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# Inequalities Pertaining to Rational Functions with Prescribed Poles

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**Abstract.** Let $\mathbf{R_n}$ be the space of rational functions with prescribed poles. If $t_1, t_2, ..., t_n$ are the zeros of $B(z) + \lambda$ and $s_1, s_2, ..., s_n$ are zeros of $B(z) - \lambda$, where $B(z)$ is the Blaschke product and $\lambda \in T$, then for $z \in T$

$$|r'(z)| \leq \frac{|B'(z)|}{2}[(\max_{1\leq k\leq n}|r(t_k)|)^2 + (\max_{1\leq k\leq n}|r(s_k)|)^2].$$

Let $r, s \in \mathbf{R_n}$ and assume $s$ has all its $n$ zeros in $D^- \cup T$ and $|r(z)| \leq |s(z)|$ for $z \in T$, then for any $\alpha$ with $|\alpha| \leq \frac{1}{2}$ and for $z \in T$

$$|r'(z) + \alpha B'(z)r(z)| \leq |s'(z) + \alpha B'(z)s(z)|.$$

In this paper, we consider a more general class of rational functions $rof \in \mathbf{R}_{m^\star n}$, defined by $(rof)(z) = r(f(z))$, where $f(z)$ is a polynomial of degree $m^\star$ and prove some generalizations of the above inequalities..

## 1. Introduction, Background and Notation

Let $\mathbf{P}_n$ be the space of complex polynomials of degree at most $n$ and $\mathbf{C}$ be the complex plane. Let $T = \{z \in \mathbf{C} : |z| = 1\}$ and $D^-$ denotes the region inside $T$ and $D^+$ denotes the region outside $T$. For $F$ defined on $T$ in the complex plane, we set

$$M(F, 1) := \sup_{z\in T}|F(z)|$$

and

$$m(F, 1) := \inf_{z\in T}|F(z)|.$$

Let $p \in \mathbf{P}_n$, then concerning the estimate of $M(p', 1)$ on $T$, we have by a famous result due to Bernstein [4]:

$$M(p', 1) \leq n\, M(p, 1). \tag{1}$$

This result is sharp and equality holds for the polynomials having all zeros at origin.

For the class of polynomials $p \in \mathbf{P}_n$, which does not vanish inside the unit disk, we have

$$M(p', 1) \leq \frac{n}{2}M(p, 1). \tag{2}$$

Equality in (2) holds for $p(z) = \lambda z^n + \mu$, $|\lambda| = |\mu| = 1$.

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Inequality (2) was conjectured by Erdös and latter on verified by Lax [7].

Let $a_1, a_2, ..., a_n$ be n given points in $D^+$. We consider the following space of rational functions $\mathbf{R_n}$ with prescribed poles:

$$\mathbf{R_n} := \mathbf{R_n}(a_1, a_2, ..., a_n) = \left[\frac{p(z)}{w(z)} : p \in \mathbf{P}_n\right],$$

where

$$w(z) = (z - a_1)(z - a_2)...(z - a_n).$$

Let

$$B(z) := \frac{z^n \overline{w(1/\bar{z})}}{w(z)} = \prod_{k=1}^{n} \frac{1 - \bar{a}_k z}{z - a_k},$$

where $B(z) \in \mathbf{R_n}$ is called Blaschke product. Note that $|B(z)| = 1$, when $z \in T$.

Li, Mohapatra and Rodriguez [9] extended Bernstein inequality to rational functions $r \in \mathbf{R_n}$ with prescribed poles $a_1, a_2, ...a_n$ replacing $z^n$ by Blaschke product $B(z)$ and proved:

**Theorem 1.1.** If $z \in T$, then for any $r \in \mathbf{R_n}$,

$$|r'(z)| \leq |B'(z)| \sup_{z \in T} |r(z)|.$$

Further more, the inequality is sharp and equality holds for $r(z) = \alpha B(z)$ with $|\alpha| = 1$.

They also proved the following result for rational functions with restricted zeros, which generalize polynomial inequality of Erdös and Lax [7].

**Theorem 1.2.** Suppose $r \in \mathbf{R_n}$ has all its zeros in $T \cup D^+$, then for $z \in T$

$$|r'(z)| \leq \frac{1}{2} |B'(z)| \sup_{z \in T} |r(z)|.$$

The inequality is sharp and equality holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta|$.

As a refinement of Theorem 1.2, Aziz and Shah [3] proved the following:

**Theorem 1.3.** If $r \in \mathbf{R_n}$ and all zeros of $r$ lie in $T \cup D^+$, then for $z \in T$

$$|r'(z)| \leq \frac{1}{2} |B'(z)| \Big[ M(r, 1) - m(r, 1) \Big],$$

where $M(r, 1) = \sup_{z \in T} |r(z)|$ and $m(r, 1) = \inf_{z \in T} |r(z)|$.

Equality holds for $r(z) = B(z) + ke^{i\alpha}, k \geq 1$ and $\alpha$ is real.

Recently Li [8] proved the following interesting result.

**Theorem 1.4.** Let $r, s \in \mathbf{R_n}$ and assume $s$ has all its $n$ zeros in $D^- \cup T$ and $|r(z)| \leq |s(z)|$ for $z \in T$, then for any $\alpha$ with $|\alpha| \leq \frac{1}{2}$ and for $z \in T$

$$|r'(z) + \alpha B'(z) r(z)| \leq |s'(z) + \alpha B'(z) s(z)|.$$

This result is sharp and equality holds for $r(z) = s(z)$.

In the literature [1-11], there exists several improvements of inequality (1). Here we mention the following improvement of inequality (1) due to Mohapatra, O'Hara and Rodriguez [10], which was also independently proved by Aziz[1].

**Theorem 1.5.** Let $p \in \mathbf{P}_n$, $t_1, t_2, ..., t_n$ be the zeros of $z^n + u$ and $s_1, s_2, ..., s_n$ be zeros of $z^n - u$, where $u \in T$, then

$$M(p', 1) \leq \frac{n}{2}[(\max_{1\leq k\leq n} |p(t_k)|)^2 + (\max_{1\leq k\leq n} |p(s_k)|)^2].$$

Recently among other things Li, Mohapatra and Rodriguez [9] proved the following Bernstein-type inequality for rational functions $r \in \mathbf{R_n}$ similar to Theorem 1.5.

**Theorem 1.6.** Let $r \in \mathbf{R_n}$. if $t_1, t_2, ..., t_n$ are the zeros of $B(z) + \lambda$ and $s_1, s_2, ..., s_n$ are zeros of $B(z) - \lambda$, where $\lambda \in T$, then for $z \in T$

$$|r'(z)| \leq \frac{|B'(z)|}{2}[(\max_{1\leq k\leq n} |r(t_k)|)^2 + (\max_{1\leq k\leq n} |r(s_k)|)^2].$$

The inequality is sharp and equality holds for $r(z) = \lambda + B(z)$ with $\lambda \in T$.

