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A Survey On Some Paranormed Sequence Spaces

Eberhard Malkowsky[a], **Feyzi Başar**[b]

[a] *Državni Univerzitet u Novom Pazaru, Vuka Karadžića bb, 36300 Novi Pazar, Serbia*
[b] *Kisikli Mah. Alim Sok. Alim Apt. No: 7/6, 34692 Üsküdar, Istanbul, Turkey*

Abstract. This paper presents a survey of most of the known fundamental results involving the sequence spaces $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, $w_0(p)$, $w(p)$ and $w_\infty(p)$, $f_0(p)$ and $f(p)$. These spaces are generalizations of the classical BK spaces ℓ_p, c_0, c and ℓ_∞, the spaces w_0^p, w^p and w_∞^p of sequences that are strongly summable to zero, strongly summable and strongly bounded with index p by the Cesàro method of order 1, and of almost null and almost convergent sequences, respectively. The results inlude the basic topological properties of the generalized spaces, the complete lists of their known α–, β–, γ–, functional and continuous duals, and the characterizations of many classes of matrix transformations between them, in particular, the complete list of characterizations of matrix transformations between the spaces $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$. Furthermore, a great number of interesting special cases are included. The presented results cover a period of four decades. They are intended to inspire the inreasing number of researchers working in related topics, and to provide them with a comprehensive collection of results they may find useful for their work.

1. Introduction and Notations

By ω, we denote the vector space of all complex valued sequences. Any vector subspace of ω is called a *sequence space*. By ℓ_∞, c, c_0 and ℓ_p for $0 < p < \infty$, we denote the classical spaces of all bounded, convergent, null and absolutely p-summable sequences, respectively; also let ϕ be the set of all finite sequences, that is, of sequences that terminate in zeros. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively, and $bv = \{x \in \omega : \sum_{k=1}^\infty |x_k - x_{k+1}| < \infty\}$ for the set of all sequences of bounded variation. Also, let e and $e^{(n)}$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$ be the sequences with $e_k = 1$ for all k and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Finally let $x^{[n]} = \sum_{k=1}^n x_k e^{(k)}$ denote the *n–section* of the sequence $x = (x_k)_{k=1}^\infty$.

A subset X of ω is said to be *normal* if $x \in X$ and $y \in \omega$ with $|y_k| \leq |x_k|$ for all k implies $y \in X$.

A subspace X of ω is said to be an FK space if it is a Fréchet space with continuous coordinates $P_n : X \to \mathbb{C}$ for all $n \in \mathbb{N}$, where $P_n(x) = x_n$ for all $x = (x_k)_{k=1}^\infty \in X$; an FK space is called a BK space if its metric is given by a norm. An FK space $X \supset \phi$ is said to have AD if ϕ is dense in X, and to have AK if every sequence $x = (x_k)_{k=1}^\infty \in X$ has a unique representation $x = \sum_{k=1}^\infty x_k e^{(k)}$. A sequence space is called an IFK (IBK) space if it can be written as the union of an increasing sequence of FK (BK) spaces. It is endowed with the inductive limit topology ([3]). We recall the definition of the inductive limit topology for the reader's convenience.

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Email addresses: Eberhard.Malkowsky@math.uni-giessen.de (Eberhard Malkowsky), fbasar@gmail.com (Feyzi Başar)

Given a set X and a family of topological spaces Y_k with functions $f_k : Y_k \to X$, the *inductive limit topology* or *final topology* $\mathcal{T}$ *on* X is the finest topology such that each function $f_k : Y_k \to (X, \mathcal{T})$ is continuous. Explicitly, the final topology can be described as follows: A subset U of X is open if and only if the pre–image $f_k^{-1}(U)$ of U is open for each k ([7]).

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex numbers, X and Y be subsets of ω, and $x \in \omega$. Then we write A_n and $A^{(k)}$ for the nth row and kth column of A, $A_n x = \sum_{k=1}^{\infty} a_{nk}x_k$ for all $n \in \mathbb{N}$ and $Ax = (A_n x)_{n=1}^{\infty}$ (provided all the series $\sum_{k=1}^{\infty} a_{nk}x_k$ converge). The sets $X_A = \{x \in \omega : Ax \in X\}$ and $M(X, Y) = \{a \in \omega : a \cdot x = (a_k x_k)_{k=1}^{\infty} \in Y$ for all $x \in X\}$ are called the *matrix domain of A in X* and the *multiplier space of X in Y*; in particular, $X^{\alpha} = M(X, \ell_1)$, $X^{\beta} = M(X, cs)$ and $X^{\gamma} = M(X, bs)$ are called the α–, β– and γ–duals of X. A subset X of ω is said to be β–reflexive if $X^{\beta\beta} = (X^{\beta})^{\beta} = X$. If a subset X of ω is a linear topological space, we denote its continuous dual by X', and if $X \supset \phi$, then

$$X^f = \{x \in \omega : x = (f(e^{(k)}))_{k=1}^{\infty} \text{ for some } f \in X'\}$$

is called the f–dual of X; X'_b denotes the space X' with the strong topology; if X is a normed sequence space then X^* denotes the space X' with the norm defined by

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} \text{ for all } f \in X'.$$

Finally, (X, Y) is the class of all matrices A such that $X \subset Y_A$; so $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

Throughout, let I, Σ and C_1 denote the matrices with the rows $I_n = e^{(n)}$, $\Sigma_n = e^{[n]} = \sum_{k=1}^{n} e^{(k)}$ and $(C_1)_n = (1/n)\Sigma_n$ for $n = 1, 2, \ldots$, that is, I, Σ and C_1 are the identity and sum matrices, and the Cesàro matrix of order 1.

We recall the definition of the concept of a paranorm.

Definition 1.1. *Let X be a real or complex linear space, g be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X, g) is called a* paranormed space *and g is a* paranorm *for X, if the following axioms are satisfied for all elements $x, y \in X$*

(PN.1) $g(\theta) = 0$ *if $x = \theta$, where θ is the zero element of X;*

(PN.2) $g(x) \geq 0$;

(PN.3) $g(-x) = g(x)$;

(PN.4) $g(x + y) \leq g(x) + g(y)$ (triangle inequality);

(PN.5) *if (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and $(x_n)_{n=1}^{\infty}$ is a sequence in X with $g(x_n - x) \to 0$ as $(n \to \infty)$ then $g(\alpha_n x_n - \alpha x) \to 0$ as $(n \to \infty)$* (continuity of multiplication by scalars).

A paranorm g is said to be total, *if $g(x) = 0$ implies $x = \theta$.*

2. Maddox's Spaces and Some Fundamental Properties

In this section, we list the fundamental properties of Maddox's sequence spaces.

2.1. Definitions and set inclusions

The definition of the spaces in this section can be found in [20, 21, 34, 39]; special cases were studied, for instance, in [4, 10, 26, 37]. We also refer to [17–19].

Let $p = (p_k)_{k=1}^{\infty}$ be a sequence of positive reals, $x \in \omega$ and $B = (b_{nk})_{n,k=1}^{\infty}$. Then we write

$$|x|^p = (|x_k|^{p_k})_{k=1}^{\infty}, \; B_n(|x|^p) = \sum_{k=1}^{\infty} b_{nk}|x_k|^{p_k} \text{ for } n \in \mathbb{N},$$

and

$$B(|x|^p) = (B_n(|x|^p)_{n=1}^\infty$$

(provided the all the series converge), and

$$[B,p]_0 = \{x \in \omega : B(|x|^p) \in c_0\},$$
$$[B,p] = \{x \in \omega : B(|x - \xi e|^p) \in c_0 \text{ for some } \xi \in \mathbb{C}\},$$

and

$$[B,p]_\infty = \{x \in \omega : B(|x|^p) \in \ell_\infty\},$$

are the sets of all sequences that are strongly B–convergent to zero, strongly B–convergent, and strongly B–bounded. If $x \in [B,p]$ then $\xi \in \mathbb{C}$ with $B(|x - \xi e|^p) \in c_0$ is referred to as a *strong B–limit*, or *$[B,p]$–limit*, of the sequence x. We write $x_k \to \xi[B,p]$ if $\lim_{n\to\infty} B_n(|x - \xi e|^p) = 0$.

Conditions for the uniqueness of the $[B,p]$–limits of sequences in $[B,p]$ and of convergent sequences were given in [20, Theorems 2 and 4] for a certain class of matrices B. Let $\mathcal{A}$ denote the class of all infinite matrices $A = (a_{nk})_{n,k=1}^\infty$ for which there exists a positive integer M such that

(i) $a_{nk} \geq 0$ for each $n \geq 1$ and for each $k > M$,

(ii) $|a_{nk}| - a_{nk} \to 0$ $(n \to \infty; 1 \leq k \leq M)$.

Two important subclasses of $\mathcal{A}$ are the nonnegative matrices, and the matrices satisfying (i) and the condition $a_{nk} \to \alpha_k \geq 0$ $(n \to \infty; 1 \leq k \leq M)$.

The following results hold.

Theorem 2.1. ([20, Theorem 2]) *Let $A \in \mathcal{A}$ and $p = (p_k)_{k=1}^\infty \in \ell_\infty$. Then the $[A,p]$ limit of a sequence in $[A,p]$ is unique if and only if at least one of the following conditions fails to hold:*

(i) $\sum_{k=1}^\infty a_{nk}$ *converges for each n,*

(ii) $\sum_{k=1}^\infty a_{nk} \to 0$ $(n \to \infty)$.

Theorem 2.2. ([20, Theorem 4]) *Let the sequence $p = (p_k)_{k=1}^\infty$ converge to a positive limit and $A \in (c_0, c_0)$, that is,*

$$\|A\| = \sup_n \|A_n\|_1 = \sup_n \sum_{k=1}^\infty |a_{nk}| < \infty \text{ and } A^{(k)} \in c_0 \text{ for each } k, \tag{1}$$

then $\lim_{k\to\infty} x_k = \xi$ implies that $x_k \to \xi[A,p]$ uniquely if and only if

$$\limsup_{n\to\infty} \left| \sum_{k=1}^\infty a_{nk} \right| > 0.$$

Now we study a few simple set inclusions.

Remark 2.3. *It is clear from the definition of the sets $[B,p]_0$, $[B,p]$ and $[B,p]_\infty$ that $[B,p]_0 \subset [B,p]$ and $[B,p]_0 \subset [B,p]_\infty$, but $[B,p]$ is not included in $[B,p]_\infty$, in general. For instance, we clearly have $e \in [\Sigma,p] \setminus [\Sigma,p]_\infty$ for arbitrary sequences p.*
If A is a nonnegative matrix with no zero columns and $p = (p_k)_{k=1}^\infty \in \ell_\infty$, then it is known ([24, p.318]) *that $[A,p] \subset [A,p]_\infty$ if and only if*

$$\|A\| = \sup_n \sum_{k=1}^\infty a_{nk} < \infty. \tag{2}$$

The following results are known.

