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On the Domain of Riesz Mean in the Space $\mathcal{L}_s$*

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Abstract. Let $0 < s < \infty$. In this study, we introduce the double sequence space $R^{qt}(\mathcal{L}_s)$ as the domain of four dimensional Riesz mean R^{qt} in the space $\mathcal{L}_s$ of absolutely s-summable double sequences. Furthermore, we show that $R^{qt}(\mathcal{L}_s)$ is a Banach space and a barrelled space for $1 \le s < \infty$ and is not a barrelled space for $0 < s < 1$. We determine the α- and $\beta(\vartheta)$-duals of the space $\mathcal{L}_s$ for $0 < s \le 1$ and $\beta(bp)$-dual of the space $R^{qt}(\mathcal{L}_s)$ for $1 < s < \infty$, where $\vartheta \in \{p, bp, r\}$. Finally, we characterize the classes $(\mathcal{L}_s : \mathcal{M}_u)$, $(\mathcal{L}_s : \mathcal{C}_{bp})$, $(R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ and $(R^{qt}(\mathcal{L}_s) : \mathcal{C}_{bp})$ of four dimensional matrices in the cases both $0 < s < 1$ and $1 \le s < \infty$ together with corollaries some of them give the necessary and sufficient conditions on a four dimensional matrix in order to transform a Riesz double sequence space into another Riesz double sequence space.

1. Introduction

We denote the set of all real or complex valued double sequences by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as *a double sequence space*. A double sequence $x = (x_{mn})$ of complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then we call that the double sequence x is *convergent in the Pringsheim's sense* to the limit l and write $p - \lim_{m,n\to\infty} x_{mn} = l$; where $\mathbb{C}$ denotes the complex field. We give the set definitions of the spaces $\mathcal{M}_u$, $\mathcal{C}_p$ and $\mathcal{L}_s$ of bounded, convergent in the Pringsheim's sense and absolutely s-summable double sequences, respectively, as follows:

$$\begin{aligned}
\mathcal{M}_u &:= \left\{x = (x_{kl}) \in \Omega : \|x\|_\infty = \sup_{k,l\in\mathbb{N}} |x_{kl}| < \infty\right\}, \\
\mathcal{C}_p &:= \left\{x = (x_{mn}) \in \Omega : \exists l \in \mathbb{C} \text{ such that } p - \lim_{m,n\to\infty} x_{mn} = l\right\}, \\
\mathcal{L}_s &:= \left\{x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^s < \infty\right\}, \quad (0 < s < \infty).
\end{aligned}$$

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$\mathcal{M}_u$ is a Banach space with the norm $\|\cdot\|_\infty$. One can easily see that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed, if we define the sequence $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} k & , \quad k \in \mathbb{N},\ l = 0, \\ l & , \quad l \in \mathbb{N},\ k = 0, \\ 0 & , \quad k, l \in \mathbb{N} \setminus \{0\} \end{cases}$$

for all $k, l \in \mathbb{N}$, then it is trivial that $x \in \mathcal{C}_p \setminus \mathcal{M}_u$, since $p - \lim_{k,l\to\infty} x_{kl} = 0$ but $\|x\|_\infty = \infty$. So, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the space of all such sequences by $\mathcal{C}_r$.

Let us consider a double sequence $x = (x_{mn})$ and define the sequence $s = (s_{mn})$ via x by $s_{mn} = \sum_{k,l=0}^{m,n} x_{kl}$ for all $m, n \in \mathbb{N}$. Then, the pair (x, s) and the sequence $s = (s_{mn})$ are called as a double series and the sequence of partial sums of the double series, respectively. Here and after, unless stated otherwise we assume that ϑ denotes any of the symbols p, bp or r. If the double sequence (s_{mn}) is convergent in the ϑ-sense, then the double series $\sum_{k,l} x_{kl}$ is said to be convergent in the ϑ-sense and it is showed that $\vartheta - \sum_{k,l} x_{kl} = \vartheta - \lim_{m,n\to\infty} s_{mn}$. Also, we find some criteria about the convergence of a double series in Limaye and Zeltser [1]. Throughout the text we use the notation $\sum_{k,l} x_{kl}$ instead of $\sum_{k,l=0}^{\infty} x_{kl}$.

By $\mathcal{L}_s$, we denote the space of absolutely s-summable double sequences defined by Başar and Sever [2]. Throughout the text, we assume that $0 < s < \infty$ and s' denotes the conjugate of s, that is, $s' = s/(s-1)$ for $1 < s < \infty$, $s' = \infty$ for $s = 1$ or $s' = 1$ for $s = \infty$. Also, by $\mathcal{L}_u$, we mean the space of absolutely convergent double series.

The *α-dual* λ^α, *$\beta(\vartheta)$-dual* $\lambda^{\beta(\vartheta)}$ with respect to the ϑ-convergence and the *γ-dual* λ^γ of a double sequence space λ are respectively defined by

$$\begin{aligned} \lambda^\alpha &:= \left\{ (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl} x_{kl}| < \infty \text{ for all } (x_{kl}) \in \lambda \right\}, \\ \lambda^{\beta(\vartheta)} &:= \left\{ (a_{kl}) \in \Omega : \vartheta - \sum_{k,l} a_{kl} x_{kl} \text{ exists for all } (x_{kl}) \in \lambda \right\}, \\ \lambda^\gamma &:= \left\{ (a_{kl}) \in \Omega : \sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl} x_{kl} \right| < \infty \text{ for all } (x_{kl}) \in \lambda \right\}. \end{aligned}$$

It is easy to see for any two spaces λ, μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(\vartheta)}$ holds while the inclusion $\lambda^{\beta(\vartheta)} \subset \lambda^\gamma$ does not hold, since the ϑ-convergence of a sequence of partial sums of a double series does not imply its boundedness.

Let λ and μ be two double sequence spaces, and $A = (a_{mnkl})$ be any four-dimensional real or complex infinite matrix. Then, we say that A defines a *matrix mapping* from λ into μ and we write $A : \lambda \to \mu$, if for every sequence $x = (x_{kl}) \in \lambda$ the A-transform $Ax = \{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and is in μ; where

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl} x_{kl} \text{ for each } m, n \in \mathbb{N}. \tag{1}$$

We define the *ϑ-summability domain* $\lambda_A^{(\vartheta)}$ of A in a space λ of double sequences by

$$\lambda_A^{(\vartheta)} := \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n\in\mathbb{N}} \text{ exists and is in } \lambda \right\}.$$

We say with the notation (1) that A maps the space λ into the space μ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four dimensional matrices, transforming the space λ into the space μ, by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (1) converges in the sense of ϑ for each $m, n \in \mathbb{N}$, i.e,

$A_{mn} \in \lambda^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}}$ for all $m, n \in \mathbb{N}$. We say that a four-dimensional matrix A is $\mathcal{C}_\vartheta$*-conservative* if $\mathcal{C}_\vartheta \subset (\mathcal{C}_\vartheta)_A$, and is $\mathcal{C}_\vartheta$*-regular* if it is $\mathcal{C}_\vartheta$-conservative and $\vartheta - \lim_A x = \vartheta - \lim_{m,n\to\infty} (Ax)_{mn} = \vartheta - \lim_{m,n\to\infty} x_{mn}$, where $x = (x_{mn}) \in \mathcal{C}_\vartheta$. In this paper, we only consider $bp-$summability domain.

Using the notation of Zeltser [3], we define the double sequences $\mathbf{e^{kl}} = \left(\mathbf{e^{kl}_{mn}}\right)$, $\mathbf{e^l}$, $\mathbf{e_k}$ and $\mathbf{e}$ by $\mathbf{e^{kl}_{mn}} = 1$ if $(k, l) = (m, n)$ and $\mathbf{e^{kl}_{mn}} = 0$ otherwise, and $\mathbf{e^l} := \sum_k \mathbf{e^{kl}}$, $\mathbf{e_k} := \sum_l \mathbf{e^{kl}}$ and $\mathbf{e} := \sum_{k,l} \mathbf{e^{kl}}$ (coordinatewise sum) for all $k, l, m, n \in \mathbb{N}$ and we denote Φ by $\Phi = span\{\mathbf{e^{kl}} : k, l \in \mathbb{N}\}$.

For all $m, n, k, l \in \mathbb{N}$, we say that $A = (a_{mnkl})$ is *a triangular matrix* if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both, [4]. Following Adams [4], we also say that a triangular matrix $A = (a_{mnkl})$ is called a *triangle* if $a_{mnmn} \neq 0$ for all $m, n \in \mathbb{N}$. Referring to Cooke [5, Remark (a), p. 22], one can conclude that every triangle matrix has an unique inverse which is also a triangle.

