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Generalized the q-Digamma and the q-Polygamma Functions via Neutrices

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Abstract. The q-digamma function $\psi_q(x)$ and the q-polygamma functions $\psi_q^{(r)}(x),\ r \in \mathbb{N} = \{1, 2, \cdots\}$ are defined for all $x > 0$ and $0 < q < 1$. In this paper, the neutrices and the neutrix limit are used to define the q-digamma function $\psi_q(x)$ and the q-polygamma functions $\psi_q^{(r)}(x),\ r \in \mathbb{N}$ for all $x \in \mathbb{R}$. Moreover, further results are given.

1. Introduction

In the second half of the twentieth century there was a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics, statistics and physics. This kind of calculus recently began to have an effective utility in quantum mechanics, because its intimate connection with the relation of commutativity and Lie algebra. Due to this significance, many of the classical facts (Stirling formula, Raabe's formula, Multiplication theorem,...,etc) about the ordinary gamma function and related functions (q-digamma, q-polygamma and q-beta functions) have been extended to the q-gamma function and related functions. Neutrices are additive groups of negligible functions that do not contain any constants except zero. Their calculus was developed by van der Corput [1] in connection with asymptotic series and divergent integrals. We note that, Jack Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamard integral to the quantum field theories, in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, and obtained finite renormalization in the loop calculations [2, 3]. Recently, the concepts of neutrix and neutrix limit have been used widely in many applications in mathematics, physics and statistics. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. Fisher used the neutrices and the neutrix limit to define gamma, beta and incomplete gamma functions [5–7]. Ozcag et al. [8, 9] applied the neutrix limit to extend the definition of the incomplete beta function and its partial derivatives for negative integers. Also the digamma function has been generalized for negative integers by Jolevska-Tuneska et al [10]. Salem and Kilicman [11] generalized the definition of polygamma functions for negative integers. The concepts of the neutrix and neutrix limit are used at the first time in the q-calculus

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field by Salem [12, 13] who applied their concepts to define the q-analogue of the gamma and the incomplete gamma functions and their derivatives for negative values of x. In continuation, we apply the concepts of the neutrix and neutrix limit to generalize the definitions of the q-digamma and the q-polygamma functions for all their real variables.

2. Some Concepts in q-Calculus and Neutrces

Throughout this paper, the quantum deformation q is taken to be $0 < q < 1$ and the definitions of q-calculus will be taken from the well known books in this field [14, 15]. For any $x \in \mathbb{C}$ and $n \in \mathbb{N}$, the basic number $[x]_q$ and the q-factorial $[n]_q!$ are defined by

$$[x]_q = \frac{1-q^x}{1-q}, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$

with $[0]_q! = 1$ and the q-shifted factorials are defined as

$$(x;q)_0 = 1, \quad (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k).$$

The limit, $\lim_{n\to\infty}(x;q)_n$, is denoted by $(x;q)_\infty$.

The exponential function e^x has many different q-extensions, one of them is defined as

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}x^n}{[n]_q!} = (-(1-q)x;q)_\infty, \qquad x \in \mathbb{C}.$$

The q-integration of Jackson is defined for a function f defined on a generic interval $[a,b]$ to be

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx$$

where

$$\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n$$

provided the sum converges absolutely.

The q-gamma function is defined as

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}E_q(-tq)d_qt, \qquad x > 0. \tag{2.1}$$

Moreover, it has the representation

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}} = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \notin \mathbb{N}_0. \tag{2.2}$$

An important function related to the q-gamma function is the so-called the q-digamma function $\psi_q(x)$ which defined as the logarithmic derivative of the q-gamma function

$$\psi_q(x) = \frac{d}{dx}(\ln \Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}. \tag{2.3}$$

The q-digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava [16] when they studied the summations of basic hypergeometric series. Some of its properties presented and proved in their work.

They proved that $\psi_q(x)$ tends to the digamma function $\psi(x)$ when letting $q \to 1$. Some properties and expansions associated with the q-digamma function have been provided by Salem [17]. Among these results, we need the q-integral representation

$$\psi_q(x) = \frac{\ln q}{1-q}\left[\gamma_q - \int_0^q \frac{1-t^{x-1}}{1-t} d_q t\right], \qquad x > 0 \tag{2.4}$$

where $\gamma_q = \frac{1-q}{\ln q}\psi_q(1)$ is the q-analogue of the Euler-Mascheroni constant, and the recursive formula

$$\psi_q(x+n) = \psi_q(x) - \ln q \sum_{k=0}^{n-1} \frac{q^{x+k}}{1-q^{x+k}}, \qquad n \in \mathbb{N}. \tag{2.5}$$

The rth derivatives of the q-digamma function are the so-called the q-polygamma functions and can be represented as

$$\psi_q^{(r)}(x) = \frac{\ln q}{1-q} \int_0^q \frac{t^{x-1} \ln^r t}{1-t} d_q t, \qquad x > 0,\ r \in \mathbb{N}. \tag{2.6}$$

A neutrix N is defined as a commutative additive group of functions $f(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some f in N, $f(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Let N be a set contained in a topological space with a limit point a which does not belong to N. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to a and we write $N - \lim\limits_{\xi \to a} f(\xi) = c$.