Theorem 2.4. ([20, Theorem 5]) *Let A be a regular matrix, that is, A satisfies the conditions in* (1) *and*

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = 1. \tag{3}$$

If $0 < p_k \le q_k$ for all k and $q_k/p_k \to \infty$ ($k \to \infty$), then $x_k \to \xi[A,q]$ does not generally imply $x_k \to \xi[A,p]$.

Theorem 2.5. ([20, p. 351]) *Let $A \in \mathcal{A}$, $\|A\| < \infty$, $0 < p_k \le q_k$ for all k, and $(q_k/p_k)_{k=1}^{\infty} \in \ell_\infty$. Then $x_k \to \xi[A,q]$ implies $x_k \to \xi[A,p]$.*

Maddox's sets are obtained as special cases of the above sets as follows:

$$\begin{aligned}
B = I: \quad & [I,p]_0 = c_0(p) = \left\{x \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\},\\
& [I,p] = c(p) = \left\{x \in \omega : \lim_{k\to\infty} |x_k - \xi|^{p_k} = 0 \text{ for some } \xi \in \mathbb{C}\right\},\\
& [I,p]_\infty = \ell_\infty(p) = \left\{x \in \omega : \sup_k |x_k|^{p_k} < \infty\right\};\\
B = C_1: \quad & [C_1,p]_0 = w_0(p) = \left\{x \in \omega : \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k}\right) = 0\right\},\\
& [C_1,p] = w(p) = \left\{\begin{array}{l} x \in \omega : \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} |x_k - \xi|^{p_k}\right) = 0 \\ \qquad\qquad \text{for some } \xi \in \mathbb{C}\end{array}\right\},\\
& [C_1,p]_\infty = w_\infty(p) = \left\{x \in \omega : \sup_n\left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k}\right) < \infty\right\};\\
B = [E,p]: \quad & [E,p]_\infty = \ell(p) = \left\{x \in \omega : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\right\}.
\end{aligned}$$

Remark 2.6. *Let B be the matrix with the rows $B_1 = e$ and $B_n = e - e^{[n-1]}$ for $n \ge 2$. Then we also have $\ell(p) = [B,p]_0$.*

The sets $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$ were defined and studied Nakano [34], Simons [39] and by Maddox [20, 21]; the sets $w_0(p)$, $w(p)$ and $w_\infty(p)$ were also defined and studied by Maddox [20]. The special cases of $c_0(1/k)$, $c(1/k)$ and $\ell_\infty(1/k)$ were studied, for instance, in [4, 10, 26, 37]. A detailed study of the topological structures of Maddox's spaces and the complete characterizations of matrix transformations between them can be found in [8] and [9].

Remark 2.7. *If $B = I$ or $A = C_1$, then $A \in \mathcal{A}$ and $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = 1$, and so the limit or strong (A,p) limit of $x \in c(p)$ or $x \in w(p)$ is unique by* Theorem 2.1.

We note that a study of the spaces

$$bs(p) = (\ell_\infty(p))_\Sigma\,,\ cs(p) = (c(p))_\Sigma,\ cs_0(p) = (c_0(p))_\Sigma \text{ and } (\ell(p))_\Sigma$$

and the determination of their α–, β– and γ–duals can be found in [5, Chapter2].
Let Δ denote the matrix of the first order differences, that is, with the rows $\Delta_1 = e^{(1)}$ and $\Delta_n = e^{(n)} - e^{(n-1)}$ for $n \ge 1$. Then various matrix transformations on the spaces $(c_0(p))_\Delta$, $(c(p))_\Delta$, $(\ell_\infty(p))_\Delta$ and $bv(p) = (\ell(p))_\Delta$ were characterized in [31] and [11].

If $p > 0$ is a constant and $p_k = p$ for all k, then the sets $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, $bs(p)$ and $cs(p)$ reduce to the familiar sets c_0, c, ℓ_∞, ℓ_p, bs and cs, respectively, and $w_0(p) = w_0^p$, $w(p) = w^p$ and $w_\infty(p) = w_\infty^p$ ([23]).

We close this subsection with some set inclusions between Maddox's spaces.

Remark 2.8. *(a) Let* $p = (p_k)_{k=1}^\infty \in \ell_\infty$*. Since the nonnegative matrices* I *and* C_1 *satisfy* (2), *it follows from* Remark 2.3 *that*

$$c_0(p) \subset c(p) \subset \ell_\infty(p) \text{ and } w_0(p) \subset w(p) \subset w_\infty(p).$$

(b) If $0 < p_k \le q_k$ *for all* k*, then obviously* $\ell(p) \subset \ell(q)$ ([39, Lemma 2]).
(c) If $p_k, q_k > 0$ *for all* k*, then* $c_0(p) \subset c_0(q)$ *and* $\ell_\infty(p) \subset \ell_\infty(q)$ *if and and only if* ([20, Lemma 1 and remark on p. 348])

$$\liminf_{k\to\infty} \frac{p_k}{q_k} > 0;$$

since obviously $I, C_1 \in \mathcal{A}$ *and* $\|I\|, \|C_1\| < \infty$*, it follows from* Theorem 2.5 *that* $c(p) \subset c(q)$ *and* $w(p) \subset w(q)$ *whenever* $0 < p_k \le q_k$ *for all* k *and* $(q_k/p_k)_{k=1}^\infty \in \ell_\infty$*.*

Necessary and sufficient conditions for $w_0(p) \subset w_0(q)$ can be found in [24]. We observe that $w_0(p) \subset w_0(q)$ if and only if $w_0(p/q) \subset w_0$, where $p/q = (p_k/q_k)_{k=1}^\infty$ and w_0 denotes $w_0(p)$ with $p_k = 1$ for all k. So it is enough to give the exact conditions for $w_0(p) \subset w_0$ when p is any positive sequence.

Theorem 2.9. ([24, Theorem 7]) *Let* $p = (p_k)_{k=1}^\infty$ *be an arbitrary positive sequence and* $N_\nu(y)$ *denote the number of integers* k *in* $[2^\nu, 2^{\nu+1} - 1]$ *such that* $p_k \ge y$*.*
Then $w_0(p) \subset w_0$ *if and only if*

(i) there exists an integer $N > 1$ *such that*

$$H_\nu = \max_{2^\nu \le k \le 2^{\nu+1}-1} N^{-1/p_k} \cdot 2^{-\nu+\nu/p_k} = O(1)$$

and

(ii) $\inf_{y>1} \left[\limsup_{\nu\to\infty} 2^{-\nu} N_\nu(y) \right] = 0.$

An analogous result holds concerning the inclusion $w_\infty(p) \subset w_\infty(q)$.

Theorem 2.10. ([28, Lemma 2 and Corollary, p. 543]) *Let* $p = (p_k)_{k=1}^\infty$ *be an arbitrary positive sequence. Then* $w_\infty(p/q) \subset w_\infty$ *if and only if*

$$H_\nu = \max_{2^\nu \le k \le 2^{\nu+1}-1} 2^{-\nu+\nu/p_k} = O(1).$$

The following special cases may be of interest.

Corollary 2.11. *Let* $p_k, q_k > 0$ *for all* k*. Then*

(i) $c_0(p) \subset c_0$ *if and only if* $p = (p_k)_{k=1}^\infty \in \ell_\infty$*;*

(ii) $c_0 \subset c_0(q)$ *if and only if* $\inf_k q_k > 0$*;*

(iii) $\ell_\infty(p) \subset \ell_\infty$ *if and only if* $\inf_k p_k > 0$*;*

(iv) $\ell_\infty \subset \ell_\infty(q)$ *if and only if* $q = (q_k)_{k=1}^\infty \in \ell_\infty$*.*

Finally we obtain for constants $p, q > 0$.

Proposition 2.12. *If* $0 < p \le q$*, then* $w_q \subset w_p$*.*

Proof. Since the inclusion is trivial for $p = q$ we assume $t = q/p > 1$. Let $x \in w_q$. Then we have by Hölder's inequality with $s = t/(t-1)$

$$\begin{aligned} 0 &\le \sum_{k=1}^{n} \frac{1}{n}|x_k - \xi|^p = \sum_{k=1}^{n} \left(\frac{1}{n}\right)^{1/t} |x_k - \xi|^p \left(\frac{1}{n}\right)^{1/s} \\ &\le \left(\sum_{k=1}^{n} \frac{1}{n}|x_k - \xi|^{pt}\right)^{1/t} \left[\sum_{k=1}^{n} \left(\frac{1}{n}\right)^{s/s}\right]^{1/s} \\ &= \left(\frac{1}{n}\sum_{k=1}^{n} |x_k - \xi|^q\right)^{1/t} \to 0 \ (n \to \infty), \end{aligned}$$

that is, $x \in w_p$. □

2.2. Linearity of the spaces

Here we study the linearity of the spaces above. It will turn out that each of those sets is a linear space if and only if the sequence p is bounded. To see this we use the following general results.

Throughout let B be a nonnegative matrix.

It is easy to show that $p \in \ell_\infty$ is a sufficient condition for $[B,p]_0$, $[B,p]$ and $[B,p]_\infty$ to be linear spaces.

Theorem 2.13. *If $p = (p_k)_{k=1}^\infty \in \ell_\infty$, then the sets $[B,p]_0$, $[B,p]$ and $[B,p]_\infty$ are linear spaces.*

Proof. We assume $p \in \ell_\infty$, and put $H = \sup_k p_k$.
We put $C = \max\{1, 2^{H-1}\}$, $L = \max\{1, |\lambda|^H\}$ and $M = \max\{1, |\mu|^H\}$ for$\lambda, \mu \in \mathbb{C}$. Since

$$|a_k + b_k|^{p_k} \le C\left(|a_k|^{p_k} + |b_k|^{p_k}\right) \text{ for each } k \in \mathbb{N} \tag{4}$$

and

$$|\lambda|^{p_k} \le L, \ |\mu|^{p_k} \le M \text{ for each } k \in \mathbb{N},$$

it follows that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} B_n\left(|\lambda x + \mu y - (\lambda\xi + \mu\eta)e|^p\right) \\ &\le C \cdot L \cdot \limsup_{n\to\infty} B_n\left(|x - \xi e|^p\right) + C \cdot M \cdot \limsup_{n\to\infty} B_n\left(|y - \eta e|^p\right). \end{aligned} \tag{5}$$

We consider the case of $[B,p]$; the cases of $[B,p]_0$ and $[B,p]_\infty$ are treated similarly.
Let $x, y \in [B,p]$. Then there exist $\xi, \eta \in \mathbb{C}$ with $\lim_{n\to\infty} B_n(|x - \xi e|^p) = 0$ and $\lim_{n\to\infty} B_n(|y - \eta e|^p) = 0$, and it follows from (5) that

$$\lim_{n\to\infty} B_n\left(|\lambda x + \mu y - (\lambda\xi + \mu\eta)e|^p\right) = 0,$$

that is, $\lambda x + \mu y \in [B,p]$. □

Remark 2.14. *Let $B = I$ or $B = C_1$. Then the $[B,p]$ limit ξ is unique for each $x \in [B,p]$ by* Remark 2.7*, that is, $\xi : [B,p] \to \mathbb{C}$ where $\xi(x)$ is such that $x - \xi(x) \cdot e \in [B,p]_0$, defines a functional, which is linear by the proof of* Theorem 2.13 *whenever $p = (p_k)_{k=1}^\infty \in \ell_\infty$.*

The following results concerning the exact conditions for the linearity of the spaces $[B,p]_0$, $[B,p]$ and $[B,p]_\infty$ can be found in [24].