Zeltser [3] essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences in her PhD thesis. Altay and Başar [6] have defined the spaces $\mathcal{BS}$ and $\mathcal{CS}_\vartheta$ of double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{C}_\vartheta$, respectively. Mursaleen and Başar [7] have introduced the spaces $\widetilde{\mathcal{M}_u}$, $\widetilde{\mathcal{C}_\vartheta}$ and $\widetilde{\mathcal{L}_s}$ of double sequences whose Cesàro transforms are in $\mathcal{M}_u$, $\mathcal{C}_\vartheta$ and $\mathcal{L}_s$, respectively. The reader can refer to Başar [8] and Mursaleen and Mohiuddine [9] for relevant terminology and required details on the double sequences and related topics.

Following Mursaleen and Başar [7] and Alotaibi and Çakan [10], Yeşilkayagil and Başar [11] have defined the double sequence spaces $R^{qt}(\mathcal{M}_u)$, $R^{qt}(\mathcal{C}_p)$, $R^{qt}(\mathcal{C}_{bp})$ and $R^{qt}(\mathcal{C}_r)$ as the domain of four dimensional Riesz mean R^{qt} in the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Also, they have characterized the matrix class $(\mathcal{M}_u : \mathcal{M}_u)$ in [12] and have introduced the some topological property of the double spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ of almost null and almost convergent double sequences, respectively, in [13].

In [14] Tuğ and Başar have introduced some new double sequence spaces $B(\mathcal{M}_u)$, $B(\mathcal{C}_\vartheta)$, and $B(\mathcal{L}_s)$ as the domain of four-dimensional generalized difference matrix $B(r, s, t, u)$ in the spaces $\mathcal{M}_u$, $\mathcal{C}_\vartheta$ and $\mathcal{L}_s$, respectively.

Let $q = (q_k)$, $t = (t_l)$ be two sequences of non-negative numbers which are not all zero and $Q_m = \sum_{k=0}^m q_k$, $q_0 > 0$, $T_n = \sum_{l=0}^n t_l$, $t_0 > 0$. Then, the Riesz mean with respect to the sequences $q = (q_k)$ and $t = (t_l)$ is defined by the matrix $R^{qt} = (r^{qt}_{mnkl})$ as follows

$$r^{qt}_{mnkl} = \begin{cases} \frac{q_k t_l}{Q_m T_n} & , \quad 0 \leq k \leq m,\ 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$. It is known by Theorem 2.8 of Yeşilkayagil and Başar [11] that the four dimensional Riesz mean R^{qt} is RH-regular if and only if $\lim_{m\to\infty} Q_m = \infty$ and $\lim_{n\to\infty} T_n = \infty$. The Riesz transform R^{qt} of a double sequence $x = (x_{kl})$ is given by

$$y_{mn} = (R^{qt}x)_{mn} = \frac{1}{Q_m T_n} \sum_{k,l=0}^{m,n} q_k t_l x_{kl} \tag{2}$$

for all $m, n \in \mathbb{N}$. Throughout the paper, we suppose that the terms of the double sequences $x = (x_{kl})$ and $y = (y_{mn})$ are connected with the relation (2) and the term with negative index is zero. If $p - \lim(R^{qt}x)_{mn} = s$, $s \in \mathbb{C}$, then the sequence $x = (x_{kl})$ is said to be Riesz convergent to s (see [10]). Note that in the case $q_k = 1$ for all k and $t_l = 1$ for all l, the Riesz mean R^{qt} is reduced to the four dimensional Cesàro mean C of order one. Let $I = (\delta_{mnkl})$ is four dimensional unit matrix, that is, $\delta_{mnkl} = \begin{cases} 1 & , \quad (m, n) = (k, l), \\ 0 & , \quad \text{otherwise} \end{cases}$. Using the equality $\delta_{mnkl} = \sum_{i,j} r_{mnij} d_{ijkl} = \frac{1}{Q_m T_n} \sum_{i,j=0}^{m,n} q_i t_j d_{ijkl}$, one can obtain by a straightforward calculation that the inverse $(R^{qt})^{-1} = (d_{mnkl})$ of the triangle matrix R^{qt} is given, as follows:

$$d_{mnkl} = \begin{cases} (-1)^{m+n-(k+l)} \frac{Q_k T_l}{q_m t_n} & , \quad m-1 \leq k \leq m,\ n-1 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m,n,k,l \in \mathbb{N}$.

In the present paper, referring Başar and Sever [2] we introduce the new space $R^{qt}(\mathcal{L}_s)$ defined by

$$R^{qt}(\mathcal{L}_s) := \left\{x = (x_{kl}) \in \Omega : \{(R^{qt}x)_{mn}\} \in \mathcal{L}_s\right\}, \quad (0 < s < \infty).$$

2. The Space $R^{qt}(\mathcal{L}_s)$ of Double Sequences

In this section, we give some results on the space $R^{qt}(\mathcal{L}_s)$.

Theorem 2.1. *The set $R^{qt}(\mathcal{L}_s)$ is the linear space with the coordinatewise addition and scalar multiplication, and the following statements hold:*

(i) If $0 < s < 1$, then $R^{qt}(\mathcal{L}_s)$ is a complete s-normed space with

$$\widetilde{\|x\|}_s = \sum_{m,n}\left|\frac{1}{Q_mT_n}\sum_{k,l=0}^{m,n} q_kt_lx_{kl}\right|^s$$

which is s-norm isomorphic to the space $\mathcal{L}_s$.

(ii) If $1 \le s < \infty$, then $R^{qt}(\mathcal{L}_s)$ is a Banach space with

$$\widehat{\|x\|}_s = \left(\sum_{m,n}\left|\frac{1}{Q_mT_n}\sum_{k,l=0}^{m,n} q_kt_lx_{kl}\right|^s\right)^{1/s} \tag{3}$$

which is norm isomorphic to the space $\mathcal{L}_s$.

Proof. Since, Part (i) can be proved in the similar way, we give the proof only for Part (ii).

The first part is a routine verification and so we omit it.

To prove the fact $R^{qt}(\mathcal{L}_s)$ is norm isomorphic to the space $\mathcal{L}_s$, we should show the existence of a linear bijection between the spaces $R^{qt}(\mathcal{L}_s)$ and $\mathcal{L}_s$. Consider the transformation U defined from $R^{qt}(\mathcal{L}_s)$ to $\mathcal{L}_s$ by $x \mapsto Ux = \{(R^{qt}x)_{mn}\}$. It is trivial that U is linear. We get from the equation

$$Ux = \begin{bmatrix} x_{00} & \frac{t_0x_{00}+t_1x_{01}}{T_1} & \frac{t_0x_{00}+t_1x_{01}+t_2x_{02}}{T_2} & \cdots \\ \frac{q_0x_{00}+q_1x_{10}}{Q_1} & \sum\limits_{k=0}^{1}\frac{q_k(t_0x_{k0}+t_1x_{k1})}{Q_1T_1} & \sum\limits_{k=0}^{1}\frac{q_k(t_0x_{k0}+t_1x_{k1}+t_2x_{k2})}{Q_1T_2} & \cdots \\ \frac{q_0x_{00}+q_1x_{10}+q_2x_{20}}{Q_2} & \sum\limits_{k=0}^{2}\frac{q_k(t_0x_{k0}+t_1x_{k1})}{Q_2T_1} & \sum\limits_{k=0}^{2}\frac{q_k(t_0x_{k0}+t_1x_{k1}+t_2x_{k2})}{Q_2T_2} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \sum\limits_{k=0}^{m}\frac{q_kx_{k0}}{Q_m} & \sum\limits_{k=0}^{m}\frac{q_k(t_0x_{k0}+t_1x_{k1})}{Q_mT_1} & \sum\limits_{k=0}^{m}\frac{q_k(t_0x_{k0}+t_1x_{k1}+t_2x_{k2})}{Q_mT_2} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} = \theta$$

that $x = \theta$ whenever $Ux = \theta$, where θ denotes the zero vector. This shows that U is injective.