In this paper, we let N be the neutrix having domain $N' = \{\epsilon : 0 < \epsilon < \infty\}$ and range N'' the real numbers, with the negligible functions being finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \qquad \ln^r \epsilon \qquad (\lambda < 0, \quad r \in \mathbb{N})$$

and all functions $o(\epsilon)$ which converge to zero in the normal sense as ϵ tends to zero [1].

3. The q-Digamma Function

It was proved in [12] that

$$\Gamma_q^{(r)}(x) = N - \lim_{\epsilon \to 0} \int_\epsilon^{\frac{1}{1-q}} t^{x-1} E_q(-tq) d_q t \tag{3.1}$$

for all $x \in \mathbb{R}$ and $r \in \mathbb{N}_0$. This proof came by applying the concepts of the neutrix and the neutrix limit to the the q-gamma function (2.1). In this section, we are seeking to apply the concepts of the neutrix and the neutrix limit on the q-digamma function (2.4) to define it for all $x \in \mathbb{R}$.

Theorem 3.1. *For all $x \in \mathbb{R}$ and $x \neq 0, -1, -2, \cdots$, we have*

$$\psi_q(x) = \frac{\ln q}{1-q}\left[\gamma_q - N - \lim_{\epsilon \to 0} \int_\epsilon^q \frac{1-t^{x-1}}{1-t} d_q t\right]. \tag{3.2}$$

Proof. Let $-n < x < -n+1$ and $n \in \mathbb{N}$, we get

$$\begin{aligned}
\int_\epsilon^q \frac{1-t^{x-1}}{1-t} d_q t &= \int_\epsilon^q \frac{1-t^{x+n-1}}{1-t} d_q t - \int_\epsilon^q \frac{1-t^n}{1-t} t^{x-1} d_q t \\
&= \int_\epsilon^q \frac{1-t^{x+n-1}}{1-t} d_q t - \int_\epsilon^q \sum_{k=0}^{n-1} t^{x+k-1} d_q t \\
&= \int_\epsilon^q \frac{1-t^{x+n-1}}{1-t} d_q t - \sum_{k=0}^{n-1} \frac{q^{x+k}}{[x+k]_q} + \sum_{k=0}^{n-1} \frac{\epsilon^{x+k}}{[x+k]_q}.
\end{aligned}$$

Taking the neutrix limit of the both sides with taking into account the equations (2.4) and (2.5) to obtain the desired result. □

Theorem 3.2. *The neutrix limit, as ϵ tends to zero, of the q-integral*

$$\int_\epsilon^q \frac{1-t^{-n-1}}{1-t} d_qt \tag{3.3}$$

exists for all $n \in \mathbb{N}_0$ and

$$N-\lim_{\epsilon\to 0}\int_\epsilon^q \frac{1-t^{-n-1}}{1-t} d_qt = (1-q)\left(1+\sum_{k=1}^{n}\frac{1}{1-q^k}\right), \qquad n\in\mathbb{N}_0. \tag{3.4}$$

Proof. When $n = 0$, using Lemma 4.3 in [12] gives

$$\int_\epsilon^q \frac{1-t^{-1}}{1-t} d_qt = -\int_\epsilon^q t^{-1} d_qt = \frac{1-q}{\ln q}(\ln q - \ln\epsilon).$$

Since $\ln\epsilon$ is negligible function, then by taking the neutrix limit gives

$$N-\lim_{\epsilon\to 0}\int_\epsilon^q \frac{1-t^{-1}}{1-t} d_qt = 1-q.$$

When $n \in \mathbb{N}$, using the geometric sequence rule and also Lemma 4.3 in [12] give

$$\begin{aligned}\int_\epsilon^q \frac{1-t^{-n-1}}{1-t} d_qt &= -\int_\epsilon^q \frac{t^{-n-1}(1-t^{n+1})}{1-t} d_qt \\ &= -\sum_{k=0}^{n-1}\int_\epsilon^q t^{k-n-1} d_qt - \int_\epsilon^q t^{-1} d_qt \\ &= -\sum_{k=0}^{n-1}\frac{q^{k-n}}{[k-n]_q} + \sum_{k=0}^{n-1}\frac{\epsilon^{k-n}}{[k-n]_q} + \frac{1-q}{\ln q}(\ln q - \ln\epsilon).\end{aligned}$$