Theorem 2.15. ([24, Theorem 1]) *The set $[B,p]_\infty$ is a linear space if and only if $\sup_{k\in S} p_k < \infty$, where $S = \{k \in \mathbb{N} : 0 < \sup_n b_{nk} < \infty\}$.*

Theorem 2.16. ([24, Theorem 2]) *Let B be a lower triangle with $B^{(k)} \in c_0$ for all k. Then $[B, p]_0$ is a linear space if and only if* $\sup_{k\in S} p_k < \infty$, *with S as in Theorem 2.15.*

Remark 2.17. *The hypothesis* $\sup_{k\in S} p_k < \infty$ *in place of $(p_k)_{k=1}^{\infty} \in \ell_\infty$ in* Theorem 2.15 *excludes the following trivial cases*

(a) If $\sup_n b_{nk} = 0$ for some $k_0 \in \mathbb{N}$ then $B^{(k_0)} = 0$, in which case p_{k_0} is not subject to any restrictions.

(b) If $B^{(k_0)} \notin \ell_\infty$ for some $k_0 \in \mathbb{N}$, then we must have $x_{k_0} = 0$ for all $x = (x_k)_{k=1}^{\infty} \in [B, p]_0$, for otherwise $b_{n,k_0}|\lambda x_{k_0}|^{p_k}$ could be arbitrarily large for $|\lambda| > 1$. Again p_{k_0} can be chosen arbitrarily. This cannot happen in view of the assumption $B^{(k)} \in c_0$ for all k.

Theorem 2.18. ([24, Theorem 3] and [27, p. 593]) *Let B be a triangle and*

$$T = \left\{k \in \mathbb{N} : 0 = \lim_{n\to\infty} b_{nk} < \sup_n b_{nk} < \infty\right\}.$$

Then $[B, p]$ is a linear space if and only if $\sup_{k\in T} p_k < \infty$.

Remark 2.19. *If $B = I$ or $B = C_1$, triangles, then clearly $T = S = \mathbb{N}$, and $B^{(k)} \in c_0$ for all k, and so Maddox's spaces are linear spaces by* Theorems 2.15, 2.16 *and* 2.18 *if and only if $p = (p_k)_{k=1}^{\infty} \in \ell_\infty$* ([20, Theorem 1], [24, Theorem 2], [27] and [25, Theorem 17, p. 190]).

2.3. Topological structures of the spaces

Here we list the fundamental topological properties of Maddox's spaces. Throughout this subsection, we assume that B is a nonnegative matrix and $p = (p_k)_{k=1}^{\infty} \in \ell_\infty$. We put $H = \sup_k p_k$ and $M = \max\{1, H\}$. If we define d on $[B, p]_0$ and $[B, p]_\infty$ by

$$d(x, y) = \sup_n \left[B_n(|x - y|^p)\right]^{1/M} = \sup_n \left(\sum_{k=1}^{\infty} b_{nk}|x_k - y_k|^{p_k}\right)^{1/M}, \tag{6}$$

then d is a natural semi–metric on both spaces; if, in addition, B satisfies the condition in (2), then (6) also defines a natural semi–metric on $[B, p]$. The triangle inequality is established as follows. We put $t_k = p_k/M \le 1$ for all k, apply the inequality in (4) with $t_k \le 1$ in place of p_k and $C = 1$, and Minkowski's inequality, and obtain for all n.

$$\begin{aligned}
(B_n(|a + c|^p))^{1/M} &= \left(\sum_{k=1}^{\infty} b_{nk}\left[|a_k + c_k|^{t_k}\right]^M\right)^{1/M} \\
&\le \left(\sum_{k=1}^{\infty} \left[b_{nk}^{1/M}\left(|a_k|^{t_k} + |c_k|^{t_k}\right)\right]^M\right)^{1/M} \\
&\le \left(\sum_{k=1}^{\infty} b_{nk}|a_k|^{t_k M}\right)^{1/M} + \left(\sum_{k=1}^{\infty} b_{nk}|c_k|^{t_k M}\right)^{1/M} \\
&\le \left(\sum_{k=1}^{\infty} b_{nk}|a_k|^{p_k}\right)^{1/M} + \left(\sum_{k=1}^{\infty} b_{nk}|c_k|^{p_k}\right)^{1/M} = (B_n(|a|^p))^{1/M} + (B_n(|c|^p))^{1/M}.
\end{aligned}$$

If we write $a_k = x_k - z_k$ and $c_k = z_k - y_k$ for all k, then it follows that

$$d(x, y) \le d(x, z) + d(z, y).$$

Remark 2.20. *(a) It follows from* Remark 2.3 *that if the matrix B satisfies the condition in* (2)*, then d in* (6) *is also a semi–metric on* $[B,p]$.
(b) It is clear that if $p=(p_k)_{k=1}^{\infty}\in\ell_\infty$ and B has no zero columns, then the semi–metrics are metrics; in particular, Maddox's spaces are metric spaces with respect to their natural metrics.

The space $[B,p]_0$ is a linear topological space.

Theorem 2.21. ([21, Theorem 1]) *For any nonnegative matrix B and any sequence $p=(p_k)_{k=1}^{\infty}\in\ell_\infty$, $[B,p]_0$ is paranormed by*

$$g(x)=d(x,0)=\sup_n\left(\sum_{k=1}^{\infty}b_{nk}|x_k|^{p_k}\right)^{1/M}\quad\text{for all } x\in[B,p]_0. \tag{7}$$

The next result gives a sufficient condition for $[B,p]_\infty$ to have a natural paranorm.

Theorem 2.22. ([21, Corollary 2]) *If B is a nonnegative matrix B and $p=(p_k)_{k=1}^{\infty}\in\ell_\infty$ is a sequence with* $0<\inf_k p_k$*, then $[B,p]_\infty$ is paranormed by g in* (7)*.*

The condition $\inf_k p_k>0$ is also necessary for $[B,p]_\infty$ to be paranormed by g when the matrix B satisfies some additional conditions.

Theorem 2.23. ([24, Theorem 4]) *Let B be a nonnegative triangle such that $B^k\in c_0$ and $M_k=\sup_n b_{nk}>0$ for each k. If $[B,p]_\infty$ is paranormed by g then* $\inf_k p_k>0$*.*

Remark 2.24. *Since, by* Remark 2.3, $[B,p]\subset[B,p]_\infty$ *if and only if the condition in* (2) *holds, the results of* Theorems 2.21 *and* 2.22 *also hold for $[B,p]$ in this case.*

Now we study the completeness of $[B,p]_0$, $[B,p]$ and $[B,p]_\infty$. We assume $p=(p_k)_{k=1}^{\infty}\in\ell_\infty$, and that the matrix B is nonnegative and has no zero columns in which case d in (6) is a metric by Remark 2.20 (b).

Theorem 2.25. *Let $p=(p_k)_{k=1}^{\infty}\in\ell_\infty$ and B be a nonnegative matrix with no zero columns.*
(a) Then $[B,p]_0$ and $[B,p]_\infty$ are complete with their natural metric defined in (6) ([24, p. 318])*.*
(b) If B satisfies the condition in (2)*, then either of the following conditions is sufficient for $[B,p]$ to be complete* ([24, Theorem 5])*:*

(i) $\limsup_{n\to\infty}\sum_{k=1}^{\infty}b_{nk}=0$*,*

(ii) $\limsup_{n\to\infty}\sum_{k=1}^{\infty}b_{nk}>0$ *and* $\inf_k p_k>0$*.*

We summarize the statements for Maddox's spaces.

Corollary 2.26. *Let $p=(p_k)_{k=1}^{\infty}\in\ell_\infty$.*

(a) Then $c_0(p)$, $w_0(p)$ and $\ell(p)$ are complete with their natural total paranorms

$$g(x)=\begin{cases}\sup_k|x_k|^{p_k/M} & \text{on } c_0(p)\\ \sup_n\left(\dfrac{1}{n}\sum\limits_{k=1}^{n}|x_k|^{p_k}\right)^{1/M} & \text{on } w_0(p)\\ \left(\sum\limits_{k=1}^{\infty}|x_k|^{p_k}\right)^{1/M} & \text{on } \ell(p).\end{cases} \tag{8}$$

(b) The spaces $\ell_\infty(p)$ and $w_\infty(p)$ are complete metric spaces with their natural metrics $d(x,y)=g(x-y)$ of Part (a); they are complete totally paranormed spaces if and only if $\inf_k p_k>0$*.*

(c) *The spaces $c(p)$ and $w(p)$ are complete metric spaces with their natural metrics $d(x,y) = g(x-y)$ of Part (a); if $\inf_k p_k > 0$ then they are complete totally paranormed spaces.*

Remark 2.27. *(a) If $p = (p_k)_{k=1}^\infty \in \ell_\infty$ and $\inf_k p_k > 0$, then the spaces $c(p)$, $\ell_\infty(p)$, $w(p)$ and $w_\infty(p)$ reduce to the well–known classical spaces c, ℓ_∞, w^p and w_∞^p.*
(b) It is usual to use the equivalent paranorm or metric

$$h(x) = \left(\sup_{\nu \geq 0} \frac{1}{2^\nu} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^{p_k}\right)^{1/M}$$

on $w_0(p)$, $w(p)$ and $w_\infty(p)$.

Remark 2.28. *The space $\ell(p)$ is an FK space with its natural paranorm; it is even a BK space if $p_k \geq 1$ for all k, with a norm given by*

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^\infty \left|\frac{x_k}{\rho}\right|^{p_k} \leq 1\right\}.$$

2.4. *Some further results*

The results in the previous subsection are not completely satisfactory in view of Corollary 2.26 (b) and (c) and Remark 2.27 (a). More satisfactory results concerning the topological structure of the spaces $c(p)$ and $\ell_\infty(p)$ were established by Grosse–Erdmann in [8]. His results are based on the theory of echelon and co–echelon spaces. The following definitions can be found in [8].