Let $y = (y_{kl}) \in \mathcal{L}_s$ and define the sequence $x = (x_{kl})$ via y by

$$x_{kl} = \frac{1}{q_kt_l}(Q_kT_ly_{kl} - Q_{k-1}T_ly_{k-1,l} - Q_kT_{l-1}y_{k,l-1} + Q_{k-1}T_{l-1}y_{k-1,l-1}) \tag{4}$$

for all $k, l \in \mathbb{N}$. Then, we have

$$\begin{aligned} Q_mT_n(R^{qt}x)_{mn} &= \sum_{k,l=0}^{m,n}(Q_kT_ly_{kl} - Q_{k-1}T_ly_{k-1,l} - Q_kT_{l-1}y_{k,l-1} + Q_{k-1}T_{k-1}y_{k-1,l-1}) \\ &= Q_0\sum_{l=0}^{n}(T_ly_{0l} - T_{l-1}y_{0,l-1}) + \sum_{l=0}^{n}(Q_1T_ly_{1l} - Q_0T_ly_{0l} - Q_1T_{l-1}y_{1,l-1} - Q_0T_{l-1}y_{0,l-1}) \\ &+ \sum_{l=0}^{n}(Q_2T_ly_{2l} - Q_1T_ly_{1l} - Q_2T_{l-1}y_{2,l-1} - Q_1T_{l-1}y_{1,l-1}) + \ldots + \\ &+ \sum_{l=0}^{n}(Q_{m-1}T_ly_{m-1,l} - Q_{m-2}T_ly_{m-2,l} - Q_{m-1}T_{l-1}y_{m-1,l-1} - Q_{m-2}T_{l-1}y_{m-2,l-1}) \\ &+ \sum_{l=0}^{n}(Q_mT_ly_{ml} - Q_{m-1}T_ly_{m-1,l} - Q_mT_{l-1}y_{m,l-1} - Q_{m-1}T_{l-1}y_{m-1,l-1}) \\ &= Q_m\sum_{l=0}^{n}(T_ly_{ml} - T_{l-1}y_{m,l-1}) = Q_mT_ny_{mn} \end{aligned}$$

and so

$$\left|(R^{qt}x)_{mn}\right| = |y_{mn}|$$

which yields that

$$\sum_{m,n}\left|(R^{qt}x)_{mn}\right|^s = \sum_{m,n}|y_{mn}|^s. \tag{5}$$

Since $y \in \mathcal{L}_s$, we have $x \in R^{qt}(\mathcal{L}_s)$, that is, U is surjective. Also, we see from (5) that U is a norm preserving transformation.

Now, we can show that $R^{qt}(\mathcal{L}_s)$ is a Banach space with the norm $\widehat{\|\cdot\|_s}$ defined by (3). To prove this, we use Part (b) of Corollary 6.3.41 in [15] which says that "*Let (X, p) and (Y, q) be semi-normed spaces and $U : (X, p) \to (Y, q)$ be an isometric isomorphism. Then, (X, p) is complete if and only if (Y, q) is complete. In particular, (X, p) is a Banach space if and only if (Y, q) is a Banach space.*" Since the transformation U defined above from $R^{qt}(\mathcal{L}_s)$ to $\mathcal{L}_s$ is an isometric isomorphism and the space $\mathcal{L}_s$ is a Banach space from Theorem 2.1 in [2], we conclude that the space $R^{qt}(\mathcal{L}_s)$ is a Banach space. This step completes the proof. □

A non-empty subset S of a locally convex space X is called *fundamental* if the closure of the linear span of S equals X, [15]. Using this definition, we define the set $S \subset \mathcal{L}_s$ as $S := \{\mathbf{e^{kl}} : k, l \in \mathbb{N}\}$. Then, we have $\Phi = spanS$. We shall show that Φ is dense in $\mathcal{L}_s$, that is, $cl\Phi = \mathcal{L}_s$. Let the relation $cl\Phi = \mathcal{L}_s$ does not hold. Hence, there exists a ball in $\mathcal{L}_s$, no matter how small, does not contain any points of Φ, i.e, there does not exist a $y \in \Phi$ such that

$$\|x - y\| \nless \varepsilon^s \tag{6}$$

for a point $x \in \mathcal{L}_s$.Then, by (6) we have that

$$\|x - y\| = \sum_{i,j}|x_{ij} - \mathbf{e^{kl}_{ij}}|^s = |x_{kl} - 1|^s \nless \varepsilon^s,$$

that is, $|x_{kl} - 1|^s \geq \varepsilon^s$. Choose $\varepsilon = 1/2$. Then, we have either $x_{kl} \leq 1/2$ or $3/2 \leq x_{kl}$ for all $k, l \in \mathbb{N}$. For both statement, we can find $x \notin \mathcal{L}_s$, a contradiction. Since $x \in \mathcal{L}_s$ is arbitrary, every ball in $\mathcal{L}_s$ contains a point of

Φ, i.e, Φ is dense in $\mathcal{L}_s$. Therefore, S is fundamental set of $\mathcal{L}_s$. Using this fact, we define the double sequence $\mathbf{b}^{(\mathbf{kl})} = \left(b_{mn}^{(\mathbf{kl})}\right)$ by

$$b_{mn}^{(\mathbf{kl})} := \begin{cases} \frac{Q_kT_l}{q_kt_l} & , \ m=k, \ n=l, \\ -\frac{Q_kT_l}{q_kt_{l+1}} & , \ m=k, \ n=l+1 \\ -\frac{Q_kT_l}{q_{k+1}t_l} & , \ m=k+1, \ n=l, \\ \frac{Q_kT_l}{q_{k+1}t_{l+1}} & , \ m=k+1, \ n=l+1, \\ 0 & , \ \text{otherwise} \end{cases} \tag{7}$$

for all $k,l,m,n \in \mathbb{N}$. Then, $\{\mathbf{b}^{(\mathbf{kl})}; k,l \in \mathbb{N}\}$ is the fundamental set of the space $R^{qt}(L_s)$; since $R^{qt}\mathbf{b}^{(\mathbf{kl})} = \mathbf{e}^{\mathbf{kl}}$ with $0 < s < \infty$.

Theorem 2.2. *If* $\left(\frac{1}{Q_mT_n}\right) \notin \mathcal{L}_s$, *then* $\mathcal{L}_s \not\subset R^{qt}(\mathcal{L}_s)$ *holds.*

Proof. Let $\left(\frac{1}{Q_mT_n}\right) \notin \mathcal{L}_s$. We take the sequence $\mathbf{e}^{\mathbf{00}}$. Obviously, $\mathbf{e}^{\mathbf{00}} \in \mathcal{L}_s$. For all $m,n \in \mathbb{N}$, we have that

$$(R^{qt}\mathbf{e}^{\mathbf{00}})_{mn} = \frac{q_0t_0}{Q_mT_n}.$$

Since $\left(\frac{1}{Q_mT_n}\right) \notin \mathcal{L}_s$, $R^{qt}\mathbf{e}^{\mathbf{00}} \notin \mathcal{L}_s$. So, $\mathbf{e}^{\mathbf{00}} \notin R^{qt}(\mathcal{L}_s)$, as desired. □

Theorem 2.3. *Let* $1 < s < r < \infty$. *Then, the inclusion* $R^{qt}(\mathcal{L}_s) \subset R^{qt}(\mathcal{L}_r)$ *strictly holds.*

Proof. Let $1 < s < r < \infty$ and $x = (x_{kl}) \in R^{qt}(\mathcal{L}_s)$. Then, the following inequality

$$\left(\sum_{m,n=0}^{i,j} \left|\frac{1}{Q_mT_n}\sum_{k,l=0}^{m,n} q_kt_lx_{kl}\right|^r\right)^{1/r} < \left(\sum_{m,n=0}^{i,j} \left|\frac{1}{Q_mT_n}\sum_{k,l=0}^{m,n} q_kt_lx_{kl}\right|^s\right)^{1/s} \tag{8}$$

holds by Jensen's inequality. Therefore, one can see by applying p-limit to (8), as $i,j \to \infty$ that $\widehat{\|x\|}_r < \widehat{\|x\|}_s < \infty$ which means that $x \in R^{qt}(\mathcal{L}_r)$, as desired.

Now, consider the sequence $x = (x_{kl})$ defined by

$$x_{kl} = \frac{1}{q_kt_l}\left\{\frac{Q_kT_l}{[(k+2)(l+2)]^{1/s}} - \frac{Q_{k-1}T_l}{[(k+1)(l+2)]^{1/s}} - \frac{Q_kT_{l-1}}{[(k+2)(l+1)]^{1/s}} + \frac{Q_{k-1}T_{l-1}}{[(k+1)(l+1)]^{1/s}}\right\} \tag{9}$$

for all $k,l \in \mathbb{N}$. Using (9), we have

$$|(R^{qt}x)_{mn}| = \frac{1}{[(m+2)(n+2)]^{1/s}}$$

and so

$$\sum_{m,n}|(R^{qt}x)_{mn}|^s = \sum_{m,n}\left\{\frac{1}{[(m+2)(n+2)]^{1/s}}\right\}^s = \sum_{m,n}\frac{1}{(m+2)(n+2)} = \infty,$$

that is, $x \notin R^{qt}(\mathcal{L}_s)$. Since $1 < s < r < \infty$, $1 < r/s$. So, we have

$$\sum_{m,n}|(R^{qt}x)_{mn}|^r = \sum_{m,n}\left\{\frac{1}{[(m+2)(n+2)]^{1/s}}\right\}^r = \sum_{m,n}\frac{1}{[(m+2)(n+2)]^{r/s}} < \infty,$$

that is, $x \in R^{qt}(\mathcal{L}_r)$. This step completes the proof. □

Let λ be a locally convex space. Then, a subset is called *barrel* if it is absolutely convex, absorbing and closed in λ. Moreover, λ is called a *barrelled space* if each barrel is a neighborhood of zero; [15, p. 336].