Taking into account the neutrix limit with noting that the second sum consists of (linear sums of ϵ^λ; $\lambda < 0$) negligible functions to obtain

$$N-\lim_{\epsilon\to 0}\int_\epsilon^q \frac{1-t^{-n-1}}{1-t} d_qt = (1-q)\left(1-\sum_{k=0}^{n-1}\frac{q^{k-n}}{1-q^{k-n}}\right).$$

Replacing $n-k$ by k to obtain the desired result. □

The two Theorems 3.1 and 3.2 suggest the following definition and corollary.

Definition 3.3. The q-digamma function $\psi_q(x)$ can be defined by

$$\psi_q(x) = \frac{\ln q}{1-q}\left[\gamma_q - N-\lim_{\epsilon\to 0}\int_\epsilon^q \frac{1-t^{x-1}}{1-t} d_qt\right] \tag{3.5}$$

for all $x \in \mathbb{R}$.

Corollary 3.4. *The value of the q-digamma function $\psi_q(x)$ at $x = 0$ is given by*

$$\psi_q(0) = \frac{\ln q}{1-q}\gamma_q - \ln q \tag{3.6}$$

and for negative integers as

$$\psi_q(-n) = \frac{\ln q}{1-q}\gamma_q - \ln q\left(1+\sum_{k=1}^{n}\frac{1}{1-q^k}\right), \qquad n\in\mathbb{N}. \tag{3.7}$$

Remark 3.5. It is worth mentioning that when $q \to 1$, the equation (3.7) approaches to

$$\psi(-n) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \qquad n \in \mathbb{N}. \tag{3.8}$$

which was proven in [10].

4. The q-Polygamma Functions

In the present section, we are seeking to define the q-polygamma functions $\psi_q^{(r)}(x)$ defined as in (2.6) for all $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The results on the q-trigamma function $\psi_q'(x)$ as $q \to 1$ were obtained in [10] and the results on $\psi^{(r)}(x),\ r \geq 2$ were shown in [11].

Lemma 4.1. ([18]) *The neutrix limit, as ϵ tends to zero, of the q-integral*

$$\int_{\epsilon}^{1} t^{\alpha-1} \ln^r t d_q t \tag{4.1}$$

exists for all values $\alpha \in \mathbb{C}$ and $r \in \mathbb{N}$ and

$$N - \lim_{\epsilon \to 0} \int_{\epsilon}^{1} t^{\alpha-1} \ln^r t d_q t = \begin{cases} \dfrac{\ln^r q}{[\alpha]_q} \displaystyle\sum_{k=1}^{r} \left(\frac{q^\alpha}{1-q^\alpha} \right)^k \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^r, & \alpha \neq 0, \\ 0, \quad \alpha = 0 \end{cases} \tag{4.2}$$

see [18]. Furthermore, if $q \to 1$, we get

$$N - \lim_{\epsilon \to 0} \int_{\epsilon}^{1} t^{\alpha-1} \ln^r t dt = \frac{(-1)^r}{\alpha^{r+1}} \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^r = \frac{(-1)^r r!}{\alpha^{r+1}}, \quad \alpha \neq 0. \tag{4.3}$$

Lemma 4.2. *The q-polygamma function $\psi_q^{(r)}(x)$ defined as in (2.6) can be expressed as*

$$\psi_q^{(r)}(x) = \frac{\ln q}{1-q} \int_0^1 \frac{t^{x-1} \ln^r t}{1-t} d_q t - \frac{q^x \ln^{r+1} q}{1-q}, \qquad x > 0,\ r \in \mathbb{N}. \tag{4.4}$$

Proof. It is not difficult to see from the definition of q-integral that

$$\int_0^{aq} f(t) d_q t = \int_0^a f(t) d_q t - a(1-q) f(a)$$

which can be exploited to complete the proof. □

Theorem 4.3. *The neutrix, limit as ϵ tends to zero, of the q-integral*

$$\int_{\epsilon}^{1} \frac{t^{x-1} \ln^r t}{1-t} d_q t \tag{4.5}$$

exists for all values of $x \in \mathbb{R}$ and $r \in \mathbb{N}$.

Proof. In this proof, we consider three cases.