We write $X_0 = c_0$ and $X_p = \ell_p$ for $1 \leq p \leq \infty$. Let $A = (a_{nk})_{n,k=1}^\infty$ be a Köthe matrix, that is, a matrix with $a_{n+1,k} \geq a_{nk} > 0$ for all $n, k \in \mathbb{N}$ ([2, 1.2]), and let $V = (1/a_{nk})_{n,k=1}^\infty$ be the associated matrix. Then the spaces

$$\lambda_p(A) = \bigcap_{n=1}^\infty \{x \in \omega : A_n \cdot x = (a_{nk}x_k)_{k=1}^\infty \in X_p\}$$

and

$$\kappa_p(V) = \bigcup_{n=1}^\infty \{x \in \omega : V_n \cdot x \in X_p\}$$

are called *echelon* and *co–echelon spaces of order p*. Echelon spaces of order 1 and co–echelon spaces of order ∞ are known as the echelon and co–echelon spaces of Köthe ([13, §30.8]). The matrix $V = (v_{nk})_{n,k=1}^\infty$ is said to be regularly decreasing if for every $n \in \mathbb{N}$ there exists some $m \geq n$ such that for all subsets K of $\mathbb{N}$,

$$\inf_{k \in K} \frac{v_{mk}}{v_{nk}} > 0 \text{ implies } \inf_{k \in K} \frac{v_{lk}}{v_{nk}} > 0 \text{ for all } l \geq m.$$

The following result is known

Theorem 2.29. ([8, Theorem 0]) *Let $p = (p_k)_{k=1}^\infty$ be a bounded sequence of positive reals. Then*

$$c_0(p) = \bigcap_{n=1}^\infty \left\{x \in \omega : \lim_{k \to \infty} |x_k| n^{1/p_k} = 0\right\}$$

and

$$\ell_\infty(p) = \bigcup_{n=1}^\infty \left\{x \in \omega : \sup_k |x_k| n^{1/p_k} < \infty\right\}.$$

Hence $c_0(p)$ is an echelon space of order 0 and $\ell_\infty(p)$ is a co–echelon space of order ∞.

Moreover ([8, Section 4]), we observe $c(p) = c_0(p) \oplus e$, and consider $c_0(p)$, $\ell_\infty(p)$ and $c(p)$ endowed with their projective and inductive limit, and limit, and direct sum topologies, respectively. The following result holds.

Theorem 2.30. ([8, Theorem 2]) *Let $p = (p_k)_{k=1}^\infty$ be a bounded sequence of positive reals. Then*
(a) $c_0(p)$ and $c(p)$ are FK spaces; they are normable if and only if $\inf_k p_k > 0$;
(b) $\ell_\infty(p)$ is a complete IBK space which is metrizable if and only if $\inf_k p_k > 0$.

Remark 2.31. *It is easy to see that $\ell(p)$ and $c(p)$ have AK.*

Remark 2.32. *As we saw in the previous subsection, Maddox studied his sequence spaces within the framework of paranormed sequence spaces. In each of the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$, he considered the function $g(x) = \sup_k |x_k|^{p_k/M}$ and introduced a topology τ_g via the corresponding metric $d(x, y) = g(x - y)$* (Remarks 2.24 and 2.27).

In $\ell_\infty(p)$, g is a paranorm and τ_g is a linear topology only in the trivial case $\inf_k p_k > 0$, when $\ell_\infty(p) = \ell_\infty$ ([39, Theorem 9]). *Indeed, by* Theorem 2.30, *the natural topology τ_g of $\ell_\infty(p)$ is not metrizable hence not paranormable unless $\ell_\infty(p) = \ell_\infty$.*

In $c_0(p)$, g is a paranorm and τ_g is an FK topology ([21, Theorem 1], [24, p. 318] and [28, Theorem 2]) *so that, by the uniqueness of FK topologies* ([41, Corollary 4.2.4]), *τ_g coincides with the projrctive limit topology for $c_0(p)$* (Theorem 2.30 (a)).

In $c(p)$, again g is a paranorm and τ_g is a linear topology only if $\inf_k p_k > 0$, when $c(p) = c$, by an argument as in [39, Theorem 9]. But, in contrast to $\ell_\infty(p)$, the natural topology of $c(p)$ can be induced by a paranorm. A convenient one is $g_1(x) = g(x - \xi e) + |\xi|$, where ξ is the unique number with $x - \xi \cdot e \in c_0(p)$.

Remark 2.33. *To the best of the authors' knowledge, no results seem to exist concerning the topologies of the spaces $w_\infty(p)$ and $w(p)$ analogous to those for $\ell_\infty(p)$ and $c(p)$ in* [8].

3. The Dual Spaces

Here we list the dual spaces of Maddox's spaces, and of $w_0(p)$, $w(p)$ and $w_\infty(p)$ in the known cases, that is, when $0 < p_k \leq 1$ for all k.

3.1. The β–duals

First we give the β–duals of Maddox's spaces. The following results are known and were also listed in [18]. Throughout, let $p = (p_k)_{k=1}^\infty$ be a positive sequence, not necessarily bounded. We need the following notations. If $p_k > 1$ then we put $q_k = p_k/(p_k - 1)$, and write

$$M(p) = \bigcup_{n=2}^\infty \left\{a \in \omega : (a_k/n)_{k=1}^\infty \in \ell(q)\right\},$$

$$M_0^{(1)}(p) = \bigcup_{n=2}^\infty \left\{a \in \omega : \left(a_k n^{-1/p_k}\right)_{k=1}^\infty \in \ell_1\right\},$$

$$M_\infty^{(1)}(p) = \bigcap_{n=2}^\infty \left\{a \in \omega : \left(a_k n^{1/p_k}\right)_{k=1}^\infty \in \ell_1\right\},$$

$$M_0^{(2)}(p) = \bigcap_{n=2}^\infty \left\{a \in \omega : \left(a_k n^{1/p_k}\right)_{k=1}^\infty \in \ell_\infty\right\}$$

and

$$M_\infty^{(2)}(p) = \bigcup_{n=2}^\infty \left\{a \in \omega : \left(a_k n^{-1/p_k}\right)_{k=1}^\infty \in \ell_\infty\right\}.$$

The first and second β–duals are known for $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$.

Theorem 3.1. *We have*

(a) $[\ell(p)]^\beta = \begin{cases} \ell_\infty(p) & \text{if } p_k \le 1 \text{ for all } k \text{ ([39, Theorem 7])} \\ M(p) & \text{if } p_k > 1 \text{ for all } k \text{ ([22, Theorem 1]);} \end{cases}$

(b) $[c_0(p)]^\beta = M_0^{(1)}(p)$ ([22, Theorem 6]);

(c) $[c(p)]^\beta = M_0^{(1)}(p) \cap cs$ ([14, Theorem 1]);

(d) $[\ell_\infty(p)]^\beta = M_\infty^{(1)}(p)$ ([15, Theorem 2]).

Theorem 3.2. *We have*

(a) ([14, Theorem 4]) $[\ell(p)]^{\beta\beta} = \begin{cases} M_\infty^{(1)}(p) & \text{if } p_k \le 1 \text{ for all } k \\ \ell(p) & \text{if } p_k > 1 \text{ for all } k; \end{cases}$

(b) $[c_0(p)]^{\beta\beta} = M_0^{(2)}(p)$ ([14, Theorem 2]);

(c) $[c(p)]^{\beta\beta} = M_\infty^{(2)}(p) + bv$ ([8, Theorem 4 (ii)]);

(d) $[\ell_\infty(p)]^\beta = M_\infty(p)$ ([15, Theorem 3]);

The first and second β–duals of $w_0(p)$, $w(p)$ and $w_\infty(p)$ in the general case are only known when $p_k \le 1$ for all n. Some results on $[w_0(p)]^\beta$ and $[w_\infty(p)]^\beta$ for some special cases of sequences $p = (p_k)_{k=1}^\infty$ with $p_k > 1$ can be found in [30]. We need the following notations:

$$\mathcal{W}_0^{(1)}(p) = \bigcup_{n=2}^\infty \left\{ a \in \omega : \sum_{\nu=0}^\infty \max_{2^\nu \le k \le 2^{\nu+1}-1} \left[(2^\nu/n)^{1/p_k} \cdot |a_k| \right] < \infty \right\},$$

$$\mathcal{W}_\infty^{(1)}(p) = \bigcap_{n=2}^\infty \left\{ a \in \omega : \sum_{\nu=0}^\infty \max_{2^\nu \le k \le 2^{\nu+1}-1} \left[(2^\nu n)^{1/p_k} \cdot |a_k| \right] < \infty \right\},$$

$$\mathcal{W}_0^{(2)}(p) = \bigcap_{n=2}^\infty \left\{ a \in \omega : \sup_{\nu \ge 0} \left[\sum_{k=2^\nu}^{2^{\nu+1}-1} (2^{-\nu} n)^{1/p_k} \cdot |a_k| \right] < \infty \right\}$$

and

$$\mathcal{W}_\infty^{(2)}(p) = \bigcup_{n=2}^\infty \left\{ a \in \omega : \sup_{\nu \ge 0} \left[\sum_{k=2^\nu}^{2^{\nu+1}-1} (2^{-\nu}/n)^{1/p_k} \cdot |a_k| \right] < \infty \right\}$$

Theorem 3.3. *Let $0 < p_k \le 1$ for all k. Then we have*

(a) $[w_0(p)]^\beta = [w(p)]^\beta = \mathcal{W}_0^{(1)}(p)$ ([17, p. 84], [15, Theorem 3]);

(b) $[w_\infty(p)]^\beta = \mathcal{W}_\infty^{(1)}(p)$ ([17, p. 84]);

(c) $[w_0(p)]^{\beta\beta} = [w(p)]^{\beta\beta} = \mathcal{W}_0^{(2)}(p)$ ([17, p. 88, 86]) ;

(d) $[w_\infty(p)]^\beta = \mathcal{W}_\infty^{(2)}(p)$ ([17, p. 88]).

The next result concerns the β–reflexivity of the sets $\ell(p)$, $\ell_\infty(p)$ and $c_0(p)$.

Theorem 3.4. *Let $p = (p_k)_{k=1}^\infty$ be a positive sequence. Then we have*

(a) ([15, Theorem 4])

$$[\ell(p)]^{\beta\beta} = \ell(p) \text{ if and only if } \begin{cases} \ell(p) = \ell_1 & \text{if } p_k \leq 1 \text{ for all } k \\ p \in \ell_\infty & \text{if } p_k > 1 \text{ for all } k; \end{cases}$$

(b) $[c_0(p)]^{\beta\beta} = c(p)$ *if and only if* $p \in c_0$ ([15, Theorem 8]);

(c) $[\ell_\infty(p)]^{\beta\beta} = \ell_\infty(p)$ *if and only if* $p \in \ell_\infty$ ([15, Theorem 5]).

Finally we consider a few special cases.