Lemma 2.4. *[17] Every Banach space and every Fréchet space is a barrelled space.*

Theorem 2.5. *The following statements hold:*

(i) Let $1 \leq s < \infty$. Then, $R^{qt}(\mathcal{L}_s)$ is a barrelled space.
(ii) Let $0 < s < 1$. Then, $R^{qt}(\mathcal{L}_s)$ is not a barrelled space.

Proof. (i) By Lemma 2.4 and Part (ii) of Theorem 2.1, we say that $R^{qt}(\mathcal{L}_s)$ is a barrelled space for $1 \leq s < \infty$.

(ii) We show that the space $\mathcal{L}_s$ is not a locally convex space for $0 < s < 1$. Let $\mathcal{U} := \{x : \|x\|_s \leq 1\}$. We shall show that $\mathcal{U}$ includes no convex neighborhood of 0. Let $\mathcal{V}$ be a convex neighborhood of 0. For some $\varepsilon > 0$, $\mathcal{V} \supset \{x : \|x\|_s \leq \varepsilon\}$. In particular, $\varepsilon^{1/s}\mathbf{e^{kl}} \in \mathcal{V}$ for each $k, l \in \mathbb{N}$. Choose integers $m, n > \frac{1}{\varepsilon^{1/2(1-s)}}$ and define the sequence $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} \frac{\varepsilon^{1/s}}{(m+1)(n+1)} & , \quad 0 \leq k \leq m \text{ and } 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then, by choosing of ε we see that $x \in \mathcal{V}$ and

$$\begin{aligned} \|x\|_s &= \sum_{k=0}^{m}\sum_{l=0}^{n}\left|\frac{\varepsilon^{1/s}}{(m+1)(n+1)}\right|^s = \frac{\varepsilon}{[(m+1)(n+1)]^s}\sum_{k=0}^{m}\sum_{l=0}^{n}1 \\ &= \frac{\varepsilon}{[(m+1)(n+1)]^s}(m+1)(n+1)] = \varepsilon[(m+1)(n+1)]^{1-s} \\ &> \varepsilon\frac{1}{\varepsilon^{1/2}}\frac{1}{\varepsilon^{1/2}} = 1. \end{aligned}$$

So, $\mathcal{V} \not\subset \mathcal{U}$. Since the space $\mathcal{L}_s$ is not a locally convex space for $0 < s < 1$, the space $R^{qt}(\mathcal{L}_s)$ is not, too. Therefore, the space $R^{qt}(\mathcal{L}_s)$ is not a barrelled space. □

A double sequence space λ is said to be *solid* if and only if

$$\widetilde{\lambda} := \{(u_{kl}) \in \Omega : \exists (x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k, l \in \mathbb{N}\} \subset \lambda,$$

[2, p. 153]. A double sequence space λ is said to be *monotone* if $xu = (x_{kl}u_{kl}) \in \lambda$ for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all double sequences of zeros and ones. If λ is monotone, then $\lambda^\alpha = \lambda^{\beta(\vartheta)}$; [3, p. 36] and λ is monotone whenever λ is solid.

Theorem 2.6. *Let $0 < s < \infty$. Then, the space $\mathcal{L}_s$ is monotone.*

Proof. Let $0 < s < \infty$, $x = (x_{kl}) \in \mathcal{L}_s$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Then, we have $|x_{kl}u_{kl}|^s = |x_{kl}|^s|u_{kl}|^s \leq |x_{kl}|^s$ for each $k, l \in \mathbb{N}$. So, we have that $\sum_{k,l}|x_{kl}u_{kl}|^s \leq \sum_{k,l}|x_{kl}|^s$, that is, $xu \in \mathcal{L}_s$. □

Theorem 2.7. *Let $0 < s < \infty$. If $\left(\frac{1}{Q_mT_n}\right) \notin \mathcal{L}_s$, then the space $R^{qt}(\mathcal{L}_s)$ is not monotone.*

Proof. Let $0 < s < \infty$ and $\left(\frac{1}{Q_mT_n}\right) \notin \mathcal{L}_s$. Choose the sequence $x = (x_{kl}) \in R^{qt}(\mathcal{L}_s)$ such that $x_{00} \neq 0$ and take the sequence $u = (u_{kl}) = \mathbf{e^{00}} \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Hence, for the sequence $z = ux = \mathbf{e^{00}}x$ we derive that

$$(R^{qt}z)_{mn} = \frac{1}{Q_mT_n}q_0t_0x_{00}.$$

Since $\left(\frac{1}{Q_mT_n}\right) \notin \mathcal{L}_s$, $R^{qt}z \notin \mathcal{L}_s$. So, $z \notin R^{qt}(\mathcal{L}_s)$, as desired. □

3. Dual Spaces

In this section, we determine the α- and $\beta(\vartheta)-$ duals of the space $\mathcal{L}_s$ in the case $0 < s \leq 1$ and $\beta(bp)-$ dual of the space $R^{qt}(\mathcal{L}_s)$ for $1 < s < \infty$.

Theorem 3.1. *Let $0 < s \leq 1$. Then, the α-dual of the space $\mathcal{L}_s$ is the space $\mathcal{M}_u$.*

Proof. Since $\mathcal{L}_u \subset \mathcal{M}_u$, $\mathcal{L}_u^\alpha = \mathcal{M}_u$ and $\mathcal{L}_s \subset \mathcal{L}_u$ for $0 < s \leq 1$, we have that $\mathcal{M}_u \subset \mathcal{L}_s^\alpha$.

Conversely, suppose that $z = (z_{kl}) \in \mathcal{L}_s^\alpha \setminus \mathcal{M}_u$. Then, $\sum_{k,l} |z_{kl}x_{kl}| < \infty$ for all $x = (x_{kl}) \in \mathcal{L}_s$ and $\sup_{k,l\in\mathbb{N}} |z_{kl}| = \infty$. Hence, there exist sequences (k_m) and (l_m) such that at least one is strictly increasing for $m \in \mathbb{N}$. So, we can take $z_{k_m l_m} > (m+1)^{2/s}$. If we define $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} (m+1)^{-2/s} & , \quad k = k_m \text{ and } l = l_m, \\ 0 & , \quad k \neq k_m \text{ or } l \neq l_m \end{cases}$$

for all $k, l, m \in \mathbb{N}$, then we have $x \in \mathcal{L}_s$. But $\sum_{k,l} |z_{kl}x_{kl}| = \sum_m |z_{k_m l_m} x_{k_m l_m}| > \sum_m 1 = \infty$, that is, $z \notin \mathcal{L}_s^\alpha$, a contradiction. Therefore, the inclusion $\mathcal{L}_s^\alpha \subset \mathcal{M}_u$ holds.

By combining the inclusions $\mathcal{M}_u \subset \mathcal{L}_s^\alpha$ and $\mathcal{L}_s^\alpha \subset \mathcal{M}_u$, we get $\mathcal{L}_s^\alpha = \mathcal{M}_u$, as desired. □

Corollary 3.2. *Let $0 < s \leq 1$. Then, the $\beta(\vartheta)$-dual of the space $\mathcal{L}_s$ is the space $\mathcal{M}_u$.*

Theorem 3.3. *Let $0 < s \leq 1$. Then, the inclusion $\{R^{qt}(\mathcal{L}_s)\}^\alpha \subset \mathcal{M}_u$ holds.*

Proof. Suppose that $z = (z_{kl}) \in \{R^{qt}(\mathcal{L}_s)\}^\alpha \setminus \mathcal{M}_u$. Then, $zx \in \mathcal{L}_u$ for all $x \in R^{qt}(\mathcal{L}_s)$. We take the sequence $\mathbf{b^{(kl)}}$ as in (7). So, we have $\sum_{m,n} |(R^{qt}b_{mn}^{\mathbf{(kl)}})|^s = \sum_{m,n} |e_{mn}^{\mathbf{kl}}|^s = 1$ for all $k, l \in \mathbb{N}$. Hence, $\mathbf{b^{(kl)}} \in R^{qt}(\mathcal{L}_s)$ and so $zx = (z_{ij}b_{ij}^{\mathbf{(kl)}}) \in \mathcal{L}_u$. With some calculation, we have following five cases;

Case 1. $z_{ij}b_{ij}^{\mathbf{(kl)}} = z_{kl}\frac{Q_kT_l}{q_kt_l}$ for $(i,j) = (k,l)$.

Case 2. $z_{ij}b_{ij}^{\mathbf{(kl)}} = -z_{k,l+1}\frac{Q_kT_l}{q_kt_{l+1}}$ for $(i,j) = (k,l+1)$.

Case 3. $z_{ij}b_{ij}^{\mathbf{(kl)}} = -z_{k+1,l}\frac{Q_kT_l}{q_{k+1}t_l}$ for $(i,j) = (k+1,l)$.