In this paper, we first prove a result for rational functions having all zeros in $D^+$, which is an improvement of a result recently proved by Li [8] and also provides a generalisation of a result earlier proved by Aziz and Shah [3]:

Now, we consider a class of rational functions $r(f(z))$, defined by

$$(rof)(z) = r(f(z)) = \frac{p(f(z))}{w(f(z))},$$

where $f(z)$ is a polynomial of degree $m^\star$ and $r(z)$ is a rational function of degree $n$, so that $rof \in \mathbf{R_{m^\star n}}$, and

$$w(f(z)) = \prod_{j=1}^{m^\star n}(z - a_j).$$

Hence, in case of Balaschke product $\mathcal{B}(z)$ is given by

$$\mathcal{B}(z) := \frac{w^*(f(z))}{w(f(z))} = \frac{z^{m^\star n}\overline{w(f(\frac{1}{\bar{z}}))}}{w(f(z))} = \prod_{j=1}^{m^\star n}\Big(\frac{1 - \bar{a}_j z}{z - a_j}\Big).$$

Now onwards, we shall always assume that all poles $a_1, a_2, ..., a_{m^\star n}$ of $r(f(z))$ lie in $D^+$. For the case when all poles are in $D^-$, we can obtain analogous results with suitable transformations.

## 2. Main Results

To prove these theorems, we make use of the following lemmas. The first two lemmas are due to Li, Mohapatra and Rodriguez [9].

**Lemma 2.1**. Suppose $\lambda \in T$. Then the equation $B(z) = \lambda$ has exactly $n$ simple roots, say $t_1, t_2, ..., t_n$ and all lie on the unit circle $T$. Moreover

$$\frac{t_k B'(t_k)}{\lambda} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2} \;\; for\; k = 1, 2, 3, ..., n. \tag{3}$$

**Lemma 2.2**. If $|x| = |y| = 1$, then

$$(x - y)^2 = -xy|x - y|^2. \tag{4}$$

**Lemma 2.3.** If $rof \in \mathbf{R}_{\mathbf{m^\star n}}$ and $z \in T$, then

$$\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) - \lambda] = \frac{\mathcal{B}(z)}{z}\sum_{k=1}^{m^\star n} c_k r(f(t_k))\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2, \tag{5}$$

where $c_k = c_k(\lambda)$ is defined for $k = 1, 2, 3, ..., m^\star n$ by

$$c_k^{-1} = \sum_{j=1}^{m^\star n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2}. \tag{6}$$

Furthermore, for $z \in T$

$$\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)} = \sum_{k=1}^{m^\star n} c_k \left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2 \tag{7}$$

and also

$$|\mathcal{B}'(z)| = \frac{z\mathcal{B}'(z)}{\mathcal{B}(z)} = \sum_{k=1}^{m^\star n} \frac{|a_k|^2 - 1}{|z - a_k|^2}, \tag{8}$$

where $t_k, k = 1, 2, 3, ..., m^\star n$ are defined in Lemma 2.1.

**Proof .** Let $q(z) = w^*(f(z)) - \lambda(w(f(z)))$. Since the solution of $\mathcal{B}(z) = \lambda$ is same as polynomial equation $w^*(f(z)) - \lambda w(f(z)) = 0$, which has degree exactly $m^\star n$, it follows that it has exactly $m^\star n$ roots counting multiplicities. If these roots are denoted by $t_1, t_2, ..., t_{m^\star n}$, then for some $K \neq 0$

$$q(z) = w(f(z))[\mathcal{B}(z) - \lambda] = K\prod_{k=1}^{m^\star n}(z - t_k).$$

For $rof = \frac{pof}{wof} \in \mathbf{R}_{\mathbf{m^\star n}}$, let $p(f(z)) = \mu z^{m^\star n} + ....$
Then

$$p(f(z)) - \frac{\mu}{K}q(z) \in \mathbf{P}_{m^\star n-1}.$$

The numbers $t_1, t_2, ..., t_{m^\star n}$ are distinct, so by Lagranges interpolation formula, we obtain

$$p(f(z)) - \frac{\mu}{K}q(z) = \sum_{k=1}^{m^\star n} \frac{p(f(t_k))q(z)}{q'(t_k)(z - t_k)}.$$

Dividing both sides by $q(z)$ and differentiating, we get

$$\left(\frac{p(f(z))}{q(z)}\right)' = -\sum_{k=1}^{m^\star n} \frac{q'(t_k)(z-t_k)(p(f(t_k)))' - p(f(t_k))q'(t_k)}{(q'(t_k)(z-t_k))^2}$$

$$= -\sum_{k=1}^{m^\star n} \frac{p(f(t_k))}{q'(t_k)(z-t_k)^2}. \tag{9}$$

Since $p(f(z)) = w(f(z))r(f(z))$ , $q(z) = w(f(z))[B(z) - \lambda]$ and $t_k$ are the roots of $\mathcal{B}(z) = \lambda$. Therefore $p(f(t_k)) = w(f(t_k))r(f(t_k))$ and $q'(t_k) = w(f(t_k))\mathcal{B}'(t_k)$. Moreover, $q(t_k) = 0$.
Using these in equation (9), we get

$$\left(\frac{r(f(z))}{\mathcal{B}(z)-\lambda}\right)' = -\sum_{k=1}^{m^\star n} \frac{r(f(t_k))}{\mathcal{B}'(t_k)(z-t_k)^2}. \tag{10}$$

This implies

$$\frac{[\mathcal{B}(z)-\lambda](r(f(z)))'-r(f(z))\mathcal{B}'(z)}{[\mathcal{B}(z)-\lambda]^2}=-\sum_{k=1}^{m^\star n}\frac{r(f(t_k))}{\mathcal{B}'(t_k)(z-t_k)^2}.$$

Multiplying both sides by $-[\mathcal{B}(z)-\lambda]^2$, we get

$$r(f(z))\mathcal{B}'(z)-f'(z)r'(f(z))[\mathcal{B}(z)-\lambda]=\sum_{k=1}^{m^\star n}\frac{r(f(t_k))[\mathcal{B}(z)-\lambda]^2}{\mathcal{B}'(t_k)(z-t_k)^2}. \tag{11}$$

For $z\in T$, $|\mathcal{B}(z)|=1$ and $|\lambda|=1$. Therefore by virtue of Lemma 2.2, we obtain $[\mathcal{B}(z)-\lambda]^2=-\mathcal{B}(z)\lambda|\mathcal{B}(z)-\lambda|^2$. Similarly, $(z-t_k)^2=-zt_k|z-t_k|^2$. Therefore it follows from equation (11)

$$\mathcal{B}'(z)r(f(z))-f'(z)r'(f(z))[\mathcal{B}(z)-\lambda]=\frac{\mathcal{B}(z)}{z}\sum_{k=1}^{m^\star n}\frac{\lambda r(f(t_k))}{t_k\mathcal{B}'(t_k)}\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2. \tag{12}$$

Using Lemma 2.1 and definition of $c_k$, we get

$$\mathcal{B}'(z)r(f(z))-f'(z)r'(f(z))[\mathcal{B}(z)-\lambda]=\frac{\mathcal{B}(z)}{z}\sum_{k=1}^{m^\star n}c_k r(f(t_k))\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2. \tag{13}$$

This proves equation (5) completely.

