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Monotone Iterative Method for a Class of Nonlinear Fractional Differential Equations on Unbounded Domains in Banach Spaces

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Abstract. In this paper, we investigate the existence of minimal nonnegative solution for a class of nonlinear fractional integro-differential equations on semi-infinite intervals in Banach spaces by applying the cone theory and the monotone iterative technique. An example is given for the illustration of main results.

1. Introduction and Terminology

Fractional differential equations are now recognized as an excellent source of models to many phenomena observed in control theory, mechanics, electricity, chemistry, biology, economics, signal and image processing, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, etc. For some recent details and examples, see [1]-[15] and the references therein.

The monotone iterative technique can be successfully applied to obtain existence results for fractional differential problems, see book [3] and papers [16]-[31]. In these papers, by employing the technique, authors obtained the existence results of fractional differential problems on bounded domains. In our paper, we also apply this technique to fractional differential problems on unbounded domains in Banach Spaces.

Boundary value problems of integer order on infinite intervals arise in the study of radially symmetric solutions of the nonlinear elliptic equations and have received considerable attention, for instance, see [32]-[40] and references therein. However, there are few papers dealing with nonlinear fractional differential equations on an unbounded domain [41]-[48]. In this paper, by using a method entirely different from the ones employed in [41]-[48], we discuss the existence of the minimal nonnegative solution on an unbounded domain in an ordered Banach space E for the following boundary value problem (BVP for short) of a fractional nonlinear integro-differential equation

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), Tu(t), Su(t)) = \theta, \quad 1 < \alpha < 2, \\ u(0) = \theta, \quad D^{\alpha-1}u(\infty) = u^*, \end{cases} \tag{1}$$

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where $t \in J = [0, +\infty)$, $f \in C[J \times P \times P \times P, P]$, P is a cone of E which defines a partial ordering in E: $x \leq y$ if and only if $y - x \in P$. D^α is the Riemann-Liouville fractional derivatives.

$$(Tu)(t) = \int_0^t k(t,s)u(s)ds, \quad (Su)(t) = \int_0^\infty h(t,s)u(s)ds.$$

$k(t,s) \in C[D, \mathbb{R}^+]$, $h(t,s) \in C[D_0, \mathbb{R}^+]$, $D = \{(t,s) \in \mathbb{R}^2 \mid 0 \leq s \leq t\}$, $D_0 = \{(t,s) \in J \times J\}$, $\mathbb{R}^+ = [0, +\infty)$. Let

$$k^* = \sup_{t\in J} \int_0^t k(t,s)ds < \infty, \quad h^* = \sup_{t\in J} \frac{1}{(1+t^{\alpha-1})} \int_0^\infty h(t,s)(1+s^{\alpha-1})ds < \infty$$

and $\lim_{t'\to t} \int_0^\infty |h(t',s) - h(t,s)|(1+s^{\alpha-1})ds = 0, \quad t, t' \in J$.

Now, we denote the space

$$FC(J,E) = \{u \in C(J,E) : \sup_{t\in J} \frac{\|u(t)\|}{1+t^{\alpha-1}} < \infty\}$$

with norm

$$\|u\|_F = \sup_{t\in J} \frac{\|u(t)\|}{1+t^{\alpha-1}}.$$

It is easy to see that $FC(J,E)$ is a Banach space. Denote $FC(J,P) = \{u \in FC(J,E) : u(t) \geq \theta, \ \forall t \in J\}$. A map $u(t) \in FC(J,P)$ with its Riemann-Liouville derivative of order α existing on J is called a nonnegative solution of (1) if $u(t) \in FC(J,P)$ satisfies (1).

2. Several Lemmas

In this section, we recall some definitions and present some preliminary lemmas.

Definition 2.1. *[1] The Riemann-Liouville fractional derivative of order δ for a continuous function f is defined by*

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\delta-1} f(s)ds, \qquad n = [\delta]+1,$$

provided the right hand side is defined pointwise on $(0,\infty)$.

Definition 2.2. *[1] The Riemann-Liouville fractional integral of order δ for a continuous function f is defined as*

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)ds, \qquad \delta > 0,$$

provided that the integral exists.