Case 4. $z_{ij}b_{ij}^{\mathbf{(kl)}} = z_{k+1,l+1}\frac{Q_kT_l}{q_{k+1}t_{l+1}}$ for $(i,j) = (k+1,l+1)$.

Case 5. $z_{ij}b_{ij}^{\mathbf{(kl)}} = 0$ for otherwise.

For example, in case 1, we write that $\left(z_{kl}\frac{Q_kT_l}{q_kt_l}\right) \in \mathcal{L}_u$ so, is in $\mathcal{M}_u$. But, we know that (Q_k) (or (T_l)) is a positive increasing sequence, that is, it is not bounded. Therefore, $(z_{kl}) \in \mathcal{M}_u$, a contradiction. Hence, the inclusion $\{R^{qt}(\mathcal{L}_s)\}^\alpha \subset \mathcal{M}_u$ holds, as desired. □

Theorem 3.4. *Let $1 < s < \infty$ and define the sets d_1, d_2 and d_3, as follows:*

$$d_1 = \left\{a = (a_{kl}) \in \Omega : \sum_{k,l} \left| Q_kT_l\Delta_{11}\left(\frac{a_{kl}}{q_kt_l}\right)\right|^{s'} < \infty\right\},$$

$$d_2 = \left\{a = (a_{kl}) \in \Omega : \sup_{n\in\mathbb{N}} \sum_{k} \left| Q_kT_n\Delta_{10}\left(\frac{a_{kn}}{q_kt_n}\right)\right|^{s'} < \infty\right\},$$

$$d_3 = \left\{a = (a_{kl}) \in \Omega : \sup_{m\in\mathbb{N}} \sum_{l} \left| Q_mT_l\Delta_{01}\left(\frac{a_{ml}}{q_mt_l}\right)\right|^{s'} < \infty \ \textit{and} \ \left(Q_mT_n\frac{|a_{mn}|}{q_mt_n}\right)^{s'} \in \mathcal{M}_u\right\}.$$

Then, $\{R^{qt}(\mathcal{L}_s)\}^{\beta(bp)} = d_1 \cap d_2 \cap d_3$.

Proof. Let $x = (x_{mn}) \in R^{qt}(\mathcal{L}_s)$. Then, there exists a double sequence $y = (y_{mn}) \in \mathcal{L}_s$ by Part (ii) of Theorem 2.1. Also, we have $s = (s_{mn})$ from (4) such that

$$s_{mn} = \sum_{k,l=0}^{m,n} x_{kl} = \sum_{k,l=0}^{m,n} \frac{1}{q_k t_l} \left(Q_k T_l y_{kl} - Q_{k-1} T_l y_{k-1,l} - Q_k T_{l-1} y_{k,l-1} + Q_{k-1} T_{l-1} y_{k-1,l-1} \right)$$

for all $m, n \in \mathbb{N}$. Now, by the generalized Abel transformation for double sequences we obtain that

$$z_{mn} = \sum_{k,l=0}^{m,n} a_{kl} x_{kl} = \sum_{k,l=0}^{m-1,n-1} s_{kl} \Delta_{11} a_{kl} + \sum_{k=0}^{m-1} s_{kn} \Delta_{10} a_{kn} + \sum_{l=0}^{n-1} s_{ml} \Delta_{01} a_{ml} + s_{mn} a_{mn} \tag{10}$$

for all $m, n \in \mathbb{N}$. With some straightforward calculation, we can rewrite the relation (10) as follows

$$\begin{aligned} z_{mn} &= \sum_{k,l=0}^{m,n} a_{kl} x_{kl} = \sum_{k,l=0}^{m-1,n-1} Q_k T_l \Delta_{11} \left(\frac{a_{kl}}{q_k t_l} \right) y_{kl} + \sum_{k=0}^{m-1} Q_k T_n \Delta_{10} \left(\frac{a_{kn}}{q_k t_n} \right) y_{kn} \\ &+ \sum_{l=0}^{n-1} Q_m T_l \Delta_{01} \left(\frac{a_{ml}}{q_m t_l} \right) y_{ml} + Q_m T_n \frac{a_{mn}}{q_m t_n} y_{mn} = (By)_{mn} \end{aligned}$$

for all $m, n \in \mathbb{N}$, where the four-dimensional matrix $B = (b_{mnkl})$ is defined by

$$b_{mnkl} = \begin{cases} Q_k T_l \Delta_{11} \left(\frac{a_{kl}}{q_k t_l} \right) & , \quad 0 \leq k \leq m-1 \text{ and } 0 \leq l \leq n-1, \\ Q_k T_n \Delta_{10} \left(\frac{a_{kn}}{q_k t_n} \right) & , \quad 0 \leq k \leq m-1 \text{ and } l = n, \\ Q_m T_l \Delta_{01} \left(\frac{a_{ml}}{q_m t_l} \right) & , \quad k = m \text{ and } 0 \leq l \leq n-1, \\ Q_m T_n \frac{a_{mn}}{q_m t_n} & , \quad k = m \text{ and } l = n, \\ 0 & , \quad \text{otherwise} \end{cases} \tag{11}$$

for all $m, n, k, l \in \mathbb{N}$. Thus, we see that $ax = (a_{mn} x_{mn}) \in \mathcal{CS}_{bp}$ whenever $x = (x_{mn}) \in R^{qt}(\mathcal{L}_s)$ if and only if $z = (z_{mn}) \in \mathcal{C}_{bp}$ whenever $y = (y_{mn}) \in \mathcal{L}_s$. This leads us to the fact that $B \in (\mathcal{L}_s : \mathcal{C}_{bp})$. Hence, from Part (ii) of Theorem 4.3, the following statement

$$\begin{aligned} &\sup_{m,n \in \mathbb{N}} \sum_{k,l} |b_{mnkl}|^{s'} \\ &= \sup_{m,n \in \mathbb{N}} \left\{ \sum_{k,l=0}^{m-1,n-1} \left| Q_k T_l \Delta_{11} \left(\frac{a_{kl}}{q_k t_l} \right) \right|^{s'} + \sum_{k=0}^{m-1} \left| Q_k T_n \Delta_{10} \left(\frac{a_{kn}}{q_k t_n} \right) \right|^{s'} + \sum_{l=0}^{n-1} \left| Q_m T_l \Delta_{01} \left(\frac{a_{ml}}{q_m t_l} \right) \right|^{s'} + \left| Q_m T_n \frac{a_{mn}}{q_m t_n} \right|^{s'} \right\} < \infty. \end{aligned}$$

holds. Therefore, we derive that

$$\begin{aligned} &\sum_{k,l} \left| Q_k T_l \Delta_{11} \left(\frac{a_{kl}}{q_k t_l} \right) \right|^{s'} < \infty, \\ &\sup_{n \in \mathbb{N}} \sum_{k} \left| Q_k T_n \Delta_{10} \left(\frac{a_{kn}}{q_k t_n} \right) \right|^{s'} < \infty, \\ &\sup_{m \in \mathbb{N}} \sum_{l} \left| Q_m T_l \Delta_{01} \left(\frac{a_{ml}}{q_m t_l} \right) \right|^{s'} < \infty, \\ &\left| Q_m T_n \frac{a_{mn}}{q_m t_n} \right|^{s'} \in \mathcal{M}_u. \end{aligned}$$

Hence, $\{R^{qt}(\mathcal{L}_s)\}^{\beta(bp)} = d_1 \cap d_2 \cap d_3$. $\square$

4. Charactarization of Some Classes of Matrix Mappings

In this section, we characterize the classes $(\mathcal{L}_s : \mathcal{M}_u)$, $(\mathcal{L}_s : \mathcal{C}_{bp})$, $(R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ and $(R^{qt}(\mathcal{L}_s) : \mathcal{C}_{bp})$ of four dimensional matrices, in the cases both $0 < s \leq 1$ and $1 < s < \infty$. We also characterize the class $(\mathcal{L}_s : \mathcal{L}_{s_1})$ of four dimensional matrices in the cases $0 < s \leq 1$ and $1 \leq s_1 < \infty$.

Theorem 4.1. *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, the following statements are satisfied:*

(i) Let $0 < s \leq 1$. Then, $A \in (\mathcal{L}_s : \mathcal{M}_u)$ if and only if

$$N = \sup_{m,n,k,l \in \mathbb{N}} |a_{mnkl}| < \infty. \tag{12}$$

(ii) Let $1 < s < \infty$. Then, $A \in (\mathcal{L}_s : \mathcal{M}_u)$ if and only if

$$M_1 = \sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}|^{s'} < \infty. \tag{13}$$

Proof. (i) Let $0 < s \leq 1$ and $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{M}_u)$. Then, Ax exists and belongs to $\mathcal{M}_u$ for all $x \in \mathcal{L}_s$, and $A_{mn} \in \mathcal{M}_u$ by Corollary 3.2 for each $m, n \in \mathbb{N}$. Therefore, we obtain for $\mathbf{e^{kl}} \in \mathcal{L}_s$ that

$$\|A\mathbf{e^{kl}}\|_\infty = \sup_{m,n \in \mathbb{N}} |a_{mnkl}| < \infty$$

for each fixed $k, l \in \mathbb{N}$. That is to say that the condition (12) is necessary.