Theorem 3.5. *Let $p = (p_k)_{k=1}^\infty$ be a positive sequence. Then we have:*

(a) *if $p_k \leq 1$ for all k, then $[\ell(p)]^\beta = \ell_\infty$ if and only if $\inf_k p_k > 0$; if $p \in \ell_\infty$, $p_k > 1$ and $q_k = p_k/(p_k - 1)$ for all k then $[\ell(p)]^\beta = \ell(q)$ if and only if $\inf_k p_k > 1$* ([22, p. 432]);

(b) *the following statements are equivalent* ([15, Theorem 8])

$$\text{(i) } \inf_k p_k > 0, \quad \text{(ii) } [c_0(p)]^\beta = \ell_1, \quad \text{(iii) } [c_0(p)]^{\beta\beta} = \ell_\infty;$$

(c) *the following statements are equivalent* ([15, Theorem 6])

$$\text{(i) } \inf_k p_k > 0, \quad \text{(ii) } [\ell_\infty(p)]^\beta = \ell_1, \quad \text{(iii) } [\ell_\infty(p)]^{\beta\beta} = \ell_\infty;$$

(d) *the following statements are eqivalent* ([15, Theorem 7])

$$\text{(i) } \inf_k p_k > 0, \quad \text{(ii) } [c(p)]^\beta = \ell_1, \quad \text{(iii) } [c(p)]^{\beta\beta} = \ell_\infty, \quad \text{(iv) } c_0 \subset c(p).$$

3.2. The α, γ–, functional and topological duals

We use the following notations: If X and Y are isomorphic linear spaces, then we write $X \equiv Y$; if X and Y are linearly homeomorphic spaces, then we write $X \sim Y$.

First, we observe that since all the spaces with the exception of $c(p)$ and $w(p)$ are normal, the α–, β– and γ– duals coincide for them.

The following results hold for the f– and continuous duals of $c_0(p)$, $\ell(p)$ and $\ell_\infty(p)$. We assume that $p = (p_k)_{k=1}^\infty$ is a positive sequence with $p \in \ell_\infty$. Since $\ell(p)$ and $c_0(p)$ are FK spaces with AK (Remark 2.31), we have $[\ell(p)]^f = [\ell(p)]^\beta \equiv [\ell(p)]'$ and $[c_0(p)]^f = [c_0(p)]^\beta \equiv [c_0(p)]'$ by [41, Theorems 7.2.7 (ii) and 7.2.9].

Before we state the next theorem we recall that if X is a linear topological space then we denote by X'_b the continuous dual of X with the strong topology. In the case of locally convex spaces the strong topology coincides with the topology of uniform convergence of bounded sets in X. Thus, for instance $f_n \to f$ in $[\ell(p)]'_b$ means that $f_n(x) \to f(x)$ uniformly on any sphere of $\ell(p)$. For instance, in the case of $\ell(p)$, we have $[\ell(p)]' \equiv M(p)$ if $p_k > 1$ for all k, and $[\ell(p)]' \equiv \ell_\infty(p)$ if $p_k \leq 1$ for all k, by Theorem 3.1 (a). We define $a^{(n)} \to a$ $(n \to \infty)$ in $M(p)$ or in $\ell_\infty(p)$ to mean that

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_k^{(n)} x_k = \sum_{k=1}^\infty a_k x_k \text{ uniformly in } x \text{ on any sphere of } \ell(p).$$

With this definition $M(p)$ and $\ell_\infty(p)$ become linear topological spaces, denoted by $\widehat{M}(p)$ and $\widehat{\ell_\infty}(p)$.

The following results hold.

Theorem 3.6. *(a) If* $1 < p_k$ *for all k, then* $[\ell(p)]_b' \sim \widehat{M}(p)$ ([22, Theorem 3]).
(b) If $1 < \inf p_k$ *and* $\ell(q)$ *has its natural paranorm topology, then* $[\ell(p)]_b' \sim \ell(q)$ ([22, Theorem 4]).
(c) If $0 < p_k \leq 1$ *for all k then* $[\ell(p)]_b' \sim \widehat{\ell_\infty}(p)$ ([22, Theorem 5]).

Only partial results seem to be known concerning the continuous duals of $w_0(p)$ and $w(p)$.

Theorem 3.7. *Let* $0 < \inf_k p_k \leq p_k \leq 1$ *for all k and*

$$\widetilde{\mathcal{W}}(p) = \left\{ a \in \omega : \sum_{\nu=0}^{\infty} \max_\nu \left| 2^{\nu/p_k} \cdot a_k \right| < \infty \right\}.$$

(a) Then we have $[w(p)]' \equiv \mathbb{C} \times \widetilde{\mathcal{W}}(p)$. *This means that, for arbitrary* $\alpha \in \mathbb{C}$, $a \in \widetilde{\mathcal{W}}(p)$ *and* $x \in w(p)$ *with* $x_k \to \xi(w(p))$,

$$f_a(x) = \alpha\xi + \sum_{k=1}^{\infty} a_k x_k, \text{ where } \alpha = f(e) - \sum_{k=1}^{\infty} f(e^{(k)}) \text{ and } a_k = f(e^{(k)}) \text{ for all } k \tag{9}$$

defines an element of $[w(p)]'$, *and conversely every element of* $[w(p)]'$ *can be represented in this form and the map* $T : \widetilde{\mathcal{W}}(p) \to [w(p)]'$ *with* $Ta = f_a$ *defined in (9) is an isomorphism* ([20, Theorem 5]).
(b) Then we have $[w_0(p)]' \equiv \widetilde{\mathcal{W}}(p)$. *This means that, for arbitrary* $a \in \widetilde{\mathcal{W}}(p)$ *and* $x \in w_0(p)$,

$$f_a(x) = \sum_{k=1}^{\infty} a_k x_k, \text{ where } a_k = f(e^{(k)}) \text{ for all } k \tag{10}$$

defines an element of $(w_0(p))'$, *and conversely every element of* $[w_0(p)]'$ *can be represented in this form and the map* $T : \widetilde{\mathcal{W}}(p) \to [w_0(p)]'$ *with* $Ta = f_a$ *defined in (10) is an isomorphism* ([20, pp. 354–355]).

Maddox also obtained the continuous dual of $w_0(p)$ for certain sequences $p = (p_k)_{k=1}^\infty \in c_0$.

Theorem 3.8. ([21, Theorem 3]) *Let Q be the set of all sequences* $p = (p_k)_{k=1}^\infty$ *of positive real numbers for which a number* $N = N(p) > 1$ *exists such that* $\sum_{k=1}^\infty N^{-1/p_k} < \infty$. *Then we have*

$$[w_0(p)]' \equiv \mathcal{S}(p) = \left\{ a \in \omega : \sup_\nu 2^\nu \max_\nu |a_k|^{p_k} < \infty \right\}.$$

Concerning the continuous duals of the other Maddox spaces the following results are known.

Theorem 3.9. ([8, Theorem 4 (i)]) *Let* $p = (p_k)_{k=1}^\infty$ *be a positive sequence with* $p \in \ell_\infty$. *Then we have* $[c_0(p)]^f = M_0^{(1)}(p)$, $[c_0(p)]_b' \sim M_0^{(1)}(p)$, $[c_0(p)]^{ff} = M_\infty^{(2)}(p)$, $\left([c_0(p)]_b'\right)_b' \sim M_\infty^{(2)}(p)$,

$$\begin{array}{lll} [\ell_\infty(p)]^f = M_\infty^{(1)}(p), & [\ell_\infty(p)]' \supset M_\infty^{(1)}(p) & \text{with} \\ & [\ell_\infty(p)]' \equiv M_\infty^{(1)}(p) & \text{if and only if } p \in c_0, \\ & & \text{in this case } [\ell_\infty(p)]_b' \sim M_\infty^{(1)}(p); \\ [\ell_\infty(p)]^{ff} = \ell_\infty(p), & \left([\ell_\infty(p)]_b'\right)_b' \sim \widehat{\ell_\infty}(p). & \end{array}$$

We also have for $c(p)$ the next results.

Theorem 3.10. ([8, Theorem 4 (ii)]) *Let* $p = (p_k)_{k=1}^\infty$ *be a positive sequence. Then we have*

(a) $[c(p)]^\alpha = M_0^{(1)}(p) \cap \ell_1$, $[c(p)]^\gamma = M_0^{(1)}(p) \cap bs$,
$[c(p)]^{\alpha\alpha} = M_\infty^{(2)}(p) + \ell_\infty$, $[c(p)]^{\gamma\gamma} = M_\infty^{(2)}(p) + bv$.

(b) *If* $p = (p_k)_{k=1}^{\infty} \in \ell_\infty$ *then*

$$\begin{array}{rclrcl} [c(p)]^f &=& M_0^{(1)}(p), & [c(p)]^{ff} &=& M_\infty^{(2)}(p), \\ [c(p)]_b' &\sim& \mathbb{C} \times M_0^{(1)}(p), & \left([c(p)]_b'\right)_b' &\sim& \mathbb{C} \times M_\infty^{(2)}(p). \end{array}$$

Remark 3.11. *In the special case where* p *is a constant sequence, that is,* $p_k = p > 0$ *for all* k*, the continuous duals of the spaces* ℓ_p $(0 < p < \infty)$, c_0 *and* w_0^p *are norm isomorphic to the* β*–duals, that is,* $\ell_p^* \sim \ell_\infty$ *for* $0 < p \le 1$, $\ell_p^* \sim \ell_q$ *for* $1 < p < \infty$ *and* $q = p/(p-1)$, *and* $(w_0^p)^* \sim (w_0^p)^\beta = \mathcal{W}^p$, *where*

$$\mathcal{W}^p = \begin{cases} \left\{a \in \omega : \|a\|_{\mathcal{W}^p} = \sum\limits_{\nu=0}^{\infty} 2^{\nu/p} \max_\nu |a_k| < \infty\right\} & (0 < p \le 1) \\ \left\{a \in \omega : \|a\|_{\mathcal{W}^p} = \sum\limits_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_\nu |a_k|^q\right)^{1/q} < \infty\right\} & (p > 1). \end{cases}$$

Also c^* *is norm isomorphic to* $\mathbb{C} \times \ell_1$ *and* $(w^p)^*$ *is norm isomorphic to* $\mathbb{C} \times \mathcal{W}^p$.

Remark 3.12. *(a) The* β*–duals of* $\Delta\ell_\infty(p) = (\ell_\infty(p))_\Delta$ *for arbitrary positive sequences* p *were determined in [33, Theorem 2]; the* β*–duals of* $\Delta c_0(p) = (c_0(p))_\Delta$, $\Delta c(p) = (c(p))_\Delta$ *and* $\Delta\ell_\infty(p)$ *for bounded positive sequences* p *were determined in [31, Theorem 1].*
(b) The results of (a) were generalized in [32]. Let u *be s sequence with no zero terms,* $m \in \mathbb{N}$ *and* Δ^m *denote the operator of the* m*–th difference, that is,* $\Delta^1 = \Delta$ *and* $\Delta^m = \Delta \circ \Delta^{m-1}$ *for* $m \ge 1$*. Given any set of sequences* X*, we write* $\Delta_u^m X = \{x \in \omega : u \cdot \Delta^m x \in X\}$*. Then the* α*–,* β*–duals and* γ*–duals of the sets* $\Delta_u^m X$*, when* X *is any of the sets* $c_0(p)$, $c(p)$ *or* $\ell_\infty(p)$*, were determined in [32, Theorems 2.1, 3.1, 3.2 and 4.1].*
(c) A detailed study of the β*–duals of the sets* $bv(p) = (\ell(p))_\Delta$ *can be found in [11].*

4. Matrix Transformations

In this section, we give a list of characterizations of matrix transformations between Maddox's sequence spaces. More general results and a comprehensive list can be found in [9].