Identity (7) follows from (5) by choosing $r(f(z))=1$.

Now, it remains to prove (8). Since

$$\mathcal{B}(z)=\frac{w^*(f(z))}{w(f(z))}=\prod_{j=1}^{m^\star n}\frac{1-\bar{a}_j z}{z-a_j}.$$

This implies

$$\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}=\sum_{j=1}^{m^\star n}\left\{\frac{-z\bar{a}_j}{1-\bar{a}_j z}-\frac{z}{z-a_j}\right\}.$$

This gives for $z\in T$

$$\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}=\sum_{j=1}^{m^\star n}\frac{|a_j|^2-1}{|z-a_j|^2}.$$

Since $|a_j|>1$, for all $1\le j\le m^\star n$ and $\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}$ is real and positive . Hence for $z\in T$, we have

$$\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}=\left|\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}\right|=|\mathcal{B}'(z)|.$$

This completes proof of the Lemma 2.3.

Next Lemma is due to Aziz and Dawood [2].

**Lemma 2.4.** Let $p(z)$ be a polynomial of degree $n$, having all zeros in $T\cup D^-$, then

$$m(p',1)\ge nm(p,1).$$

**Lemma 2.5.** Let $r(f(z)) \in R_{m^\star n}$, then for $z \in T$

$$|r'(f(z))| + |r^{*\prime}(f(z))| \leq \frac{|\mathcal{B}'(z)|}{m^\star m(f,1)} \sup_{z\in T} |r(f(z))|, \tag{14}$$

where $r^*(f(z)) = \mathcal{B}(z)\overline{r(f(\frac{1}{\bar{z}}))}$.
The result is sharp and equality holds for $r(f(z)) = a\mathcal{B}(z)$ with $a \in T$ where $f(z) = z^{m^\star}$.

**Proof** . By equations (5) and (7), we have for $z \in T$

$$\begin{aligned}
|\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) - \lambda]| &= \left|\frac{\mathcal{B}(z)}{z}\sum_{k=1}^{m^\star n} c_k r(f(t_k))\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2\right| \\
&\leq \left|\frac{\mathcal{B}(z)}{z}\right|\sum_{k=1}^{m^\star n}|r(f(t_k))|\left|c_k\right|\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2\Bigg| \\
&\leq \max_{z\in T}|r(f(z))|\left|\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}\right| \\
&= |\mathcal{B}'(z)|\sup_{z\in T}|r(f(z))|.
\end{aligned}$$

Choosing argument of $\lambda$ suitably, we get

$$|\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))\mathcal{B}(z)| + |f'(z)r'(f(z))| \leq |\mathcal{B}'(z)|\sup_{z\in T}|r(f(z))|. \tag{15}$$

Since

$$r^*(f(z)) = \mathcal{B}(z)\overline{r(f(\frac{1}{\bar{z}}))}.$$

So that

$$(r^*(f(z)))' = \mathcal{B}'(z)\overline{r(f(\frac{1}{\bar{z}}))} - \frac{1}{z^2}\mathcal{B}(z)\overline{r'(f(\frac{1}{\bar{z}}))f'(\frac{1}{\bar{z}})}.$$

This implies

$$\left|(r^*(f(z)))'\right| = \left|\mathcal{B}'(z)\overline{r(f(\frac{1}{\bar{z}}))} - \frac{1}{z^2}\mathcal{B}(z)\overline{r'(f(\frac{1}{\bar{z}}))f'(\frac{1}{\bar{z}})}\right|.$$

Therefore for $z \in T$, we have

$$\left|(r^*(f(z)))'\right| = \left|z\frac{\mathcal{B}'(z)}{\mathcal{B}(z)}\overline{r(f(z))} - \overline{zr'(f(z))f'(z)}\right|.$$

Also using the fact that $\frac{zB'(z)}{\mathcal{B}(z)}$ is real, we obtain

$$\begin{aligned}
\left|(r^*(f(z)))'\right| &= \left|z\frac{\mathcal{B}'(z)}{\mathcal{B}(z)}r(f(z)) - zr'(f(z))f'(z)\right| \\
&= |\mathcal{B}'(z)r(f(z)) - r'(f(z))f'(z)\mathcal{B}(z)|.
\end{aligned} \tag{16}$$

Hence we have from inequality (15)

$$|r^{*\prime}(f(z))f'(z)| + |r'(f(z))f'(z)| \leq |\mathcal{B}'(z)|\sup_{z\in T}|r(f(z))|.$$

This gives by use of Lemma 2.4

$$|r'(f(z))| + |r^{*\prime}(f(z))| \leq \frac{|\mathcal{B}'(z)|}{m^\star m(f,1)} \sup_{z\in T} |r(f(z))|,$$

where $r^*(f(z)) = \mathcal{B}(z)\overline{r(f(\frac{1}{\bar{z}}))}$.

**Lemma 2.6.** Suppose $t_1, t_2, ..., t_{m^\star n}$ are the zeros of $\mathcal{B}(z) - \lambda$ and $s_1, s_2, ..., s_{m^\star n}$ are zeros of $\mathcal{B}(z) + \lambda$, where $\lambda \in T$, then for $rof \in \mathcal{R}_{\Updownarrow^\star\backslash}$ and $z \in T$

$$|r'(f(z))|^2 + |r'^*(f(z))|^2 \leq \frac{|\mathcal{B}'(z)|^2}{2(m^\star m(f,1))^2}[(\max_{1\leq k\leq m^\star n} |r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n} |r(f(s_k))|)^2].$$

The inequality is sharp and equality holds for $r(f(z)) = u\mathcal{B}(z)$ with $u \in T$.