Conversely, suppose that (12) holds and take any $x = (x_{kl}) \in \mathcal{L}_s$. Then, $A_{mn} \in \mathcal{M}_u$ by Corollary 3.2 for each $m, n \in \mathbb{N}$ which implies the existence of Ax. Let $m, n \in \mathbb{N}$ be fixed. Then, since

$$\begin{aligned}
\left|\sum_{k,l} a_{mnkl}x_{kl}\right|^s &\leq \left(\sum_{k,l} |a_{mnkl}||x_{kl}|\right)^s \\
&\leq \left(\sup_{k,l \in \mathbb{N}} |a_{mnkl}|\right)^s \left(\sum_{k,l} |x_{kl}|\right)^s \\
&\leq \left(\sup_{k,l \in \mathbb{N}} |a_{mnkl}|\right)^s \sum_{k,l} |x_{kl}|^s
\end{aligned}$$

one can obtain by taking supremum over $m, n \in \mathbb{N}$ that

$$\|Ax\|_\infty = \sup_{m,n \in \mathbb{N}} \left|\sum_{k,l} a_{mnkl}x_{kl}\right| \leq N \left(\|x\|_s\right)^{1/s}.$$

This shows the sufficiency of the condition (12).

(ii) Let $1 < s < \infty$ and $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{M}_u)$. Then, Ax exists and is in $\mathcal{M}_u$ for all $x \in \mathcal{L}_s$. We assume that $M_1 = \infty$. Then, we may choose the sequences (m_i), (k_i), (n_j) and (l_j) in $\mathbb{N}$ with $k_i < k_{i+1}$ and $l_j < l_{j+1}$ for all $i, j \in \mathbb{N}$ such that

$$|a_{m_i n_j k_i l_j}|^{s'} > (ij)^{s'}. \tag{14}$$

Let us define the double sequence $x = (x_{kl}) \in \mathcal{L}_s$ by

$$x_{kl} := \begin{cases} sgn\,(a_{m_i n_j kl}) & , \quad k = k_i \text{ and } l = l_j, \\ 0 & , \quad otherwise \end{cases}$$

for all $k, l \in \mathbb{N}$. Since $s' > 1$, using the inequality (14) we see that

$$|(Ax)_{m_i n_j}| = \left|\sum_{k,l} a_{m_i n_j kl}x_{kl}\right| = \left|a_{m_i n_j k_i l_j}x_{k_i l_j}\right| = \left|a_{m_i n_j k_i l_j}\right| > ij$$

and so,

$$\sup_{i,j\in\mathbb{N}} |(Ax)_{m_i n_j}| > \infty,$$

a contradiction. Therefore, the condition (13) is necessary.

Conversely, suppose that (13) holds and take any $x = (x_{kl}) \in \mathcal{L}_s$. Then Ax exists, since $A_{mn} \in \mathcal{L}_{s'}$ for each $m, n \in \mathbb{N}$ by Theorem 2.7 in [2]. Therefore, we obtain by Hölder's inequality that

$$\begin{aligned} \|Ax\|_\infty &= \sup_{m,n\in\mathbb{N}} \left| \sum_{k,l} a_{mnkl} x_{kl} \right| \\ &\leq \sup_{m,n\in\mathbb{N}} \left(\sum_{k,l} |a_{mnkl}|^{s'} \right)^{1/s'} \left(\sum_{k,l} |x_{kl}|^s \right)^{1/s} \\ &< M_1 \|x\|_s, \end{aligned}$$

as desired.

This completes the proof. □

Theorem 4.2. *Let* $0 < s \leq 1$ *and* $1 \leq s_1 < \infty$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{L}_{s_1})$ *if and only if*

$$\sup_{k,l\in\mathbb{N}} \sum_{m,n} |a_{mnkl}|^{s_1} < \infty. \tag{15}$$

Proof. Let $0 < s \leq 1$, $1 \leq s_1 < \infty$ and $A \in (\mathcal{L}_s : \mathcal{L}_{s_1})$. Then, Ax exists and belongs to $\mathcal{L}_{s_1}$ for all $x \in \mathcal{L}_s$, and $A_{mn} \in \mathcal{M}_u$ by Corollary 3.2 for each $m, n \in \mathbb{N}$. Therefore, we obtain for $\mathbf{e^{kl}} \in \mathcal{L}_s$ that

$$\|A\mathbf{e^{kl}}\|_{s_1} = \left(\sum_{m,n} |a_{mnkl}|^{s_1} \right)^{1/s_1} < \infty$$

for each fixed $k, l \in \mathbb{N}$. That is to say that the condition (15) is necessary.

Conversely, suppose that the condition (15) is satisfied and take any $x = (x_{kl}) \in \mathcal{L}_s$. Then, $A_{mn} \in \mathcal{M}_u$ by Corollary 3.2 for each $m, n \in \mathbb{N}$ which implies the existence of Ax. Then,

$$\begin{aligned} \left(\sum_{m,n=0}^{i,j} |(Ax)_{mn}|^{s_1} \right)^{1/s_1} &= \left(\sum_{m,n=0}^{i,j} \left| \sum_{k,l} a_{mnkl} x_{kl} \right|^{s_1} \right)^{1/s_1} \\ &\leq \sum_{k,l} \left(\sum_{m,n=0}^{i,j} |a_{mnkl} x_{kl}|^{s_1} \right)^{1/s_1} \\ &= \sum_{k,l} \left[|x_{kl}| \left(\sum_{m,n=0}^{i,j} |a_{mnkl}|^{s_1} \right)^{1/s_1} \right] \\ &\leq \sup_{k,l\in\mathbb{N}} \left(\sum_{m,n=0}^{i,j} |a_{mnkl}|^{s_1} \right)^{1/s_1} \sum_{k,l} |x_{kl}| < \infty. \end{aligned}$$

Since $i, j \in \mathbb{N}$'s are arbitrary, we obtain that $\|Ax\|_{s_1} < \infty$, as desired. □

Theorem 4.3. *Let* $A = (a_{mnkl})$ *be any four dimensional matrix. Then, the following statements hold:*

(i) Let $0 < s \leq 1$. *Then,* $A \in (\mathcal{L}_s : \mathcal{C}_{bp})$ *if and only if (12) holds and there exists* $(\alpha_{kl}) \in \Omega$ *such that*

$$bp - \lim_{m,n\to\infty} a_{mnkl} = \alpha_{kl}. \tag{16}$$

(ii) Let $1 < s < \infty$. Then, $A \in (\mathcal{L}_s : \mathcal{C}_{bp})$ if and only if (13) and (16) hold.

Proof. (i) Let $0 < s \leq 1$ and suppose that $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{C}_{bp})$. Then, since the inclusion $\mathcal{C}_{bp} \subset \mathcal{M}_u$ holds, the necessity of the condition (12) is obtained from Part (i) of Theorem 4.1. Besides, since Ax exists and belongs to $\mathcal{C}_{bp}$ for every $x \in \mathcal{L}_s$ by hypothesis, this also holds for $\mathbf{e^{kl}} \in \mathcal{L}_s$ which gives that $A\mathbf{e^{kl}} = (a_{mnkl})_{m,n\in\mathbb{N}} \in \mathcal{C}_{bp}$ for each fixed $k, l \in \mathbb{N}$. Hence, the condition (16) is necessary.

Conversely, suppose that (12) and (16) hold, and $x = (x_{kl})$ be any sequence in the space $\mathcal{L}_s$. Then, since $A_{mn} \in \mathcal{L}_s^{\beta(\vartheta)}$ for each $m, n \in \mathbb{N}$, Ax exists. Therefore, we get by (16) for each fixed $k, l \in \mathbb{N}$ with (12) that

$$|\alpha_{kl}| = bp - \lim_{m,n\to\infty} |a_{mnkl}| \leq \sup_{m,n\in\mathbb{N}} |a_{mnkl}|$$

which gives that $(\alpha_{kl}) \in \mathcal{M}_u$. Hence, the series $\sum_{k,l} \alpha_{kl}x_{kl}$ converges for every $x \in \mathcal{L}_s$.