Given a sequence $p = (p_k)_{k=1}^{\infty}$ of positive real numbers we write $K_1 = \{k \in \mathbb{N} : p_k \le 1\}$, $K_2 = \{k \in \mathbb{N} : p_k > 1\}$ and $q_k = p_k/(p_k - 1)$ for $k \in K_2$.

4.1. Matrix transformations involving $\ell(p)$

Many of the results in this subsection can also be found in a more general form in the next subsection.

A theorem concerns matrix transformations between $\ell(p)$, ℓ_∞, c_0 and ℓ_1.

Theorem 4.1. ([9, Proposition 3.2]) *Let* $p = (p_k)_{k=1}^{\infty} \in \ell_\infty$ *be a sequence of positive real numbers.*
(a) Then the necessary and sufficient conditions for $A \in (\ell(p), Y)$ *where* $Y = \ell_\infty, c_0, \ell_1$ *can be read from the following table*

From $\ell(p)$ *to*	ℓ_∞	c_0	ℓ_1
	1.	**2.**	**3.**

where

1. (1.1∗) $\begin{cases} \sup_n \sup_{k \in K_1} |a_{nk}|^{p_k} < \infty \\ \sup_n \sum_{k \in K_2} |a_{nk} M^{-1}|^{q_k} < \infty \text{ for some } M \in \mathbb{N} \end{cases}$

2. (1.1∗) *and* (2.1∗) $\lim_{n\to\infty} a_{nk} = 0$ *for all* $k \in \mathbb{N}$

3. (3.1∗) $\begin{cases} \sup\{|\sum_{n \in N} a_{nk}|^{p_k} : k \in K_1,\ N \subset \mathbb{N} \text{ finite}\} < \infty \\ \sup\{\sum_{k \in K_2} |\sum_{n \in N} a_{nk} M^{-1}|^{q_k} : N \subset \mathbb{N} \text{ finite}\} < \infty \\ \qquad \text{for some } M \in \mathbb{N}. \end{cases}$

(b) If $p_k \geq 1$ for all $k \in \mathbb{N}$, then the necessary and sufficient conditions for $A \in (X, \ell(p))$ when $X = \ell_\infty, c_0, \ell_1$ can be read from the following table

From	ℓ_∞	c_0	ℓ_1	*to* $\ell(p)$
	4.	**5.**	**6.**	

where

4., 5. (4.1∗) $\sup\{\sum_{n=1}^{\infty} |\sum_{k\in K} a_{nk}|^{p_n} : K \subset \mathbb{N} \textit{ finite}\} < \infty$

6. (6.1∗) $\sup_k \sum_{n=1}^{\infty} |a_{nk}|^{p_n} < \infty$.

Remark 4.2. *(a) The characterization of the class $(\ell(p), \ell_\infty)$ can also be found in [15, Theorem 1] or [29, Theorem 4].*
(b) The necessary and sufficient condition for $A \in (\ell(p), c)$ are (1.1∗) *and*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \textit{ exists for all } k \in \mathbb{N} \text{ ([15, Corollary, p. 101]).} \tag{11}$$

(c) Since $c_0 \subset c \subset \ell_\infty$ it follows from **4.** *and* **5.** *in Theorem 4.1 that $(c, \ell(p)) = (c_0, \ell(p)) = (\ell_\infty, (\ell(p))$ for $p_k \geq 1$ $(k = 1, 2, \dots)$.*

The following results are due to Maddox and Willey [29], and generalize some of the results in the previous theorem.

Theorem 4.3. *Let $p = (p_k)_{k=1}^{\infty}$ and $q = (q_n)_{n=1}^{\infty}$ be bounded sequences of positive real numbers, and $s_k = p_k/(p_k - 1)$ if $p_k > 1$. Then we have*
(a) ([29, Theorems 5 (i) and 7]) *$A \in (\ell(p), \ell_\infty(q))$ if and only if*

$$\begin{cases} \sup_n \left(\sup_k |a_{nk}| M^{-1/p_k}\right)^{q_n} < \infty \textit{ for some } M > 1 & (0 < p_k \leq 1) \\ \sup_n \sum_{k=1}^{\infty} |a_{nk}|^{s_k} M^{-s_k/q_n} < \infty \textit{ for some } M > 1 & (p_k > 1). \end{cases} \tag{12}$$

(b) [29, Theorem 5 (ii) and (iii)]) *Let $0 < p_k \leq 1$ for all k. Then we have $A \in (\ell(p), c_0(q))$ if and only if*

$$\lim_{n\to\infty} |a_{nk}|^{q_n} = 0 \textit{ for every } k \in \mathbb{N} \tag{13}$$

and

$$\lim_{M\to\infty} \limsup_{n\to\infty} \left(\sup_k |a_{nk}| M^{-1/p_k}\right)^{q_n} = 0; \tag{14}$$

$A \in (\ell(p), c(q))$ if and only if

$$\sup_{n,k} |a_{nk}| M^{-1/p_k} < \infty \tag{15}$$

and there exist $\alpha_1, \alpha_2, \dots \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0 \textit{ for every } k \in \mathbb{N} \tag{16}$$

and

$$\lim_{M\to\infty} \limsup_{n\to\infty} \left(\sup_k |a_{nk} - \alpha_k| M^{-1/p_k}\right)^{q_n} = 0. \tag{17}$$

(c) ([29, Theorems 8 and 9]) *Let $p_k > 1$ for all k. Then we have $A \in (\ell(p), c_0(q))$ if and only if the condition in (13) is satisfied and*

$$\lim_{B\to\infty} \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} |a_{nk}|^{s_k} D^{s_k/q_n} B^{-s_k} \right)^{q_n} = 0 \text{ for every } D \geq 1 \tag{18}$$

holds; $A \in (\ell(p), c(q))$ if and only if

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|^{s_k} B^{-s_k} < \infty \text{ for some } B > 1, \tag{19}$$

and there exist $\alpha_1, \alpha_2, \dots \in \mathbb{C}$ such that (16) holds and

$$\lim_{B\to\infty} \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} |a_{nk} - \alpha_k|^{s_k} D^{s_k/q_n} B^{-s_k} \right)^{q_n} = 0 \text{ for all } D \geq 1. \tag{20}$$

Remark 4.4. *The following special case is also known* ([29, Theorem 6]). *Let $0 < p_k \leq 1$ for all k and $q = (q_n)_{n=1}^{\infty} \in c_0$ then $A \in (\ell(p), c_0(q))$ if and only if the condition in (17) is satisfied.*

Now we give the characterizations of $(\ell_\infty(p), \ell_\infty)$, $(\ell_\infty(p), c)$ and $(w(p), c)$.

Theorem 4.5. *(a) Let $p_k > 0$ for every k. Then $A \in (\ell_\infty(p), \ell_\infty)$ if and only if*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| N^{1/p_k} < \infty \text{ for every integer } N > 1 \text{ ([15, Theorem 3]).} \tag{21}$$

(b) Let $p_k > 0$ for every k. Then $A \in (\ell_\infty(p), c)$ if and only if

$$\sum_{k=1}^{\infty} |a_{nk}| N^{1/p_k} \text{ converges uniformly in } n, \text{ for all integers } N > 1 \tag{22}$$

and (11) holds ([15, Corollary, p. 102]).
(c) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then $A \in (w(p), c)$ if and only if

$$\sup_n \sum_{\nu=0}^{\infty} \max_\nu \left((2^\nu B^{-1})^{1/p_k} |a_{nk}| \right) < \infty \text{ for some integer } B > 1, \tag{23}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \alpha \text{ exists} \tag{24}$$

and (11) holds ([15, Theorem 5]).

We mention a few more characterizations.

Remark 4.6. *(a) The characterization of the class $(c_0(p), c_0(p'))$ for arbitrary positive sequences $p = (p_k)_{k=1}^{\infty}$ and bounded poitive sequences $p' = (p'_k)_{k=1}^{\infty}$ can be found in [26, Theorem 1]; the special cases $p_k = p'_k$ for $k = 1, 2, \dots$, and $p_k = 1$ for all k and $p' \in c_0$ are due to Brown [4] and Roles [38], respectively. Lascarides [14, Corollary 2] gave the characterization of the class $(c_0(p), c)$, and the special case of $(c_0(1/k), c)$ is due to Rao [37, Theorem (III)].*
(b) The characterizations of the classes $(c_0(p), \ell_\infty(p))$ and $(c(p), c)$ are due to Lascarides [14, Theorems 10 and 9]: $A \in (c_0(p), \ell_\infty(p))$ if and only if

$$\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| B^{-1/p_k} \right)^{p_n} < \infty \text{ for some constant } B > 1; \tag{25}$$

$A \in (c(p), c)$ if and only if the conditions in (25) with $p_n = 1$ for all n, (11) and (24) are satisfied. When $p = (p_k)_{k=1}^{\infty} \in Q$ with Q defined as in Theorem 3.8, then $A \in (c_0(p), \ell_\infty(p))$ if and only if

$$\sup_{n,k} |a_{nk}|^{1/(1/p_k+1/p_n)} < \infty \text{ ([14, Theorem 11]).}$$

This generalizes the result for the characterization of the class $(c_0(1/k), \ell_\infty(1/k))$ by Rao ([37, Theorem (V)]).

At the end of this subsection, we mention the characterizations of the classes (bv, Y) where $bv = bv(e)$, and Y is any of the spaces $\ell(q)$, $\ell_\infty(q)$, $c_0(q)$ and $c(q)$ ([40]), and more generally, characterizations of some matrix transformations on $bv(p)$ ([11]).