**Proof .** Since $t_1, t_2, ..., t_{m^\star n}$ are the zeros of $\mathcal{B}(z) - \lambda$, $\lambda \in T$, therefore by Lemma 2.3, we obtain for $z \in T$

$$\begin{aligned}|\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) - \lambda]| &= |\frac{\mathcal{B}(z)}{z}\sum_{k=1}^{m^\star n} c_k r(f(t_k))\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2| \\ &\leq \sum_{k=1}^{m^\star n} |c_k||r(f(t_k))|\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2 \\ &\leq \sum_{k=1}^{m^\star n} c_k\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2 \max_{1\leq k\leq m^\star n}|r(f(t_k))|. \end{aligned} \tag{17}$$

Since for $z \in T, |\mathcal{B}(z)| = 1$ and $c_k \geq 0,\ k = 1, 2, ..., m^\star n$. Inequality (17) in conjunction with (7) and (8) gives

$$\begin{aligned}|\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) - \lambda]| &\leq \sum_{k=1}^{m^\star n} c_k\left|\frac{\mathcal{B}(z)-\lambda}{z-t_k}\right|^2 \max_{1\leq k\leq m^\star n}|r(f(t_k))| \\ &= |\mathcal{B}'(z)| \max_{1\leq k\leq m^\star n}|r(f(t_k))|. \end{aligned} \tag{18}$$

Replace $\lambda$ by $-\lambda$ and noting that $s_1, s_2, ..., s_{m^\star n}$ are the zeros of $\mathcal{B}(z) + \lambda$, we get from (18)

$$|\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) + \lambda]| \leq |\mathcal{B}'(z)| \max_{1\leq k\leq m^\star n}|r(f(s_k))|. \tag{19}$$

From (18) and (19), it follows that

$$\begin{aligned}&|\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) - \lambda]|^2 + |\mathcal{B}'(z)r(f(z)) - f'(z)r'(f(z))[\mathcal{B}(z) + \lambda]|^2 \\ &\qquad \leq |\mathcal{B}'(z)|^2\Big[(\max_{1\leq k\leq m^\star n}|r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n}|r(f(s_k))|)^2\Big]. \end{aligned} \tag{20}$$

Using Lemma 2.4 and the identity

$$|c + d|^2 + |c - d|^2 = 2|c|^2 + 2|d|^2$$

in (20), with $c = \mathcal{B}'(z)r(f(z)) - f'(z)\mathcal{B}(z)r'(f(z))$ and $d = \lambda f'(z)r'(f(z))$, we get for $z \in T$

$$2[|r'^*(f(z))|^2 + |r'(f(z))|^2] \leq \frac{|\mathcal{B}'(z)|^2}{(m^\star m(f,1))^2}[(\max_{1\leq k\leq m^\star n}|r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n}|r(f(s_k))|)^2].$$

This gives

$$|r'(f(z))|^2 + |r'^*(f(z))|^2 \leq \frac{|\mathcal{B}'(z)|^2}{2(m^\star m(f,1))^2}[(\max_{1\leq k\leq m^\star n}|r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n}|r(f(s_k))|)^2].$$

This completes the proof of the Lemma 2.6.

**Lemma 2.7**. Let $rof \in \mathbf{R}_{\mathbf{m^\star n}}$. If all zeros of $r(f(z))$ lie in $T \cup D^+$, then for $z \in T$ and $r(f(z)) \neq 0$

$$Re\left(\frac{z(r(f(z)))'}{r(f(z))}\right) \leq \frac{1}{2}|\mathcal{B}'(z)|. \tag{21}$$

**Proof**. If $p(z)$ has $n'$ zeros and $f(z)$ has $m^\star$ zeros, then $p(f(z))$ has $m^\star n'$ zeros. Let $b_1, b_2, ..., b_{m^\star n'}$ be the zeros of $p(f(z))$, $m^\star n' \leq m^\star n$.
Now

$$r(f(z)) = \frac{p(f(z))}{w(f(z))}.$$

This gives

$$z\frac{(r(f(z)))'}{r(f(z))} = \sum_{j=1}^{m^\star n'} \frac{z}{z-b_j} - \sum_{j=1}^{m^\star n} \frac{z}{z-a_j}. \tag{22}$$

Since all zeros of $p(f(z))$ lie in $T \cup D^+$, therefore for $z \in T$ with $z \neq b_j$, $1 \leq j \leq m^\star n'$. we have

$$\left|\frac{z}{z-b_j}\right| \leq \left|\frac{z}{z-b_j} - 1\right|, \ \ for\ j = 1, 2, 3..., m^\star n'. \tag{23}$$

Using the fact that $Re(z) \leq \frac{1}{2}$ if and only if $|z| \leq |z-1|$, we get

$$Re\left(\frac{z}{z-b_j}\right) \leq \frac{1}{2}, \ \ for\ \ j = 1, 2, ..., m^\star n'.$$

Therefore (22) yields

$$\begin{aligned} Re\left(z\frac{(r(f(z)))'}{r(f(z))}\right) &\leq \sum_{j=1}^{m^\star n'} \frac{1}{2} - \sum_{j=1}^{m^\star n} Re\left(\frac{z}{z-a_j}\right) \\ &\leq \sum_{j=1}^{m^\star n} Re\left(\frac{1}{2} - \frac{z}{z-a_j}\right) \\ &= \sum_{j=1}^{m^\star n} \frac{|a_j|^2-1}{2|z-a_j|^2}. \end{aligned}$$

This in conjunction with equation (8) gives

$$Re\left(z\frac{(r(f(z)))'}{r(f(z))}\right) \leq \frac{1}{2}|\mathcal{B}'(z)|.$$

This completes the proof of Lemma 2.7.

**Remark 2.8.** Setting $f(z) = z$, Lemma 2.7 yields result of Li, Mohapatra and Rodriguez ([9], Lemma 4).

We now prove the following results.

**Theorem 2.9.** Let $rof \in \mathbf{R_{m^\star n}}$ and all zeros of $r(f(z))$ lie in $T \cup D^+$. If $t_1, t_2, ..., t_{m^\star n}$ are the zeros of $\mathcal{B}(z) + \lambda$ and $s_1, s_2, ..., s_{m^\star n}$ are zeros of $\mathcal{B}(z) - \lambda$, where $\lambda \in T$, then for $z \in T$

$$|r'(f(z))| \leq \frac{|\mathcal{B}'(z)|}{2m^\star m(f,1)}[(\max_{1\leq k\leq m^\star n}|r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n}|r(f(s_k))|)^2]^{\frac{1}{2}},$$

where $m(f, 1) = \inf_{z\in T}|f(z)|$.

The inequality is sharp and equality holds for $r(f(z)) = \lambda + \mathcal{B}(z)$ with $\lambda \in T$.