Additionally, for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|a_{mnkl} - \alpha_{kl}| < \varepsilon$ for all $m, n > n_0$ by (16). Then, we obtain that

$$\begin{aligned}
\left|\sum_{k,l} a_{mnkl}x_{kl} - \sum_{k,l} \alpha_{kl}x_{kl}\right|^s &= \left|\sum_{k,l}(a_{mnkl} - \alpha_{kl})x_{kl}\right|^s \\
&\leq \left[\sum_{k,l} |(a_{mnkl} - \alpha_{kl})x_{kl}|\right]^s \\
&< \varepsilon^s \left(\sum_{k,l} |x_{kl}|\right)^s \\
&< \varepsilon^s \sum_{k,l} |x_{kl}|^s .
\end{aligned}$$

This shows that $bp - \lim_{m,n\to\infty}(Ax)_{mn} = \sum_{k,l} \alpha_{kl}x_{kl}$, as desired.

(ii) Let $s > 1$. Since the necessity of the conditions can be easily seen in the similar way used in Part (i), we omit the details.

It is obtained with (13) for all $i, j \in \mathbb{N}$ that

$$\sum_{k,l=0}^{i,j} |\alpha_{kl}|^{s'} = bp - \lim_{m,n\to\infty} \sum_{k,l=0}^{i,j} |a_{mnkl}|^{s'} \leq \sup_{m,n\in\mathbb{N}} \sum_{k,l=0}^{i,j} |a_{mnkl}|^{s'} < \infty. \tag{17}$$

This means that $(\alpha_{kl}) \in \mathcal{L}_{s'}$. Hence, the double series $\sum_{k,l} \alpha_{kl}x_{kl}$ converges for every $x \in \mathcal{L}_s$.

For any given $\varepsilon > 0$, let us choose fixed $k_0, l_0 \in \mathbb{N}$ such that

$$\sum_{k,l=0,l_0+1}^{k_0,\infty} |x_{kl}|^s + \sum_{k,l=k_0+1,0}^{\infty,l_0} |x_{kl}|^s + \sum_{k,l=k_0+1,l_0+1}^{\infty} |x_{kl}|^s < \left(\frac{\varepsilon}{12M_1^{1/s'}}\right)^s. \tag{18}$$

Then, there exist an $n_0 \in \mathbb{N}$ by (16) such that

$$\left|\sum_{k,l=0}^{k_0,l_0}(a_{mnkl} - \alpha_{kl})x_{kl}\right| < \frac{\varepsilon}{2} \tag{19}$$

for every $m, n > n_0$. Therefore, by applying Hlder's inequality with using relations (17)-(19) we have that

$$\left|\sum_{k,l} a_{mnkl}x_{kl} - \sum_{k,l} \alpha_{kl}x_{kl}\right| = \left|\sum_{k,l}(a_{mnkl} - \alpha_{kl})x_{kl}\right| < \varepsilon$$

for all sufficiently large m, n. Hence, $Ax \in \mathcal{C}_{bp}$.

This step completes the proof. □

Theorem 4.4. *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, the following statements hold:*

(i) Let $0 < s \leq 1$. Then, $A \in (R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ if and only if

$$\sup_{m,n,k,l\in\mathbb{N}} \left| Q_k T_l \frac{a_{mnkl}}{q_k t_l} \right| < \infty, \tag{20}$$

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} \left| Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \right| < \infty, \tag{21}$$

$$\lim_{k\to\infty} Q_k T_l \Delta_{01}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) = 0 \text{ for each } l \in \mathbb{N}, \tag{22}$$

$$\lim_{l\to\infty} Q_k T_l \Delta_{10}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) = 0 \text{ for each } k \in \mathbb{N}. \tag{23}$$

(ii) Let $1 < s < \infty$. Then, $A \in (R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ if and only if the conditions (22)-(23) hold and

$$\sup_{m,n,\in\mathbb{N}} \sum_{k,l} \left| Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \right|^{s'} < \infty. \tag{24}$$

Proof. (i) Let $0 < s \leq 1$ and $x = (x_{mn}) \in R^{qt}(\mathcal{L}_s)$. Then, there exists a sequence $y = (y_{mn}) \in \mathcal{L}_s$. For the $(i, j)th$ rectangular partial sum of the series $\sum_{k,l} a_{mnkl}x_{kl}$, we have

$$\begin{aligned} (Ax)_{mn}^{[i,j]} &= \sum_{k,l=0}^{i,j} a_{mnkl}x_{kl} = \sum_{k,l=0}^{i-1,j-1} s_{kl}\Delta_{11}^{kl}a_{mnkl} + \sum_{k=0}^{i-1} s_{kj}\Delta_{10}^{kj}a_{mnkj} \\ &+ \sum_{l=0}^{j-1} s_{il}\Delta_{01}^{il}a_{mnil} + s_{ij}a_{mnij} \end{aligned}$$

for all $m, n \in \mathbb{N}$, where $s_{mn} = \sum_{i,j=0}^{m,n} x_{ij}$. Now, using the relation (4) we derive that

$$\begin{aligned} (Ax)_{mn}^{[i,j]} &= \sum_{k,l=0}^{i,j} a_{mnkl}x_{kl} = \sum_{k,l=0}^{i-1,j-1} Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) y_{kl} + \sum_{k=0}^{i-1} Q_k T_j \Delta_{10}^{kj} \left(\frac{a_{mnkj}}{q_k t_j} \right) y_{kj} \\ &+ \sum_{l=0}^{j-1} Q_i T_l \Delta_{01}^{il} \left(\frac{a_{mnil}}{q_i t_l} \right) y_{il} + Q_i T_j \frac{a_{mnij}}{q_i t_j} y_{ij} \end{aligned} \tag{25}$$

for all $m, n, i, j \in \mathbb{N}$. Define the matrix $B_{mn} = \left(b_{ijkl}^{mn}\right)$ by

$$b_{ijkl}^{mn} = \begin{cases} Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) & , \ 0 \leq k \leq i-1 \text{ and } 0 \leq l \leq j-1 \\ Q_k T_j \Delta_{10}^{kj} \left(\frac{a_{mnkj}}{q_k t_j} \right) & , \ 0 \leq k \leq i-1 \text{ and } l = j \\ Q_i T_l \Delta_{01}^{il} \left(\frac{a_{mnil}}{q_i t_l} \right) & , \ k = i \text{ and } 0 \leq l \leq j-1 \\ Q_i T_j \frac{a_{mnij}}{q_i t_j} & , \ k = i \text{ and } l = j \\ 0 & , \ \text{otherwise.} \end{cases} \tag{26}$$

Therefore, (25) can be written as $(Ax)_{mn}^{[i,j]} = (B_{mn}y)_{[i,j]}$. Then, the bp-convergence of the rectangular partial sums $(Ax)_{mn}^{[i,j]}$ for all $m, n \in \mathbb{N}$ and for all $x \in R^{qt}(\mathcal{L}_s)$ is equivalent to the statement that $B_{mn} \in (\mathcal{L}_s : \mathcal{C}_{bp})$ and hence the conditions

$$\sup_{k,l\in\mathbb{N}} \left| Q_k T_l \frac{a_{mnkl}}{q_k t_l} \right| < \infty, \tag{27}$$

$$\sum_{k,l} \left| Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \right| < \infty, \tag{28}$$

$$\lim_{k\to\infty} Q_k T_l \Delta_{01}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) = 0 \ \text{ for each } l \in \mathbb{N}, \tag{29}$$

$$\lim_{l\to\infty} Q_k T_l \Delta_{10}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) = 0 \ \text{ for each } k \in \mathbb{N} \tag{30}$$

must be satisfied for every fixed $m, n \in \mathbb{N}$.

If we take bp-limit in the terms of the matrix $B_{mn} = \left(b_{ijkl}^{mn}\right)$ as $i, j \to \infty$, we have

$$bp - \lim_{i,j\to\infty} b_{ijkl}^{mn} = Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right). \tag{31}$$

Using the relation (31) we can define a four dimensional matrix $B = (b_{mnkl})$ by

$$b_{mnkl} = Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \tag{32}$$

for all $m, n, k, l \in \mathbb{N}$. So, by the relations (25), (29), (30) and (31) we have

$$bp - \lim_{i,j\to\infty} (Ax)_{mn}^{[i,j]} = bp - \lim(By)_{mn}.$$

Thus, it is seen by combining the fact "$A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ if and only if $B \in (\mathcal{L}_s : \mathcal{M}_u)$" with Part (i) of Theorem 4.1 that

$$\sup_{m,n,k,l\in\mathbb{N}} \left| Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \right| < \infty. \tag{33}$$

Therefore, from the conditions (27)-(33), we see that $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ if and only if the conditions (20)-(23) hold.

(ii) Let $1 < s < \infty$. With the similar way used in the proof of Part (i), we have the bp-convergence of the rectangular partial sums $(Ax)_{mn}^{[i,j]}$ for all $m, n \in \mathbb{N}$ and for all $x \in R^{qt}(\mathcal{L}_s)$ is equivalent to the statement that $B_{mn} \in (\mathcal{L}_s : \mathcal{C}_{bp})$ and hence the conditions (21)-(23) and

$$\sum_{k,l} \left| Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \right|^{s'} < \infty \tag{34}$$

must be satisfied for every fixed $m, n \in \mathbb{N}$. Also, by the definition of the matrix $B_{mn} = \left(b_{ijkl}^{mn}\right)$ in (26) we have the relation (32).