Theorem 4.7. ([40, Theorems 1–4]) *Let $q = (q_n)_{n=1}^{\infty}$ be a bounded sequence of positive real numbers. Then the necessary and sufficient conditions for $A \in (bv, Y)$ where $Y = \ell(q), \ell_\infty(q), c_0(q), c(q)$ can be read from the following table*

From bv to	$\ell(q)$	$\ell_\infty(q)$	$c_0(q)$	$c(q)$
	7.	**8.**	**9.**	**10.**

where

7. (7.1∗) $\sup_j \sum_{n=1}^{\infty} \left| \sum_{k=j}^{\infty} a_{nk} \right|^{q_n} < \infty$

8. (8.1∗) $\sup_n \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} \right| M^{-1} \right)^{q_n} < \infty$ *for some $M > 1$*

9. (9.1∗) $\lim_{M\to\infty} \limsup_{n\to\infty} \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} \right| M^{-1} \right)^{q_n} = 0,$

(9.2) $\lim_{n\to\infty} \left| \sum_{k=j}^{\infty} a_{nk} \right|^{q_n} = 0$ *for each j*

10. (10.1∗) $\sup_{n,j} \left| \sum_{k=j}^{\infty} a_{nk} \right| < \infty,$

there exist $\alpha_1, \alpha_2 \cdots \in \mathbb{C}$ such that

(10.2) ∗ $\lim_{M\to\infty} \limsup_{n\to\infty} \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} - \alpha_j \right| M^{-1} \right)^{q_n} = 0,$

(10.3∗) $\lim_{n\to\infty} \left| \sum_{k=j}^{\infty} a_{nk} - \alpha_j \right|^{q_n} = 0$ *for each j.*

Remark 4.8. *The results in Theorem 4.7 are contained as special cases in [11, Theorems 3.1 and 3.2] where, for bounded sequences p and s of positive real numbers, the characterizations were established of the classes $(bv(p), \ell(s))$ for $p_k \leq 1$ and $s_k \geq 1$, $(bv(p), \ell_\infty(p))$, $(bv(p), c_0(s))$ and $(bv(p), c(p))$, and of $(bv(p), \ell_1)$, $(bv(p), \ell_\infty)$, $(bv(p), c_0)$ and $(bv(p), c)$.*

Remark 4.9. *If T is an arbitrary triangle then the characterization of any class (X, Y_T) can be reduced to that of (X, Y) by the trivial observation that $A \in (X, Y_T)$ if and only if $B = T \cdot A \in (X, Y)$.*

4.2. Matrix transformations between (co–)echelon spaces

The general results in this subsection are due to Grosse–Erdmann [9], and give the characterizations of matrix mappings between general (co–)echelon spaces.

Let $X_0 = c_0$ and $X_p = \ell_p$ for $1 \leq p \leq \infty$.

Theorem 4.10. ([9, Theorem 4.1]) *Let $B = (b_{nk})_{n,k=1}^\infty$ and $C = (c_{nk})_{n,k=1}^\infty$ be Köthe matrices, V and W be their associated matrices and $p, q \in \{0\} \cup [1, \infty]$. Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix. Then we have*
(a) $A \in (\kappa_p(V), \lambda_q(C))$ if and only if

$$(a_{nk}c_{Ln}b_{Mk})_{n,k=1}^\infty \in (X_p, X_q) \text{ for all } L, M \in \mathbb{N};$$

(b) $A \in (\kappa_p(V), \kappa_q(W))$ if and only if

$$\text{for all } M \in \mathbb{N} \text{ there exists } L \in \mathbb{N} \text{ with } (a_{nk}b_{Mk}/c_{Ln})_{n,k=1}^\infty \in (X_p, X_q).$$

If $(p, q) \neq (\infty, 0)$, then
(c) $A \in (\lambda_p(B), \lambda_q(C))$ if and only if

$$\text{for all } L \in \mathbb{N} \text{ there exists } M \in \mathbb{N} \text{ with } (a_{nk}c_{Ln}/b_{Mk})_{n,k=1}^\infty \in (X_p, X_q);$$

(d) $A \in (\lambda_p(B), \kappa_q(W))$ if and only if

$$\text{there exist } L, M \in \mathbb{N} \text{ with } (a_{nk}/(c_{Ln}b_{Mk}))_{n,k=1}^\infty \in (X_p, X_q).$$

Now we consider the case $(p, q) = (\infty, 0)$ which has been excluded in Theorem 4.10 (c), (d).

Theorem 4.11. ([9, Theorems 4.3 and 4.4]) *We use the notations of Theorem 4.10. Then we have*
(a) $A \in (\lambda_\infty(B), \lambda_0(C))$ if and only if

$$\begin{gathered}\text{for all } L \in \mathbb{N},\ (a_{nk}c_{Ln})_{k=1}^\infty \in \kappa_1(V) \text{ for all } n \in \mathbb{N} \\ \text{and} \\ (a_{nk}c_{Ln})_{k=1}^\infty \to 0 \text{ in } \kappa_1(V) \text{ as } n \to \infty;\end{gathered}$$

(b) $A \in (\lambda_\infty(B), \kappa_0(W))$ if and only if

$$\begin{gathered}\text{there exists } L \in \mathbb{N} \text{ with } (a_{nk}/c_{Ln})_{k=1}^\infty \in \kappa_1(V) \text{ for all } n \in \mathbb{N} \\ \text{and} \\ (a_{nk}/c_{Ln})_{k=1}^\infty \to 0 \text{ in } \kappa_1(V) \text{ as } n \to \infty.\end{gathered}$$

If V is regularly decreasing, then we have
(c) $A \in (\lambda_\infty(B), \lambda_0(C))$ if and only if

$$\begin{gathered}\text{for all } L \in \mathbb{N} \text{ there exists } M \in \mathbb{N} \text{ with } \sum_{k=1}^\infty |a_{nk}|\frac{c_{Ln}}{b_{Mk}} < \infty \text{ for all } n \in \mathbb{N} \\ \text{and} \\ \lim_{n\to\infty} \sum_{k=1}^\infty |a_{nk}|\frac{c_{Ln}}{b_{Mk}} = 0;\end{gathered}$$

(d) $A \in (\lambda_\infty(B), \lambda_0(C))$ if and only if

$$\begin{gathered}\text{there exist } L, M \in \mathbb{N} \text{ with } \sum_{k=1}^\infty \frac{|a_{nk}|}{c_{Ln}b_{Mk}} < \infty \text{ for all } n \in \mathbb{N} \\ \text{and} \\ \lim_{n\to\infty} \sum_{k=1}^\infty \frac{|a_{nk}|}{c_{Ln}b_{Mk}} = 0.\end{gathered}$$

The characterizations of matrix transformations between any two of the spaces $\ell(p), c_0(p), c(p)$ and $\ell_\infty(p)$ can be found in [9], of course, with the exception of $(\ell(p), \ell(q))$ in the general case; even the classical pair (ℓ_p, ℓ_q) is an open problem if $1 < p, q < \infty$ and $(p, q) \neq (2, 2)$. Also, in the case of $(E, \ell(q))$ it has to be assumed that $q_k \geq 1$ for all k. Let L and M denote natural numbers, N and K be finite subsets of $\mathbb{N}$, and α and α_k be complex numbers.

Remark 4.12. *(a) The class* $(\ell(p), \ell_1)$ *was characterized in Theorem 4.1* **3.**.
(b) ([9, Theorem 5.1 **0**]) *If* $p_k \leq 1$ *and* $q_k \geq 1$ *for all* $k \in \mathbb{N}$*, then* $A \in (\ell(p), \ell(q))$ *if and only if*

$$\sup_k \sum_{n=1}^{\infty} \left|a_{nk} M^{-1/p_k}\right|^{q_n} < \infty \text{ for some } M,$$

or equivalently,

$$\lim_{M\to\infty} \sup_k \sum_{n=1}^{\infty} \left|a_{nk} M^{-1/p_k}\right|^{q_n} = 0.$$

In the following theorem conditions in $(\times *)$ and $(\times * *)$ are equivalent.

Theorem 4.13. ([9, Theorem 5.1]) *Let* $p = (p_k)_{k=1}^{\infty}$ *and* $q = (q_k)_{k=1}^{\infty}$ *be sequences of positive real numbers and* $p \in \ell_\infty$*;* $q \in \ell_\infty$ *is only assumed in* **11.–13.**. *Then the necessary and sufficient conditions for* $A \in (X, Y)$*, where* X *and* Y *are any of the spaces* $\ell(p), c_0(p), c(p)$ *and* $\ell_\infty(p)$*, can be read from the following table*

X \ Y	$\ell(q)$, $(q_k \geq 1)$	$c_0(q)$	$c(q)$	$\ell_\infty(q)$
$\ell(p)$	**unknown**	**14.**	**18.**	**12.**
$c_0(p)$	**11.**	**15.**	**19.**	**23.**
$c(p)$	**12.**	**16.**	**20.**	**24.**
$\ell_\infty(p)$	**13.**	**17.**	**21.**	**25.**

where

11. (11.1∗) $\sup_K \sum_{n=1}^{\infty} \left|\sum_{k\in K} a_{nk} M^{-1/p_k}\right|^{q_n} < \infty$ *for some* M*,*

(11.1 ∗ ∗) $\lim_{M\to\infty} \sup_K \sum_{n=1}^{\infty} \left|\sum_{k\in K} a_{nk} M^{-1/p_k}\right|^{q_n} = 0$

12. (11.1∗) *and* (12.1∗) $\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} a_{nk}\right|^{q_n} < \infty$

13. (13.1∗) $\sup_K \sum_{n=1}^{\infty} \left|\sum_{k\in K} a_{nk} M^{1/p_k}\right|^{q_n} < \infty$ *for all* M*;*

14. (14.1∗), (14.2∗) *and* (14.3∗)*, where*

(14.1∗) $\lim_{n\to\infty} |a_{nk}|^{q_n} = 0$ *for all* k*,*

(14.2∗) $\sup_n \sup_{k\in K_1} \left|a_{nk} L^{1/q_n}\right|^{p_k} < \infty$ *for all* L*,*

(14.2 ∗ ∗) $\lim_{M\to\infty} \sup_n \sup_{k\in K_1} \left|a_{nk} M^{-1/p_k}\right|^{q_n} = 0$*,*

(14.3∗) *for all* L *there exists* M *with* $\sup_n \sum_{k\in K_2} \left|a_{nk} L^{1/q_n} M^{-1}\right|^{p'_k} < \infty$*,*

(14.3 ∗ ∗) $\lim_{M\to\infty} \sup_n \sum_{k\in K_2} \left|a_{nk} L^{1/q_n} M^{-1}\right|^{p'_k} = 0$ *for all* L

15. (14.1∗) *and* (15.1∗)*, where*

(15.1∗) *for all* L *there exists* M *with* $\sup_n L^{1/q_n} \sum_{k=1}^{\infty} |a_{nk}| M^{-1/p_k} < \infty$*,*

(15.1 ∗ ∗) $\lim_{M\to\infty} \sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| M^{-1/p_k}\right)^{q_n} = 0$

16. (14.1∗), (15.1∗) *and* (16.1∗), *where*

(16.1∗) $\lim_{n\to\infty}\left|\sum_{k=1}^{\infty} a_{nk}\right|^{q_n} = 0$

17. (17.1∗) $\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}|a_{nk}|M^{1/p_k}\right)^{q_n} = 0$ *for all* M

18. (18.1∗), (18.2∗) *and there exists a sequence* $(\alpha_k)_{k=1}^{\infty}$ *such that* (18.3∗), (18.4∗) *and* (18.5∗), *where*

(18.1∗) $\sup_n \sup_{k\in K_1}|a_{nk}|^{p_k} < \infty$,

(18.2∗) $\sup_n \sum_{k\in K_2}\left|a_{nk}M^{-1}\right|^{p'_k} < \infty$ *for some* M,

(18.2 ∗ ∗) $\lim_{M\to\infty}\sup_n \sum_{k\in K_2}\left|a_{nk}M^{-1}\right|^{p'_k} = 0$,

(18.3∗) $\lim_{n\to\infty}|a_{nk}-\alpha_k|^{q_n} = 0$ *for all* k,

(18.4∗) $\sup_n \sup_{k\in K_1}\left(|a_{nk}-\alpha_k|\,L^{1/q_n}\right)^{p_k} < \infty$ *for all* L,