**Proof.** By hypothesis all the zeros of $r(f(z))$ lie in $T \cup D^+$, therefore by Lemma 2.7, we have, for the points on $T$, which are not the zeros of $r(f(z))$

$$Re\Big(\frac{z(r(f(z)))'}{r(f(z))}\Big) \leq \frac{1}{2}|\mathcal{B}'(z)|. \tag{24}$$

Now by virtue of (8), $|\mathcal{B}'(z)| > 0$, so that (24) can be written as

$$Re\Big(\frac{z(r(f(z)))'}{|\mathcal{B}'(z)|r(f(z))}\Big) \leq \frac{1}{2}.$$

This implies, for $z \in T$, which are not the zeros of $r(f(z))$

$$\Big|\frac{z(r(f(z)))'}{|\mathcal{B}'(z)|r(f(z))}\Big| \leq |\frac{z(r(f(z)))'}{|\mathcal{B}'(z)|r(f(z))} - 1|.$$

Equivalently

$$|z(r(f(z)))'| \leq |z(r(f(z)))' - |\mathcal{B}'(z)|r(f(z))|. \tag{25}$$

(25) is trivially true for points $z \in T$ which are zeros of $r(f(z))$. Therefore, it follows that for $z \in T$

$$|(r(f(z)))'| \leq |z(r(f(z)))' - |\mathcal{B}'(z)|r(f(z))|. \tag{26}$$

Since for $z \in T$, $|\mathcal{B}(z)| = 1$ and by (8), we have

$$|\mathcal{B}'(z)| = \frac{z\mathcal{B}'(z)}{\mathcal{B}(z)},$$

therefore from (26), we get for $z \in T$

$$|f'(z)r'(f(z))| \leq |\mathcal{B}(z)(r(f(z)))' - \mathcal{B}'(z)r(f(z))| = |f'(z)r'^*(f(z))|. \tag{27}$$

Inequality (27) with the help of Lemma 2.6, for $z \in T$ implies

$$\begin{aligned} 2|r'(f(z))|^2 &\leq |r'(f(z))|^2 + |r'^*(f(z))|^2 \\ &\leq \frac{|\mathcal{B}'(z)|^2}{2(m^\star m(f,1))^2}[(\max_{1\leq k\leq m^\star n}|r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n}|r(f(s_k))|)^2]. \end{aligned}$$

Equivalently

$$|r'(f(z))| \leq \frac{|\mathcal{B}'(z)|}{2m^\star m(f,1)}[(\max_{1\leq k\leq m^\star n}|r(f(t_k))|)^2 + (\max_{1\leq k\leq m^\star n}|r(f(s_k))|)^2]^{\frac{1}{2}}.$$

This proves the Theorem 2.9 completely.

**Remark 2.10.** Taking $f(z) = z$ and noting that $m^\star = 1, m(f, 1) = \inf_{z\in T}|f(z)| = 1$, the result of Aziz and Shah [3] follows immediately.

Next, we prove the following more general result, which generalizes some of the polynomial inequalities earlier proved by Aziz and Dawood [2].

**Theorem 2.11.** Let $rof \in \mathbf{R_{m^\star n}}$ and all zeros of $r(f(z))$ lie in $T \cup D^+$. Then for $z \in T$

$$|r'(f(z))| \leq \frac{|\mathcal{B}'(z)|}{2m^\star m(f,1)}[M(rof,1) - m(rof,1)].$$

**Proof.** By hypothesis $r(f(z))$ does not vanish in $D^-$. Suppose first that $r(f(z))$ does not vanish for $z \in T$, so that all zeros of $r(f(z))$ lie in $D^+$ and

$$\inf_{z\in T}|r(f(z))| = m(r(f),1) > 0.$$

Let $\alpha \in \mathbf{C}$ (field of complex numbers) such that $|\alpha| < 1$, then for $z \in T$

$$|\alpha m(rof,1)\mathcal{B}(z)| < |r(f(z))|.$$

Therefore by Rouche's theorem, it follows that

$$R(z) = \alpha m(rof,1)\mathcal{B}(z) + r(f(z))$$

does not vanish in $D^-$. This is true even if $m(r(f),1) = 0$. Hence in any case all the zeros of $R(z)$ lie in $T \cup D^+$. So that by Lemma 2.7, we have for $z \in T$, such that $R(z) \neq 0$

$$Re\left(\frac{zR'(z)}{|\mathcal{B}'(z)|R(z)}\right) \leq \frac{1}{2}.$$

Hence as in proof of Theorem 2.9, it follows that for $z \in T$

$$|R'(z)| \leq |\mathcal{B}(z)R'(z) - \mathcal{B}'(z)R(z)| = |(R^*(z))'|, \tag{28}$$

where

$$\begin{aligned} R^*(z) &= \mathcal{B}(z)\overline{R(1/\bar{z})} \\ &= \mathcal{B}(z)\overline{r(f(1/\bar{z}))} + \bar{\alpha}m(rof,1) \\ &= r^*(f(z)) + \bar{\alpha}m(rof,1). \end{aligned} \tag{29}$$

Therefore (28) implies for $z \in T$

$$|f'(z)r'(f(z)) + \alpha m(rof,1)\mathcal{B}'(z)| \leq |f'(z)r'^{*}(f(z))|. \tag{30}$$

Choosing argument of $\alpha$ suitably , we get

$$|f'(z)r'(f(z))| + |\alpha|m(rof,1)|\mathcal{B}'(z)| \leq |f'(z)r'^{*}(f(z))|.$$

Letting $|\alpha| \to 1$, we get for $z \in T$

$$|f'(z)r'(f(z))| \leq |f'(z)||r'^{*}(f(z))| - m(rof,1)|\mathcal{B}'(z)|. \tag{31}$$

Also by Lemma 2.3, we have for $z, \lambda \in T$

$$|\mathcal{B}'(z)r(f(z)) - \mathcal{B}(z)f'(z)r'(f(z)) + \lambda f'(z)r'(f(z))| \leq |\mathcal{B}'(z)|M(rof,1).$$

Choosing argument of $\lambda$ suitably, we get

$$|\mathcal{B}'(z)r(f(z)) - \mathcal{B}(z)f'(z)r'(f(z))| + |\lambda||f'(z)r'(f(z))| \leq |\mathcal{B}'(z)|M(rof,1).$$

Letting $|\lambda| \to 1$, we get for $z \in T$

$$|\mathcal{B}'(z)r(f(z)) - \mathcal{B}(z)f'(z)r'(f(z))| + |f'(z)r'(f(z))| \leq |\mathcal{B}'(z)|M(rof, 1).$$

This gives for $z \in T$

$$[|r'^{*}(f(z))| + |r'(f(z))|]|f'(z)| \leq |\mathcal{B}'(z)|M(rof, 1).$$

Now by using Lemma 2.4, we get for $z \in T$

$$|r'^{*}(f(z))| + |r'(f(z))| \leq \frac{|\mathcal{B}'(z)|}{m^{\star}m(f,1)} \sup_{z\in T} |r(f(z))|. \tag{32}$$

From (31), we have for $z \in T$

$$2|r'(f(z))||f'(z)| \leq |f'(z)||r'^{*}(f(z))| + |f'(z)||r'(f(z))| - m(rof, 1)|\mathcal{B}'(z)|. \tag{33}$$

Using (32) in (33), we get for $z \in T$

$$|r'(f(z))| \leq \frac{|\mathcal{B}'(z)|}{2m^{\star}m(f,1)}[M(rof, 1) - m(rof, 1)].$$

This proves the Theorem 2.11 completely.

**Remark 2.12.** Choosing $f(z) = z$, Theorem 2.11 reduces to the result earlier proved by Aziz and Shah [3].

If $r(f(z))$ has no poles, we obtain the following result.