The first case: When $-n < x < -n+1$ and $n \in \mathbb{N}$, using the geometric sequence sum rule gives

$$\begin{aligned}\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t} d_q t &= \int_\epsilon^1 \frac{t^{x-1}\ln^r t(1-t^n)}{1-t} d_q t + \int_\epsilon^1 \frac{t^{x+n-1}\ln^r t}{1-t} d_q t \\ &= \sum_{k=0}^{n-1}\int_\epsilon^1 t^{x+k-1}\ln^r t d_q t + \int_\epsilon^1 \frac{t^{x+n-1}\ln^r t}{1-t} d_q t.\end{aligned}$$

Taking the neutrix limit as $\epsilon \to 0$ yields

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t} d_q t = \sum_{k=0}^{n-1}\left(N-\lim_{\epsilon\to 0}\int_\epsilon^1 t^{x+k-1}\ln^r t d_q t\right) + \int_0^1 \frac{t^{x+n-1}\ln^r t}{1-t} d_q t.$$

Lemma 4.1 tells that the neutrix limit of the q-integral

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 t^{x+k-1}\ln^r t d_q t$$

exists for all $-n < x < -n+1$ and so does the q-integral (4.5).

The second case: When $x = 0$, we get

$$\int_\epsilon^1 \frac{t^{-1}\ln^r t}{1-t} d_q t = \int_\epsilon^1 t^{-1}\ln^r t d_q t + \int_\epsilon^1 \frac{\ln^r t}{1-t} d_q t. \tag{4.6}$$

Also, Lemma 4.1 tells that the neutrix limit of the first q-integral equals zero and so we get

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{-1}\ln^r t}{1-t} d_q t = \int_0^1 \frac{\ln^r t}{1-t} d_q t.$$

The last case: When $x = -n$ and $n \in \mathbb{N}$, we find

$$\begin{aligned}\int_\epsilon^1 \frac{t^{-n-1}\ln^r t}{1-t} d_q t &= \int_\epsilon^1 \frac{t^{-n-1}(1-t^{n+1})\ln^r t}{1-q} d_q t + \int_\epsilon^1 \frac{\ln^r t}{1-t} d_q t \\ &= \sum_{k=0}^{n}\int_\epsilon^1 t^{k-n-1}\ln^r t d_q t + \int_\epsilon^1 \frac{\ln^r t}{1-t} d_q t \\ &= \sum_{k=0}^{n-1}\int_\epsilon^1 t^{k-n-1}\ln^r t d_q t + \int_\epsilon^1 t^{-1}\ln^r t d_q t + \int_\epsilon^1 \frac{\ln^r t}{1-t} d_q t.\end{aligned}$$

Taking the neutrix limit as $\epsilon \to 0$ yields

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{-n-1}\ln^r t}{1-t} d_q t = \sum_{k=0}^{n-1}\left(N-\lim_{\epsilon\to 0}\int_\epsilon^1 t^{k-n-1}\ln^r t d_q t\right) + \int_0^1 \frac{\ln^r t}{1-t} d_q t. \tag{4.7}$$

Using Lemma 4.1 ends the proof. □

The above theorem leads us to introduce the following definition.

Definition 4.4. The q-polygamma function $\psi_q^{(r)}(x)$ can be defined by

$$\psi_q^{(r)}(x) = \frac{\ln q}{1-q}\left(N-\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t} d_q t - q^x \ln^r q\right) \tag{4.8}$$

for all $x \in \mathbb{R}$ and $r \in \mathbb{N}$.

Theorem 4.5. *The value of q-polygamma function $\psi_q^{(r)}(x)$ at $x = 0$ is given by*

$$\psi_q^{(r)}(0) = \psi_q^{(r)}(1) - \ln^{r+1} q \tag{4.9}$$

and for negative integers as

$$\psi_q^{(r)}(-n) = \psi_q^{(r)}(1) - q^{-n}[n+1]_q \ln^{r+1} q - \sum_{k=1}^{n} \frac{q^k \ln^{r+1} q}{1-q^k} \sum_{j=1}^{r} \left(\frac{1}{1-q^k}\right)^j \sum_{i=1}^{j} (-1)^i \binom{j}{i} i^r. \tag{4.10}$$

In particular, when $r = 1$, we have

$$\psi_q'(-n) = \psi_q'(1) - q^{-n}[n+1]_q \ln^2 q + \sum_{k=1}^{n} \frac{q^k \ln^2 q}{(1-q^k)^2}. \tag{4.11}$$

Proof. In view of (4.2), (4.4), (4.6) and (4.7), we can easily get the proof. □

Remark 4.6. When $q \to 1$, we get

$$\psi^{(r)}(0) = \psi^{(r)}(1) \tag{4.12}$$

and for negative integers as

$$\psi^{(r)}(-n) = \psi^{(r)}(1) + \sum_{k=1}^{n} \frac{1}{k^{r+1}}. \tag{4.13}$$

which were obtained in [11].

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