Thus, it is seen by combining the fact "$A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ if and only if $B \in (\mathcal{L}_s : \mathcal{M}_u)$" with Part (ii) of Theorem 4.1 that

$$\sup_{m,n,\in\mathbb{N}} \sum_{k,l} \left| Q_k T_l \Delta_{11}^{kl} \left(\frac{a_{mnkl}}{q_k t_l} \right) \right|^{s'} < \infty. \tag{35}$$

Also, the condition (35) contains the conditions (21) and (34). Therefore, we see that $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{M}_u)$ if and only if the conditions (22)-(24) hold. This completes the proof. □

Since Theorems 4.5 and 4.6 can be proved in a similar way to that used in the proof of Theorem 4.4, we give them without proof.

Theorem 4.5. *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, the following statements hold:*

(i) Let $0 < s \leq 1$. Then, $A \in (R^{qt}(\mathcal{L}_s) : \mathcal{C}_{bp})$ if and only if the conditions (20)-(23) hold and

$$\exists(\alpha_{kl}) \in \Omega \text{ such that } bp-\lim_{m,n\to\infty} Q_k T_l \Delta_{11}^{kl}\left(\frac{a_{mnkl}}{q_k t_l}\right) = \alpha_{kl}. \tag{36}$$

(ii) Let $1 < s < \infty$. Then, $A \in (R^{qt}(\mathcal{L}_s) : \mathcal{C}_{bp})$ if and only if the conditions (22)-(24) and (36) hold.

Theorem 4.6. *Let $0 < s < 1$ and $1 < s_1 < \infty$. Then, $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{L}_{s_1})$ if and only if the conditions (21)-(23) hold and*

$$\sup_{k,l\in\mathbb{N}} \left| Q_k T_l \frac{a_{mnkl}}{q_k t_l} \right| < \infty, \tag{37}$$

$$\sup_{k,l\in\mathbb{N}} \sum_{m,n} \left| Q_k T_l \Delta_{11}^{kl}\left(\frac{a_{mnkl}}{q_k t_l}\right) \right|^{s_1} < \infty. \tag{38}$$

Theorem 4.7. *Let λ, μ be two double sequence spaces, $A = (a_{mnkl})$ be any four dimensional matrix and $B = (b_{mnij})$ also be a four dimensional triangle matrix such that $b_{mnij} = 0$ if $i > m$ and $j > n$ for all $m, n, i, j \in \mathbb{N}$. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

Proof. Suppose that λ, μ are two double sequence spaces, $A = (a_{ijkl})$ is any four dimensional matrix and $B = (b_{mnij})$ is also a four dimensional triangle matrix such that $b_{mnij} = 0$ if $i > m$ and $j > n$ for all $m, n, i, j \in \mathbb{N}$. Let $x = (x_{kl}) \in \lambda$. Then, since the equality

$$\sum_{i,j=0}^{m,n} b_{mnij} \sum_{k,l=0}^{r,t} a_{ijkl} x_{kl} = \sum_{k,l=0}^{r,t}\left(\sum_{i,j=0}^{m,n} b_{mnij} a_{ijkl}\right) x_{kl} \tag{39}$$

holds for all $m, n, r, t \in \mathbb{N}$ one can obtain by letting $r, t \to \infty$ in (39) that $B(Ax) = (BA)x$. Therefore, it is immediate that $Ax \in \mu_B$ whenever $x \in \lambda$ if and only if $(BA)x \in \mu$ whenever $x \in \lambda$.

This completes the proof. □

Now, we define the four dimensional matrices $C = (c_{mnkl})$, $D = (d_{mnkl})$ and $E = (e_{mnkl})$ by

$$c_{mnkl} = \sum_{i,j=0}^{m,n} a_{ijkl}, \quad d_{mnkl} = \sum_{i,j=0}^{m,n} \frac{a_{ijkl}}{(m+1)(n+1)} \text{ and } e_{mnkl} = \sum_{i,j=0}^{m,n} \frac{q_i t_j a_{ijkl}}{Q_m T_n}$$

for all $m, n, k, l \in \mathbb{N}$.

One can derive several new results from Theorems 4.1-4.7.

Corollary 4.8. *Let $0 < s \leq 1$. Then, the following statements hold:*

(i) $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{BS})$ if and only if (12) holds with c_{mnkl} instead of a_{mnkl}.

(ii) $A = (a_{mnkl}) \in (\mathcal{L}_s : \widetilde{\mathcal{M}_u})$ if and only if (12) holds with d_{mnkl} instead of a_{mnkl}.

(iii) $A = (a_{mnkl}) \in (\mathcal{L}_s : R^{qt}(\mathcal{M}_u))$ if and only if (12) holds with e_{mnkl} instead of a_{mnkl}.

(iv) $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{CS}_{bp})$ if and only if (12) and (16) hold with c_{mnkl} instead of a_{mnkl}.

(v) $A = (a_{mnkl}) \in (\mathcal{L}_s : \widetilde{\mathcal{C}_{bp}})$ if and only if (12) and (16) hold with d_{mnkl} instead of a_{mnkl}.

(vi) $A = (a_{mnkl}) \in (\mathcal{L}_s : R^{qt}(\mathcal{C}_{bp}))$ if and only if (12) and (16) hold with e_{mnkl} instead of a_{mnkl}.

Corollary 4.9. *Let $1 < s < \infty$. Then, the following statements hold:*

(i) $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{BS})$ if and only if (13) holds with c_{mnkl} instead of a_{mnkl}.

(ii) $A = (a_{mnkl}) \in (\mathcal{L}_s : \widetilde{\mathcal{M}_u})$ *if and only if (13) holds with* d_{mnkl} *instead of* a_{mnkl}.
(iii) $A = (a_{mnkl}) \in (\mathcal{L}_s : R^{qt}(\mathcal{M}_u))$ *if and only if (13) holds with* e_{mnkl} *instead of* a_{mnkl}.
(iv) $A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{CS}_{bp})$ *if and only if (13) and (16) hold with* c_{mnkl} *instead of* a_{mnkl}.
(v) $A = (a_{mnkl}) \in (\mathcal{L}_s : \widetilde{\mathcal{C}_{bp}})$ *if and only if (13) and (16) hold with* d_{mnkl} *instead of* a_{mnkl}.
(vi) $A = (a_{mnkl}) \in (\mathcal{L}_s : R^{qt}(\mathcal{C}_{bp}))$ *if and only if (13) and (16) hold with* e_{mnkl} *instead of* a_{mnkl}.

Corollary 4.10. *Let* $0 < s \leq 1$. *Then, the following statements hold:*
(i) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{BS})$ *if and only if (20)-(23) hold with* c_{mnkl} *instead of* a_{mnkl}.
(ii) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{CS}_{bp})$ *if and only if (20)-(23) and (36) hold with* c_{mnkl} *instead of* a_{mnkl}.
(iii) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \widetilde{\mathcal{M}_u})$ *if and only if (20)-(23) hold with* d_{mnkl} *instead of* a_{mnkl}.
(iv) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \widetilde{\mathcal{C}_{bp}})$ *if and only if (20)-(23) and (36) hold with* d_{mnkl} *instead of* a_{mnkl}.
(v) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : R^{qt}(\mathcal{M}_u))$ *if and only if (20)-(23) hold with* e_{mnkl} *instead of* a_{mnkl}.
(vi) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : R^{qt}(\mathcal{C}_{bp}))$ *if and only if (20)-(23) and (36) hold with* e_{mnkl} *instead of* a_{mnkl}.

Corollary 4.11. *Let* $1 < s < \infty$. *Then, the following statements hold:*
(i) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{BS})$ *if and only if (22)-(24) hold with* c_{mnkl} *instead of* a_{mnkl}.
(ii) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \mathcal{CS}_{bp})$ *if and only if (22)-(24) and (36) hold with* c_{mnkl} *instead of* a_{mnkl}.
(iii) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \widetilde{\mathcal{M}_u})$ *if and only if (22)-(24) hold with* d_{mnkl} *instead of* a_{mnkl}.
(iv) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : \widetilde{\mathcal{C}_{bp}})$ *if and only if (22)-(24) and (36) hold with* d_{mnkl} *instead of* a_{mnkl}.
(v) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : R^{qt}(\mathcal{M}_u))$ *if and only if (22)-(24) hold with* e_{mnkl} *instead of* a_{mnkl}.
(vi) $A = (a_{mnkl}) \in (R^{qt}(\mathcal{L}_s) : R^{qt}(\mathcal{C}_{bp}))$ *if and only if (22)-(24) and (36) hold with* e_{mnkl} *instead of* a_{mnkl}.

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