**Corollary 2.13.** Let $pof \in \mathbf{P}_{m^{\star}n}$ and all zeros of $p(f(z))$ lie in $T \cup D^{+}$. Then for $z \in T$

$$|p'(f(z))| \leq \frac{n}{2m^{\star}m(f,1)}[M(pof, 1) - m(pof, 1)].$$

**Remark 2.14.** By Taking $f(z) = z$ in Corollary 2.13, we get result of Aziz and Dawood [2].

**Lemma 2.15.** Let $rof \in \mathbf{R_{m^{\star}n}}$. If all zeros of $r(f(z))$ lie in $T \cup D^{-}$, then for $z \in T$ and $r(f(z)) \neq 0$, we have

$$Re\left(\frac{z(r(f(z)))'}{r(f(z))}\right) \geq \frac{1}{2}|\mathcal{B}'(z)|. \tag{34}$$

**Proof.** Let $m^{\star}n'$ and $m^{\star}n$ be respectively the number of zeros and poles of $r(f(z))$. By hypothesis all zeros of $r(f(z))$ lie in $T \cup D^{-}$ and $z \in T$ with $z \neq b_j$ , $1 \leq j \leq m^{\star}n'$.
Then as in Lemma 2.7, we obtain

$$Re\left(\frac{z}{z-b_j}\right) \geq \frac{1}{2} \ \ for \ \ j = 1, 2, ..., m^{\star}n'.$$

Using equation (22), we get

$$\begin{aligned}
Re\left(z\frac{(r(f(z)))'}{r(f(z))}\right) &\geq \sum_{j=1}^{m^{\star}n'} \frac{1}{2} - \sum_{j=1}^{m^{\star}n} Re\left(\frac{z}{z-a_j}\right) \\
&= \sum_{j=1}^{m^{\star}n} Re\left(\frac{1}{2} - \frac{z}{z-a_j}\right) - \frac{(m^{\star}n - m^{\star}n')}{2} \\
&= \sum_{j=1}^{m^{\star}n} \frac{|a_j|^2 - 1}{2|z-a_j|^2} - \frac{m^{\star}}{2}(n-n')
\end{aligned}$$

$$= \frac{1}{2}\Big\{|\mathcal{B}'(z)| - m^\star(n-n')\Big\}.$$

This in particular gives

$$Re\Big(\frac{z(r(f(z)))'}{r(f(z))}\Big) \geq \frac{1}{2}|\mathcal{B}'(z)|.$$

This completes the proof of Lemma 2.15.

**Remark 2.16.** Lemma 2.15 gives a result of Li, Mohapatra and Rodriguez ([9], Lemma 4), after setting $f(z) = z$.

**Theorem 2.17.** Let $rof \in \mathbf{R_{m^\star n}}$ and all zeros of $r(f(z))$ lie in $T \cup D^-$. Then for $z \in T$

$$|r\prime(f(z))| \geq \frac{|\mathcal{B}'(z)|}{2m^\star M(f,1)}[M(rof,1) + m(rof,1)],$$

where $M(f,1) = \sup_{z\in T} |f(z)|$.

**Proof.** By hypothesis $r(f(z))$ has exactly $m^\star n$ zeros in $T \cup D^-$. First suppose $r(f(z))$ has no zero on $T$, then $m(rof,1) > 0$. Let $\alpha \in \mathbf{C}$ such that $|\alpha| < 1$ and $z \in T$ , then we have

$$|\alpha m(rof,1)| < |r(f(z))|.$$

Therefore by Rouche's theorem, it follows that

$$R(z) = \alpha m(rof,1) + r(f(z))$$

has $m^\star n$ zeros in $D^-$. In case $R(z)$ has a zero on $T$, then $m(rof,1) = 0$ and in this case

$$R(z) = r(f(z))$$

has all zeros in $T \cup D^-$. Applying Lemma 2.15 to $R(z)$, with $m^\star n = m^\star n'$, we get for $z \in T$ and $R(z)$ does not vanish on $T$

$$Re\Big(\frac{zR'(z)}{R(z)}\Big) \geq \frac{|\mathcal{B}'(z)|}{2}.$$

This gives for $z \in T$

$$|R'(z)| \geq \frac{|\mathcal{B}'(z)||R(z)|}{2}.$$

Equivalently, for $z \in T$

$$|r'(f(z))||f'(z)| \geq \frac{|\mathcal{B}'(z)||\alpha m(rof,1) + r(f(z))|}{2}. \tag{35}$$

(35) in conjunction with (1), yields

$$|r'(f(z))| \geq \frac{|\mathcal{B}'(z)||\alpha m(rof,1) + r(f(z))|}{2m^\star M(f,1)}. \tag{36}$$

Choosing argument of $\alpha$ suitably on the right hand side of (36), we get

$$|r'(f(z))| \geq \frac{|\mathcal{B}'(z)|[|\alpha|m(rof,1) + |r(f(z))|]}{2m^\star M(f,1)}.$$

Letting $|\alpha| \to 1$

$$|r'(f(z))| \geq \frac{|\mathcal{B}'(z)|[m(rof,1) + |r(f(z))|]}{2m^\star M(f,1)}.$$

This proves the Theorem 2.17 completely.

Setting $f(z) = z$, Theorem 2.17 yields the following result of Aziz and Shah [3].

**Corollary 2.18.** Let $r \in \mathbf{R_n}$ and all zeros of $r(z)$ lie in $T \cup D^-$. Then for $z \in T$

$$|r'(z)| \geq \frac{|B'(z)|}{2}[M(r,1) + m(r,1)].$$

**Remark 2.19.** Suppose $r(z)$ has no poles, then we get result of Aziz and Dawood[2].

**Lemma 2.20.** Suppose $0 \leq \alpha < 2\pi$, then the following hold: (i) $\mathcal{B}(z) + ke^{i\alpha}$ has all its zeros in $T \cup D^+$, for all $k \geq 1$. (ii) $\mathcal{B}(z) + ke^{i\alpha}$ has all its zeros in $T \cup D^-$, for all $k \leq 1$.

**Proof.** We have

$$\mathcal{B}(z) = \frac{w^*(p(z))}{w(p(z))} = \prod_{j=1}^{m^\star n}\left(\frac{1 - \bar{a}_j z}{z - \bar{a}_j}\right), \quad |a_j| > 1, j = 1, 2, ..., m^\star n.$$

By assumption all the zeros of $w(z)$ lie in $D^-$, therefore, the function $\frac{w^*(p(z))}{w(p(z))}$ is analytic in $T \cup D^-$. Moreover for $z \in T$

$$|\frac{w^*(p(z))}{w(p(z))}| = 1. \tag{37}$$

Since $\frac{w^*(p(z))}{w(p(z))}$ is not a constant, therefore, it follows from (37) by maximum modulus principle that for $z \in D^-$

$$|\mathcal{B}(z)| = |\frac{w^*(p(z))}{w(p(z))}| < 1. \tag{38}$$

Replacing $z$ by $\frac{1}{z}$, we obtain for $z \in D^+$

$$|\frac{w^*(p(z))}{w(p(z))}| = |z^n \frac{\overline{w^*(p(\frac{1}{\bar{z}}))}}{z^n \overline{w(p(\frac{1}{\bar{z}}))}}| < 1. \tag{39}$$

From (38) and (39), it follows that $W = \mathcal{B}(z)$ maps $|z| < 1$ into $|W| < 1$ and $|z| > 1$ (such that $\mathcal{B}(z)$ is finite) into $|W| > 1$. If $z_0$ is a zero of $\mathcal{B}(z) - ke^{i\alpha}$, then $|\mathcal{B}(z_0)| = |k|$. Thus $|k| \leq 1$ implies $|z_0| \leq 1$ and $|k| \geq 1$ implies $|z_0| \geq 1$. This proves the Lemma completely.