For the forthcoming analysis, we need the following assumptions:

(H_1) there exist nonnegative functions $a(t), b(t) \in C(J, \mathbb{R}^+)$ and positive constants c_1, c_2, c_3 such that

$$\|f(t,u,v,w)\| \leq a(t) + b(t)(c_1\|u\| + c_2\|v\| + c_3\|w\|), \ t \in J, u, v, w \in P.$$

Furthermore, we set $a^* = \int_0^\infty a(t)dt < \infty$, $b^* = \int_0^\infty (1+t^{\alpha-1})b(t)dt < \infty$.

(H_2) $f(t,u,v,w)$ is increasing in $u, v, w \in P$, that is,

$$f(t,u,v,w) \leq f(t,\overline{u},\overline{v},\overline{w}), \quad t \in J, \ \overline{u} \geq u \geq \theta, \ \overline{v} \geq v \geq \theta, \ \overline{w} \geq w \geq \theta.$$

Lemma 2.3. *Assume* (H_1) *holds. Then* $u(t) \in FC(J,P)$ *with its Riemann-Liouville derivative of order* α *existing on* J *is called a nonnegative solution of problem (1) if and only if* $u(t) \in FC(J,P)$ *is a solution of the integral equation*

$$u(t) = \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s) f(s,u(s),(Tu)(s),(Su)(s))ds, \tag{2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t, \\ t^{\alpha-1}, & 0 \le t \le s. \end{cases} \tag{3}$$

Proof. We can reduce (1) to the integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),(Tu)(s),(Su)(s))ds, \tag{4}$$

where constants $c_1, c_2 \in \mathbb{R}$.

By the conditions $u(0) = \theta$ and $D^{\alpha-1}u(\infty) = u^*$, we can get

$$c_1 = \frac{1}{\Gamma(\alpha)}\Big(\int_0^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + u^*\Big), \qquad c_2 = 0.$$

Substituting c_1 and c_2 into (4), we have

$$\begin{aligned} u(t) =& \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty f(s,u(s),(Tu)(s),(Su)(s))ds \\ &- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),(Tu)(s),(Su)(s))ds \\ =& \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s) f(s,u(s),(Tu)(s),(Su)(s))ds, \end{aligned} \tag{5}$$

where $G(t,s)$ is defined by 3. The converse follows by direct computation.
□

Remark 2.4. *Notice that* $G(t,s) \ge 0$ *and* $\dfrac{G(t,s)}{1+t^{\alpha-1}} < \dfrac{1}{\Gamma(\alpha)}$.

Define the operator A by

$$(Au)(t) = \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s) f(s,u(s),(Tu)(s),(Su)(s))ds. \tag{6}$$

Lemma 2.5. *If* (H_1) *is satisfied, then the operator* A *is from* $FC(J,P)$ *to* $FC(J,P)$.

Proof. Let $u(t) \in FC(J,P)$, that is $u(t) \ge \theta$ and $\|u\|_F < \infty$. Since $f \in C[J \times P \times P \times P, P]$ and $G(t,s) > 0$, therefore $(Au)(t) \ge \theta$. By the condition (H_1), we have

$$\begin{aligned} \|(Au)(t)\| &\le \frac{\|u^*\| t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s) \|f(s,u(s),(Tu)(s),(Su)(s))\| ds \\ &\le \frac{\|u^*\| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1+t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty [a(s) + b(s)(c_1\|u(s)\| + c_2\|(Tu)(s)\| + c_3\|(Su)(s)\|)]ds \\ &\le \frac{\|u^*\| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1+t^{\alpha-1}}{\Gamma(\alpha)} [a^* + b^*(c_1 + c_2 k^* + c_3 h^*)\|u\|_F], \end{aligned} \tag{7}$$

which implies that

$$\|Au\|_F = \sup_{t\in J} \frac{\|(Au)(t)\|}{1+t^{\alpha-1}} < \infty,$$

that is, A is $FC(J,P) \to FC(J,P)$. This completes the proof. □

3. Main Results

Theorem 3.1. *Let P be a fully regular cone and the assumptions (H_1), (H_2) are satisfied. Furthermore,*

$$r = \frac{b^*(c_1 + c_2 k^* + c_3 h^*)}{\Gamma(\alpha)} < 1. \tag{8}$$

Then there exists a nondecreasing sequence $\{u_n\} \subset FC(J, P)$ converges uniformly on J to the minimal solution $\overline{u}$. That is, for any solution $u(t)$ of (1), we have $u(t) \geq \overline{u}(t), \forall t \in J$. Moreover, $\|\overline{u}\|_F \leq \frac{d}{1-r}$, where $d = \frac{\|u^*\|_F + a^*}{\Gamma(\alpha)}$ and $a^*, b^*, k^*, h^*, c_1, c_2, c_3$ are given by (H_1) and (H_2).