(18.4 ∗ ∗) $\lim_{M\to\infty}\sup_n \sup_{k\in K_1}\left(|a_{nk}-\alpha_k|\,M^{-1/p_k}\right)^{p_k} = 0$,

(18.5∗) *for all* L *there exists* M *such that*

$$\sup_n \sum_{k\in K_2}\left(|a_{nk}-\alpha_k|\,L^{1/q_n}M^{-1}\right)^{p'_k} < \infty,$$

(18.5 ∗ ∗) $\lim_{M\to\infty}\sup_n \sum_{k\in K_2}\left(|a_{nk}-\alpha_k|\,L^{1/q_n}M^{-1}\right)^{p'_k} = 0$ *for some* L

19. (19.1∗) *and there exists a sequence* $(\alpha_k)_{k=1}^{\infty}$ *such that* (18.3∗) *and* (19.2∗), *where*

(19.1∗) $\sup_n \sum_{k=1}^{\infty}|a_{nk}|M^{-1/p_k} < \infty$ *for some* M,

(19.1 ∗ ∗) $\lim_{M\to\infty}\sup_n \sum_{k=1}^{\infty}|a_{nk}|M^{-1/p_k} = 0$,

(19.2∗) *for all* L *there exists* M *such that*

$$\sup_n L^{1/q_n}\sum_{k=1}^{\infty}|a_{nk}-\alpha_k|M^{-1/p_k} < \infty,$$

(19.2 ∗ ∗) $\lim_{M\to\infty}\left(\sup_n \sum_{k=1}^{\infty}|a_{nk}-\alpha_k|M^{-1/p_k}\right)^{q_n} = 0$

20. (19.1∗), *there exists* α *such that* (20.1∗) *and there exists a sequence* $(\alpha_k)_{k=1}^{\infty}$ *such that* (18.3∗) *and* (19.2∗), *where*

(20.1∗) $\lim_{n\to\infty}|a_{nk}-\alpha|^{q_n} = 0$

21. (21.1∗) *and there exists a sequence* $(\alpha_k)_{k=1}^{\infty}$ *such that* (21.2∗), *where*

(21.1∗) $\sup_n \sum_{k=1}^{\infty}|a_{nk}|M^{1/p_k} < \infty$ *for all* M,

(21.2∗) $\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}|a_{nk}-\alpha_k|M^{1/p_k}\right)^{q_n} = 0$ *for all* M

22. (22.1∗) *and* (22.2∗), *where*

(22.1∗) $\sup_n \sup_{k\in K_1}\left|a_{nk}L^{-1/q_n}\right|^{p_k} < \infty$ *for some* L,

(22.1 ∗ ∗) $\sup_n \sup_{k\in K_1}\left|a_{nk}M^{-1/p_k}\right|^{q_n} < \infty$ *for some* M,

(22.2∗) $\sup_n \sum_{k\in K_2}\left|a_{nk}L^{-1/q_n}\right|^{p'_k} < \infty$ *for some* L

23. (23.1∗) $\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| M^{-1/p_k} \right)^{q_n} < \infty$ *for some M*

24. (23.1∗) *and* (24.1∗) $\sup_n \left| \sum_{k=1}^{\infty} a_{nk} \right|^{q_n} < \infty$

25. (25.1∗) $\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| M^{1/p_k} \right)^{1/q_n} < \infty$ *for all M.*

Remark 4.14. *Let p and q be bounded positive sequences.*
(a) The characterizations of the classes (X, Y) for $X = \Delta\ell_\infty(p), \Delta c_0(p), \Delta c(p)$ and $Y = \ell_\infty(q), c_0(q), c(q)$ were established in [31, Theorem 4].
(b) The characterizations of the classes $(\Delta_u^m X, Y)$ for $X = \ell_\infty(p), c_0(p), c(p)$ and $Y = \ell_\infty(q), c_0(q), c(q)$ were established in [32, Theorems and Corollaries 3.1–3.3]

5. The Paranormed Space of Almost Convergent Sequences

In this section, the indices of sequences start with 0.

The concept of *almost convergence* arises from a generalization of that of convergence. Banach [1] proved the existence of a functional L on ℓ_∞ satisfying the following conditions for all $x, y \in \ell_\infty$ and all scalars λ and μ

(i) $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$
(ii) $x_k \geq 0$ for all k implies $L((x_k)_{k=0}^{\infty}) \geq 0$
(iii) $L((x_{k+n})_{k=0}^{\infty}) = L((x_k)_{k=0}^{\infty})$ for all $n \in \mathbb{N}$
(iv) $L(e) = 1$.

Lorentz [16] defined a *Banach limit* to be any functional on ℓ_∞ satisfying the condition in (i)–(iv), and called a bounded sequence $x = (x_k)_{k=0}^{\infty}$ to be *almost convergent to s* if $L(x) = s$ for every Banach limit L; s is then called the *generalized limit of x*; this is denoted by $Lim x_n = s$. Lorentz proved that

$$\text{Lim } x_n = s \text{ if and only if } \lim_{m\to\infty} t_{mn}(x) = \frac{1}{m+1} \sum_{k=0}^{m} x_{k+m} = s \text{ uniformly in } n.$$

The sets of all sequences that are almost convergent or almost convergent to zero are denoted by f or f_0; some authors write $\hat{c} = f$ and $\hat{c}_0 = f_0$.

Again let $p = (p_k)_{k=0}^{\infty}$ be a sequence of positive real numbers. S. Nanda [35] generalized the sets f_0 and f as follows:

$$f_0(p) = \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} |t_{mn}(x)|^{p_m} = 0 \ \text{ uniformly in } n \right\},$$

$$f(p) = \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} |t_{mn}(x) - l|^{p_m} = 0 \text{ uniformly in } n \right\}.$$

The following set inclusions between the spaces $c_0(p)$, $f_0(p)$, $c(p)$ and $f(p)$ hold.

Theorem 5.1. ([35, Proposition 1]) *The following inclusions hold:*

$$c_0(p) \subset f_0(p), \quad c(p) \subset f(p) \ \text{ and } \ f_0(p) \subset f(p).$$

Theorem 5.2. ([35, Proposition 2]) *If $0 < p_m \leq q_m < \infty$, then the inclusions $f_0(p) \subset f_0(q)$ and $f(p) \subset f(q)$ hold.*

The following results are known concerning the topological structures of the spaces $f_0(p)$ and $f(p)$. Again we assume that $p = (p_k)_{k=0}^{\infty}$ is a bounded sequence of positive real numbers, and $M = \max\{1, \sup_k p_k\}$.

Theorem 5.3. ([35, Theorem 1]) *The space $f_0(p)$ is a complete linear topological space with respect to the paranorm g defined by*

$$g(x) = \sup_{m,n} \left|t_{m,n}(x)\right|^{p_m/M}.$$

If $\inf_m p_m > 0$, then $f(p)$ is a complete linear topological space with respect to the paranorm g.

We recall that for $r > 0$, a non-empty subset Y of a linear space X is said to be absolutely r-convex if $x, y \in Y$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^r + |\beta|^r \le 1$ together imply that $\alpha x + \beta y \in Y$. A linear topological space X is said to be r-convex (cf. [28]) if every neighbourhood of $\theta \in X$ contains an absolutely r-convex neighbourhood of $\theta \in X$. The next result is known.

Theorem 5.4. ([35, Proposition 3]) *The spaces $f_0(p)$ and $f(p)$ are 1-convex.*

Several classes of matrix transformations into the normed or paranormed spaces of almost convergent or almost null sequences were characterized in [35, 36]. Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix. Then we write

$$a(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} \text{ for all } n,k,m.$$

Theorem 5.5. ([35, Theorems 2, 3, 5, 6]) *We have*
(a) $A \in (c_0(p), f_0(p))$ if and only if
(i) there exists an integer $B > 1$ such that

$$C_n = \sup_m \left(\sum_{k=0}^{\infty} |a(n,k,m)| B^{-1/p_k} \right)^{p_m} < \infty \text{ for all } n,$$

$$\lim_{m\to\infty} |a(n,k,m)|^{p_m} \text{ uniformly in } n. \tag{ii}$$

(b) $A \in (c(p), f)$ if and only if
(i) there exists an integer $B > 1$ such that

$$D_n = \sup_m \sum_{k=0}^{\infty} |a(n,k,m)| B^{-1/p_k} < \infty \text{ for all } n,$$

$$\lim_{m\to\infty} a(n,k,m) = \alpha_k \text{ uniformly in } n \text{ for each } k, \tag{ii}$$

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} a(n,k,m) = \alpha \text{ uniformly in } n. \tag{iii}$$

(c) $A \in (\ell_\infty(p), f)$ if and only if the condition (b.ii) is satisfied and

$$\sup_m \sum_{k=0}^{\infty} |a(n,k,m)| < \infty \text{ for all } n, \tag{ii}$$

(iii) for all integers $N > 1$

$$\lim_{m\to\infty} |a(n,k,m) - \alpha_k| N^{1/p_k} = 0 \text{ uniformly in } n.$$

(d) $A \in (\ell_1, f_0)$ if and only if

$$\sup_{n,k} |a(n,k,m)| < \infty \text{ for all } n, \tag{i}$$

$$\lim_{m\to\infty} a(n,k,m) = 0 \text{ uniformly in } n. \tag{ii}$$

Remark 5.6. *(a) The characterizations of the classes $(c_0(p), f)$ and $(\ell_\infty(p), f_0)$ can be found in the [35, Corollaries p. 180 and 181] as special cases of [35, Theorems 3 and 5].*
(b) Parts (b) and (c) of Theorem 5.5 generalize the characterizations of the classes (c, f) and (ℓ_∞, f) established in [12, Theorem 3.1] and [6, Theorem 2.1].

We close with the characterizations of the classes $(\ell(p), f)$ and $(w(p), f)$ in [36, Theorems 1 and 5].

Theorem 5.7. *We have*
(a) $A \in (\ell(p), f)$ if and only if the condition in Theorem 5.7 (b.ii) holds and
(i) there exists an integer $B > 1$ such that for all n

$$\sup_m \sum_{k=0}^{\infty} |a(n,k,m)|^{q_k} B^{-q_k} < \infty \quad \text{if } p_k < 1 \text{ and } q_k = p_k/(p_k - 1) \text{ for all } k$$

$$\sup_{m,k} |a(n,k,m)|^{p_k} < \infty \quad \text{if } 0 < p_k \leq 1 \text{ for all };$$

(b) if $0 < p_k \leq 1$ for all k then $A \in (w(p), f)$ if and only if the conditions in Theorem 5.7 (b.ii) and (b.iii) hold and
(i) there exists $B < 1B$such that

$$C_n = \sup_m \sum_{\nu=0}^{\infty} \max_\nu \left(\left(2^\nu B^{-1} \right)^{1/p_k} |a(n,k,m)| \right) < \infty \text{ for all } n.$$

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