Next Lemma is due to Li[8].

**Lemma 2.21.** Let A and B be any two complex numbers, then (i) If $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers $\delta$ satisfying $|\delta| < 1$. (ii) Conversely, if $A \neq \delta B$ for all complex numbers $\delta$ satisfying $|\delta| < 1$, then $|A| \geq |B|$.

**Theorem 2.22.** Let $rof, sof \in \mathbf{R_{m^\star n}}$ and assume $s(f(z))$ has all its zeros in $T \cup D^-$ and for $z \in T$

$$|r(f(z))| \leq |s(f(z))|.$$

Then for any $\rho \in \mathbf{C}$ with $|\rho| \leq \frac{1}{2}$ and for $z \in T$

$$|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)r(f(z))| \leq |f'(z)s'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)s(f(z))|,$$

where $M(f,1) = \sup_{z\in T}|f(z)|$.
This result is sharp and equality holds for $r(f(z)) = s(f(z))e^{i\gamma}$, $0 \le \gamma < 2\pi$.
**Proof.** First of all we suppose that no zero of $s(f(z))$ are on the unit circle and therefore all zeros of $s(f(z))$ are in unit disk.

Let $\alpha \in \mathbf{C}$ be such that $|\alpha| < 1$. By hypothesis $|r(f(z))| \le |s(f(z))|$ for $z \in T$. Therefore for $z \in T$ and $|\alpha| < 1$

$$|\alpha r(f(z))| < |s(f(z))|.$$

But $s(f(z))$ has $m^\star n$ zeros in $D^-$. By Rouche's theorem, $\alpha r(f(z)) + s(f(z))$ has the same number of zeros in $D^-$ as $s(f(z))$. Thus, $\alpha r(f(z)) + s(f(z))$ also has $m^\star n$ zeros in $D^-$. By special case of Lemma 2.15, for $z \in T$, we get

$$|\alpha f'(z)r'(f(z)) + f'(z)s'(f(z))| \ge \frac{1}{2m^\star M(f,1)}|\mathcal{B}'(z)||\alpha r(f(z)) + s(f(z))|.$$

Now, note that $\mathcal{B}'(z) \neq 0$. So, the right hand side is non zero. Thus, by using (i) of Lemma 2.21, we have, for all $\beta$ satisfying $|\beta| < 1$ and for $z \in T$,

$$\alpha f'(z)r'(f(z)) + f'(z)s'(f(z)) \neq \frac{1}{2m^\star M(f,1)}\beta\mathcal{B}'(z)\Big[\alpha r(f(z)) + s(f(z))\Big].$$

Equivalently, for $z \in T$, $|\alpha| < 1, |\beta| < 1$,

$$\alpha\Big[f'(z)r'(f(z)) - \frac{\beta}{2m^\star M(f,1)}\mathcal{B}'(z)r(f(z))\Big] \neq -\Big[f'(z)s'(f(z)) - \frac{\beta}{2m^\star M(f,1)}\mathcal{B}'(z)s(f(z))\Big].$$

Taking $\rho := -\frac{\beta}{2}$ with $|\rho| < \frac{1}{2}$, yields

$$|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)r(f(z))| \le |f'(z)s'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)s(f(z))|.$$

Finally using continuity in zeros and $\rho$, we can obtain the inequality when some zeros of $s(f(z))$ lie on the unit circle and for $|\rho| \le \frac{1}{2}$.

**Remark 2.23.** By taking $f(z) = z$, we get result of Li[8].
Letting $\rho \to 0$ in Theorem 2.22, we obtain the following result of Li ([8] Theorem 3.1)

**Corollary 2.24.** Suppose $rof, sof \in \mathbf{R_{m^\star n}}$ and assume $s(f(z))$ has all its zeros in $T \cup D^-$ such that for $z \in T$

$$|r(f(z))| \le |s(f(z))|.$$

Then for any $z \in T$

$$|r'(f(z)) \le |s'(f(z))|.$$

If $s(f(z))$ and $r(f(z))$ has no poles, then from Theorem 2.22, we obtain the following:

**Corollary 2.25.** If $pof, qof \in \mathbf{P}_{m^\star n}$ and assume $q(f(z))$ has all the zeros in $T \cup D^-$ such that for $z \in T$

$$|p(f(z))| \le |q(f(z))|.$$

Then for any $\rho \in \mathbf{C}$ with $|\rho| \le \frac{1}{2}$ and for $z \in T$

$$|f'(z)p'(f(z)) + \frac{\rho n}{M(f,1)}p(f(z))| \le |f'(z)q'(f(z)) + \frac{\rho n}{M(f,1)}q(f(z))|,$$

where $M(f,1) = \sup_{z\in T}|f(z)|$.

Choosing $f(z) = z$, in Corollary 2.25 we get:

**Corollary 2.26.** If $p, q \in \mathbf{P}_n$ and assume $q(z)$ has all the zeros in $T \cup D^-$ such that for $z \in T$

$$|p(z)| \leq |q(z)|.$$

Then for any $\rho \in \mathbf{C}$ with $|\rho| \leq \frac{1}{2}$ and for $z \in T$

$$|p'(z) + \rho np(z)| \leq |q'(z) + \rho nq(z)|.$$

Letting $\rho \to 0$ in Corollary 2.26, we get the following inequality of Bernstein [4].

**Corollary 2.27.** Let $p, q \in \mathbf{P}_n$ and assume $q(z)$ has all the zeros in $T \cup D^-$ such that for $z \in T$

$$|p(z)| \leq |q(z)|.$$

Then for any for $z \in T$

$$|p'(z)| \leq |q'(z)|.$$

**Theorem 2.28.** Suppose $rof \in \mathbf{R_{m^\star n}}$ and $rof$ does not vanish in $D^- \cup T$, then for $z \in T$ and $|\rho| \leq \frac{1}{2}$

$$\begin{aligned} &|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)r(f(z))| \\ &\quad \leq |f'(z)r'^{*}(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)r^*(f(z))| - m(rof,1)|\mathcal{B}'(z)|\{|1 + \frac{[\rho\mathcal{B}(z) - |\rho|]}{m^\star M(f,1)}|\} \end{aligned} \tag{40}$$

and

$$\begin{aligned} &|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}'(z)r(f(z))| \\ &\quad \leq \frac{1}{2}|\mathcal{B}'(z)|\Big[(1 + \frac{2}{m^\star M(f,1)}|\rho|)M(rof,1) - (1 - \frac{2}{m^\star M(f,1)}|\rho|)m(rof,1)\Big], \end{aligned} \tag{41}$$

where $r^*(f(z)) = \mathcal{B}(z)\overline{r(f(1/\bar{z}))}$.