Proof. Let $u_0(t) = \theta$, $u_n(t) = Au_{n-1}(t), n = 1, 2, 3, \cdots$, where

$$u_n(t) = \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s) f(s, u_{n-1}(s), (Tu_{n-1})(s), (Su_{n-1})(s)) ds. \tag{9}$$

By Lemma 2.5, we have $u_n(t) \in FC(J, P)$. Thus $u_n(t) \geq \theta$.

On one hand, using (H_2) and the fact that $f \in C[J \times P \times P \times P, P]$ and $G(t, s) \geq 0$, we have

$$\theta = u_0(t) \leq u_1(t) \leq u_2(t) \leq \cdots \leq u_n(t) \leq \cdots, \quad t \in J. \tag{10}$$

Now, from the iteration formula (9), we can get

$$\begin{aligned} \|u_n\|_F &= \|Au_{n-1}\|_F \leq d + r\|u_{n-1}\|_F \leq d + r(d + r\|u_{n-2}\|_F) \\ &\leq d + r(d + r(d + r\|u_{n-3}\|_F)) \leq \cdots \leq d(1 + r + r^2 + r^3 + \cdots + r^n\|u_0\|_F) \\ &\leq \frac{d}{1-r}, \quad n = 1, 2, 3, \cdots, \end{aligned} \tag{11}$$

where r and d are given in the statement of Theorem 3.1.

It follows from (11) and the fully regularity of P that

$$\lim_{n\to\infty} u_n(t) = \overline{u}(t), \ t \in J. \tag{12}$$

Since $u_n(t) \in FC(J, P)$ and $FC(J, P)$ is a closed convex set in space $C(J, E)$, therefore, by (11), we have $\overline{u} \in FC(J, P)$ and $\|\overline{u}\|_F \leq \frac{d}{1-r}$.

Moreover, we have

$$f(s, u_n(s), (Tu_n)(s), (Su_n)(s)) \to f(s, \overline{u}(s), (T\overline{u})(s), (S\overline{u})(s)) \tag{13}$$

and

$$\begin{aligned} &\|f(s, u_n(s), (Tu_n)(s), (Su_n)(s)) - f(s, \overline{u}(s), (T\overline{u})(s), (S\overline{u})(s))\| \\ &\leq 2a(s) + 2b(s)(c_1 + c_2 k^* + c_3 h^*)\frac{d}{1-r}, \ s \in J, \ n = 1, 2, \cdots. \end{aligned} \tag{14}$$

Taking the limit $n \to \infty$ in (9), and using (13) and (14), we obtain

$$\overline{u}(t) = \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s) f(s, \overline{u}(s), (T\overline{u})(s), (S\overline{u})(s)) ds, \tag{15}$$

which, by Lemma 2.3, implies that $\overline{u} \in FC(J, P)$ is a nonnegative solution of problem (1).

Finally, we prove the minimal property of the solution $\overline{u}(t)$. Let $u(t) \in FC(J, P)$ be any solution of (1). By Lemma 2.3, $u(t)$ satisfies (2). Also, we have that $u(t) \geq \theta = u_0(t)$ for $t \in J$. Assuming that $u(t) \geq u_{n-1}(t)$ holds for $t \in J$, it follows from (2), (9) and (H_2) that $u(t) \geq u_n(t)$. Hence, by induction, taking the limit $n \to \infty$, we get $u(t) \geq \overline{u}(t)$ for $t \in J$. This implies that $\overline{u}(t)$ is the minimal solution of (1). This completes the proof. □

Theorem 3.2. *Let P be a regular cone and the conditions* $(H_1),(H_2)$ *hold. If there exists a* $w \in FC(J,P)$ *with its Riemann-Liouville derivative of order* α *existing on J such that*

$$\begin{cases} D^{\alpha}w(t) + f(t, w(t), Tw(t), Sw(t)) \leq \theta, \\ w(0) = \theta, \quad D^{\alpha-1}w(\infty) \geq u^*, \end{cases} \tag{16}$$

then (1) has a minimal nonnegative solution $\overline{u}$*. Moreover,* $\overline{u} \in FC(J,P)$ *and* $\overline{u} \leq w(t), \forall t \in J$.

