh(kT)\psi_2\Big\}\sinh(kt). \tag{13}$$

Substituting the found result into (8), we get the formal solution to problem (4) - (7).

## 6. A generalized Solution of Problem (4) - (7)

Consider problem (4) - (7) in sense of a generalized solution. The most suitable notion for demonstrating conditions of stability is the notion of a strong solution.

**Definition 6.1.** *The function $u(x,t) \in L_2(\Omega)$ is called a strong solution to problem (4) - (7), if there exists the sequence of the smooth functions $u_n \in C^4(\overline{\Omega})$, such that $u_n \to u$ takes place in $L_2(\Omega)$ for $n \to \infty$ and*

$$u_n(x,0) \to \varphi_1, \ (u_n)_t(x,0) \to \varphi_2, \ (\Delta u_n)(x,T) \to \psi_1 \ \ ((\Delta u_n)_t)(x,T) \to \psi_2 \tag{14}$$

*in $L_2(0,\pi)$.*

As the required sequence $u_n$, we choose a sequence of partial sums of the Fourier series:

$$u_n(x,t) = \sum_{k=1}^{n} \omega_k(x) v_k(t). \tag{15}$$

If $\varphi_i \in L_2(0,\pi)$, $\psi_i \in L_2(0,\pi)$, $i = 1,2$, then fulfillment (14) is obvious. Consequently the existence of the strong solution to problem (4) - (7) is equivalent to the convergence of the sequence $u_n$ in $L_2(\Omega)$.

By virtue of the Parseval equality, the convergence of the sequence $u_n$ in $L_2(\Omega)$ is equivalent to the convergence of the numerical series

$$\sum_{k=1}^{\infty} \|v_k(t)\|^2_{L_2(0,T)} < \infty. \tag{16}$$

## 7. A Criterion of Existence of a Solution to Problem (4) - (7)

The main result of the paper is the following theorem:

**Theorem 7.1.** *Let $\varphi_i \in L_2(0,\pi)$, $\psi_i \in L_2(0,\pi)$, $i = 1,2$. A strong solution to problem* (4) - (7) *exists iff the numerical series converge*

$$\sum_{k=1}^{\infty} \frac{1}{k^3} e^{2kT} \left| k\varphi_{1k} + \varphi_{2k} \right|^2 < \infty, \tag{17}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^7} e^{4kT} \left| k\psi_{1k} - \psi_{2k} \right|^2 < \infty. \tag{18}$$

*Proof.* The solution (13) is represented in the form

$$\begin{aligned} v_k(t) = {} & \frac{1}{8k^3} e^{kT} \left( e^{kt} + (2kt+1)e^{-kt} \right)(k\psi_1 - \psi_2) + \frac{1}{2k} e^{kt}(k\varphi_1 + \varphi_2) + \\ & + \frac{1}{8k^3} e^{-kT} \left( (2kt-1)e^{kt} + e^{-kt} \right)(k\psi_1 + \psi_2) + \frac{1}{2k} e^{-kt}(k\varphi_1 - \varphi_2). \end{aligned} \tag{19}$$

Taking into account that

$$\left\| e^{kt} + (2kt+1)e^{-kt} \right\|^2 = \frac{1}{2k} \left\{ e^{2kT} + 4(k^2T^2 + kT + 1) - (4k^2T^2 + 8kT + 5)e^{-2kT} \right\},$$

$$\left\| e^{kt} \right\|^2 = \frac{1}{2k} \left\{ e^{2kT} - 1 \right\},$$

from (19) we obtain that conditions (17) and (18) are necessary and sufficient for fulfillment of (16). □

## 8. Conclusion

One problem for the biharmonic equation is considered in the paper. It is shown that this problem is ill-posed. The example of the instability of solution to the problem is shown. Herewith the boundary conditions are given on the whole boundary, and the operator corresponding to the boundary value problem is symmetric and positive. The criterion on the existence in terms of the Fourier coefficients of the boundary data is proved for the introduced definition of the strong solution. The necessary condition for the existence of the solution is the tendency to zero with the exponential speed.

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