**Proof.** Suppose first that $rof \in \mathbf{R_{m^\star n}}$ has all its zeros in $D^+$, then

$$m(rof,1) = \inf_{z\in T}|r(f(z))| > 0.$$

Let $\alpha$ be a complex number with $|\alpha| < 1$, so that for $z \in T$

$$|\alpha m(rof,1)\mathcal{B}(z)| < |r(f(z))|.$$

Therefore by Rouche's theorem, it follows that

$$R(z) = \alpha m(rof,1)\mathcal{B}(z) + r(f(z))$$

has all the zeros in $D^+$.
If

$$\begin{aligned} R^*(z) &= \mathcal{B}(z)\overline{R(1/\bar{z})} \\ &= \mathcal{B}(z)\overline{r(f(1/\bar{z}))} + \bar{\alpha}m(rof,1) \\ &= r^*(f(z)) + \bar{\alpha}m(rof,1), \end{aligned}$$

then all the zeros of $R^*(z)$ lie in $D^-$ and hence applying Theorem 2.22 to $R(z)$, we have for $z \in T$ and $|\rho| \leq \frac{1}{2}$

$$|R^{'}(z) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)R(z)| \leq |(R^*(z))' + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)R^*(z)|. \tag{42}$$

Substitute for $R(z)$, inequality (42) gives

$$|f'(z)r^{'}(f(z)) + \alpha m(rof,1)\mathcal{B}^{'}(z) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z)) + \alpha\frac{\rho}{m^\star M(f,1)}m(rof,1)\mathcal{B}^{'}(z)\mathcal{B}(z)|$$

$$\leq |f'(z)(r\prime^*(f(z))) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r^*(f(z)) + \bar{\alpha}\frac{\rho}{m^\star M(f,1)}m(rof,1)\mathcal{B}^{'}(z)|. \tag{43}$$

Choosing argument of $\alpha$ in left hand side of (43) suitably, we get for $z \in T$

$$|f'(z)r^{'}(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))| + m(rof,1)|\alpha||\mathcal{B}^{'}(z)||1 + \frac{\rho}{m^\star M(f,1)}\mathcal{B}(z)|$$

$$\leq |f'(z)r^*\prime(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r^*(f(z))| + \frac{m(rof,1)}{m^\star M(f,1)}|\alpha||\rho||\mathcal{B}^{'}(z)|.$$

This gives by letting $|\alpha| \to 1$

$$|f'(z)r^{'}(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))|$$

$$\leq |f'(z)r^*\prime(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r^*(f(z))| - m(rof,1)|\mathcal{B}^{'}(z)|\{|1 + \frac{\rho\mathcal{B}(z) - |\rho|}{m^\star M(f,1)}|\},$$

for $z \in T, |\rho| \leq \frac{1}{2}$.

Adding $|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))|$ on both sides, we get for $z \in T$

$$2|f'(z)r^{'}(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))|$$

$$\leq |f'(z)r^*\prime(f(z))| + \frac{|\rho|}{m^\star M(f,1)}|\mathcal{B}^{'}(z)||r^*(f(z))| + |f'(z)r'(f(z))| + \frac{|\rho|}{m^\star M(f,1)}|\mathcal{B}^{'}(z)||r(f(z))|$$

$$- m(rof,1)|\mathcal{B}^{'}(z)|\{|1 + \frac{\rho}{m^\star M(f,1)}\mathcal{B}(z)| - \frac{|\rho|}{m^\star M(f,1)}\}.$$

Equivalently

$$2|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))|$$

$$\leq |f'(z)r^*\prime(f(z))| + |f'(z)r'(f(z))| + |\mathcal{B}^{'}(z)|\Big[\frac{|\rho|}{m^\star M(f,1)}|r^*(f(z))| + \frac{|\rho|}{m^\star M(f,1)}|r(f(z))|$$

$$- m(rof,1)\{|1 + \frac{\rho\mathcal{B}(z)}{m^\star M(f,1)}| - \frac{|\rho|}{m^\star M(f,1)}\}\Big]. \tag{44}$$

Inequality (44) by using Lemma 2.5, yields for $z \in T$

$$2|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))|$$

$$\leq |\mathcal{B}^{'}(z)|\sup_{z\in T}|r(f(z))| + |\mathcal{B}^{'}(z)|\Big[\frac{|\rho|}{m^\star M(f,1)}|r^*(f(z))| + \frac{|\rho|}{m^\star M(f,1)}|r(f(z))|$$

$$- m(rof,1)\{1 - \frac{|\rho|}{m^\star M(f,1)}|\mathcal{B}(z)| - \frac{|\rho|}{m^\star M(f,1)}\}\Big].$$

Using the fact that $|r^*(f(z))| = |r(f(z))|$ for $z \in T$, we get

$$|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))| \leq \frac{|\mathcal{B}^{'}(z)|}{2}\Big[\sup_{z\in T}|r(f(z))| + 2\frac{|\rho|}{m^\star M(f,1)}|r(f(z))|$$

$$- \inf_{z\in T}|r(f(z))|\{1 - 2\frac{|\rho|}{m^\star M(f,1)}\}\Big].$$

In Particular

$$|f'(z)r'(f(z)) + \frac{\rho}{m^\star M(f,1)}\mathcal{B}^{'}(z)r(f(z))| \leq \frac{|\mathcal{B}^{'}(z)|}{2}\Big[(1 + 2\frac{|\rho|}{m^\star M(f,1)})\sup_{z\in T}|rof| - (1 - 2\frac{|\rho|}{m^\star M(f,1)})\inf_{z\in T}|rof|\Big].$$

In case $r(f(z))$ has a zero on $T$, then inequality (40) of Theorem follows from special case of Theorem 2.22 and also inequality (41) follows from Theorem 2.22 by the same technique as used in the proof of the Theorem above.
Hence the Theorem is completely proved.

The case when a zero of $r(f(z))$ lie on $T_1$, can also follow by using argument of continuity.

**Remark 2.29.** If $f(z) = z$, Theorem 2.28 yields result of Irshad, Liman and Shah [6].

**Remark 2.30 .** The inequality (40) above is a generalization of refinement of a result of Li [8, Cor. 3.6 ], where as if we let $|\rho| \to 0$ in (41), we get the generalization of result of Aziz and Shah [3], Theorem 1.3.

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