Proof. From (16) and Lemma 2.3, we have

$$\begin{aligned} w(t) &\geq \frac{u^* t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^{\infty} G(t,s) f(s, w(s), (Tw)(s), (Sw)(s))ds \\ &= (Aw)(t), \end{aligned} \tag{17}$$

where $G(t,s)$ is given by 3 and the operator A is defined by (6).

Let $u_0(t) = \theta,\ u_n(t) = (Au_{n-1})(t), n = 1, 2, 3 \cdots$. As in the proof of Theorem 3.1, (10) holds. Furthermore, $u_0(t) \leq w(t)$. Assuming $u_{n-1}(t) \leq w(t)$ for $t \in J$, we find by (H_3) and (17) that

$$u_n(t) = (Au_{n-1})(t) \leq (Aw)(t) \leq w(t), \ \forall t \in J.$$

Also, from (10), (17) and the regularity of P, (12) holds and we have that $\overline{u}(t) \leq w(t), \forall t \in J$. Thus, it follows that

$$\|\overline{u}(t)\| \leq N\|w(t)\|, \forall t \in J,$$

where N denotes the normal constant of the cone P. As in the proof of Theorem 3.1, it can be shown that $\{u_n(t)\}$ converges to $\overline{u}(t)$ uniformly on J. Hence $\overline{u}(t) \in FC(J,P)$. Further, we have that $\overline{u}(t)$ satisfies (15) and $\overline{u}$ is the minimal nonnegative solution of (1). This completes the proof. □

Concluding Remarks. It is imperative to note that our method of proof is entirely different from the one employed in ([41]-[48]). In case f does not depend on Volterra integral operator $Tu(t)$ and Fredholm integral operator $Su(t)$, our problem reduces to the one considered in [44], where the existence of the solution for the problem (1) with nonlinearity $f(t, u(t))$ was shown by requiring a condition of the form:

(H) there exists a nonnegative function $l(t) \in L^1(J)$ such that $\alpha(f(t, B)) \leq l(t)\alpha(B), \quad t \in J$, where B is any bounded subset of E and $\int_0^{\infty}(1 + t^{\alpha-1})l(t)dt < \Gamma(\alpha)$.

In the present work, we not only remove the condition (H) on f, but also obtain minimal nonnegative solution of the problem (1). Thus, our results generalize and improve the work presented in [44].

4. Example

Consider the problem

$$\begin{cases} D^{\frac{3}{2}}u_n(t) + \dfrac{e^{-t}u_{n+1}}{5(1+\sqrt{t})^5} + \dfrac{e^{-2t}\sqrt{1+2u_n+u_{2n+1}}}{2^{n+3}(1+\sqrt{t})^3} + \dfrac{e^{-3t}}{10(1+\sqrt{t})^2}\Big[1 + \displaystyle\int_0^t e^{-(t+1)s}u_{2n}(s)ds\Big]^{\frac{1}{5}} \\ \qquad + \dfrac{e^{-2t}}{2^{n+2}(1+\sqrt{t})}\Big[\displaystyle\int_0^{\infty} \dfrac{u_n(s)}{1+t+s^2}ds\Big]^{\frac{1}{3}} = \theta, \qquad 0 \leq t < \infty, \\ u_n(0) = \theta, \quad D^{\frac{1}{2}}u_n(\infty) = \dfrac{1}{n^3}, \end{cases} \tag{18}$$

We will prove that the problem (18) has a minimal nonegative solution $u_n(t)$ satisfying $\sum_{n=1}^{\infty} u_n(t) < \infty$ for $t \geq 0$.

Let $E = l^1 = \{u = (u_1, \cdots, u_n, \cdots) : \sum_{n=1}^{\infty} |u_n| < \infty\}$ with norm $\|u\| = \sum_{n=1}^{\infty} |u_n|$ and $P = \{u = (u_1, \cdots, u_n, \cdots) \in l^1 : u_n \geq 0, n = 1, 2, \cdots\}$. Then P is a normal cone in E. Since l^1 is weakly complete, from Theorem 2.2 in [49], the normality of P implies the regularity of P, it's easy to show that P is fully regular.

Now (18) can be considered as a boundary value problem of form (1) in E, where $u = (u_1, \cdots, u_n, \cdots), v = (v_1, \cdots, v_n, \cdots), w = (w_1, \cdots, w_n, \cdots), k(t,s) = e^{-(t+1)s}, h(t,s) = (1+t+s^2)^{-1}, f = (f_1, \cdots, f_n, \cdots)$,

$$f_n(t,u,v,w) = \frac{e^{-t}u_{n+1}}{5(1+\sqrt{t})^5} + \frac{e^{-2t}\sqrt{1+2u_n+u_{2n+1}}}{2^{n+3}(1+\sqrt{t})^3} + \frac{e^{-3t}}{10(1+\sqrt{t})^2}\Big(1+v_{2n}\Big)^{\frac{1}{5}} + \frac{e^{-2t}}{2^{n+2}(1+\sqrt{t})}w_n^{\frac{1}{3}}, \tag{19}$$

and $\theta = \{0, \cdots, 0, \cdots\}$, $u^* = \{1, \cdots, \frac{1}{n^3}, \cdots\}$.

Clearly, $f \in C(J \times P \times P \times P, P)$, where $J = [0, \infty)$, and $\theta, u^* \in P$. Then, the condition (H_2) holds.

Note that

$$k^* = \sup_{t \in J} \int_0^t e^{-(t+1)s} ds = \sup_{t \in J} \frac{1 - e^{-(t+1)t}}{t+1} ds \le 1, \quad h^* = \sup_{t \in J} \int_0^\infty (1+t+s^2)^{-1} ds \le \int_0^\infty (1+s^2)^{-1} ds = \frac{\pi}{2}.$$

and

$$\lim_{t' \to t} \int_0^\infty \Big| \frac{1}{1+t'+s^2} - \frac{1}{1+t+s^2} \Big| ds = \lim_{t' \to t} \int_0^\infty \frac{|t'-t|}{(1+t+s^2)(1+t'+s^2)} ds = 0, \quad t', t \in J.$$

By a simple computation, we have

$$\begin{aligned} 0 \le f_n(t,u,v,w) \le & \frac{e^{-t}u_{n+1}}{5(1+\sqrt{t})^5} + \frac{e^{-2t}}{2^{n+3}(1+\sqrt{t})^3}\Big(1+u_n+\frac{1}{2}u_{2n+1}\Big) + \frac{e^{-3t}}{10(1+\sqrt{t})^2}\Big(1+\frac{1}{5}v_{2n}\Big) \\ & + \frac{e^{-2t}}{2^{n+2}(1+\sqrt{t})}\Big(\frac{2}{3}+\frac{1}{3}w_n\Big) \\ \le & \frac{e^{-2t}}{10(1+\sqrt{t})} + \frac{e^{-t}}{(1+\sqrt{t})}\Big[\frac{1}{5}u_{n+1} + \frac{1}{2^{n+3}}u_n + \frac{1}{2^{n+4}}u_{2n+1} + \frac{1}{50}v_{2n} + \frac{1}{3 \times 2^{n+2}}w_n\Big]. \end{aligned}$$

So,

$$\begin{aligned} \|f(t,u,v,w)\| \le & \frac{e^{-2t}}{10(1+\sqrt{t})} + \frac{e^{-t}}{(1+\sqrt{t})}\Big[\frac{1}{5}\|u\| + \frac{1}{8}\|u\| + \frac{1}{16}\|u\| + \frac{1}{50}\|v\| + \frac{1}{12}\|w\|\Big] \\ = & \frac{e^{-2t}}{10(1+\sqrt{t})} + \frac{e^{-t}}{(1+\sqrt{t})}\Big[\frac{31}{80}\|u\| + \frac{1}{50}\|v\| + \frac{1}{12}\|w\|\Big]. \end{aligned}$$

and $a^* = \int_0^\infty a(t)dt = \int_0^\infty \frac{e^{-2t}}{10(1+\sqrt{t})}dt \le \frac{1}{20}$, $b^* = \int_0^\infty (1+\sqrt{t})b(t)dt = \int_0^\infty e^{-t}dt = 1$. Then (H_1) holds.

In addition,

$$r = \frac{b^*(c_1 + c_2k^* + c_3h^*)}{\Gamma(\alpha)} \le \frac{\frac{31}{80} + \frac{1}{50} + \frac{\pi}{24}}{\frac{\sqrt{\pi}}{2}} < 0.607915 < 1.$$

Hence, all conditions of Theorem 3.1 hold. Thus, our conclusion follows from Theorem 3.1.

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