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Fractional Differential Equations with Nonlocal (Parametric Type) Anti-Periodic Boundary Conditions

Ravi P. Agarwal[a,b], Bashir Ahmad[b], Juan J. Nieto[c,b]

[a]*Department of Mathematics, Texas A&M University, Kingsville, TX 78363-8202, USA*
[b]*NAAM-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[c]*Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain*

Abstract. In this paper, we introduce a new concept of nonlocal anti-periodic boundary conditions and solve fractional and sequential fractional differential equations supplemented with these conditions. The anti-periodic boundary conditions involve a nonlocal intermediate point together with one of the fixed end points of the interval under consideration, and accounts for a flexible situation concerning anti-periodic phenomena. The existence results for the given problems are obtained with the aid of standard fixed point theorems. Some examples illustrating the main results are also discussed. The paper concludes with several interesting observations.

1. Introduction

The topic of boundary value problems is an important and interesting field of research. In recent years, boundary value problems of fractional differential equations and inclusions involving different kinds of boundary conditions such as multipoint and nonlocal conditions have extensively been investigated by several researchers and a variety of results can be found in the works [1]-[11].

Anti-periodic boundary conditions appear in a variety of situations of applied problems. As a matter of fact, many numerical problems (in the study of modes) converge faster when anti-periodic boundary conditions are used instead of periodic boundary conditions [12]. The classical as well as fractional antiperiodic boundary conditions have been considered by several authors ([13]-[18]). However, the concept of parametric (nonlocal) anti-periodic boundary conditions has not been addressed yet.

In this paper, we consider nonlocal (parametric type) anti-periodic conditions involving a nonlocal intermediate point $0 < a < T$ and the right end point $(t = T)$. This gives rise to a new kind of anti-periodic conditions: $x(a) = -x(T)$, $x'(a) = -x'(T)$. As a first problem, we consider an anti-periodic boundary value problem of Caputo type fractional differential equations given by

$$ {}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \ T > 0, \ 1 < q \le 2, \tag{1} $$

$$ x(a) = -x(T), \qquad x'(a) = -x'(T), \ 0 < a < T, \tag{2} $$

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Email addresses: Ravi.Agarwal@tamuk.edu (Ravi P. Agarwal), bashirahmad_qau@yahoo.com (Bashir Ahmad), juanjose.nieto.roig@usc.es (Juan J. Nieto)

where ${}^cD^q$ denotes the Caputo fractional derivative of order q and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$. is a given function. In the second problem, we replace Caputo type fractional differential equation (1) with a sequential fractional differential equation (see Section 4).

Here we remark that the interval $[0,T]$ can be replaced with an interval of the form $(-\infty, T]$ with a in it. This means that the anti-periodic phenomena can start from an arbitrary point in $(-\infty, T)$.

The paper is organized as follows. In Section 2, we solve a linear fractional differential equation of fractional order $q \in (1,2]$ subject to nonlocal classical and fractional anti-periodic boundary conditions. Section 3 presents uniqueness results for nonlinear fractional-order differential equations supplemented with nonlocal classical and fractional anti-periodic boundary conditions. In Section 4, we extend our study carried out in Section 3 to nonlinear sequential fractional differential equations. Section 5 contains some illustrating examples while Section 6 concludes our work.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [19].

Definition 2.1. *The Riemann-Liouville fractional integral of order ν for $g \in L^1(0,T)$ is defined as*

$$I^\nu h(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{g(s)}{(t-s)^{1-\nu}} ds, \quad \nu > 0.$$

Definition 2.2. *Let $h : [0,\infty) \to \mathbb{R}$ be such that $h \in AC^n[0,T]$. Then the Caputo derivative of fractional order ν for h is defined as*

$${}^cD^\nu h(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^{(n)}(s) ds, \quad n-1 < \nu < n, n = [\nu]+1,$$

where $[\nu]$ denotes the integer part of the real number ν.

2.1. Nonlocal classical anti-periodic boundary conditions

In this subsection, we consider a linear variant of the problem (1)-(2) and prove the following result that will be used to define an operator equation for the given problem.

Lemma 2.1. For any $1 < q \le 2$ and $g \in C[0,T]$, the integral solution of the equation ${}^cD^q x(t) = g(t)$ with the boundary conditions (2) is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - \frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s)ds + \int_0^a \frac{(a-s)^{q-1}}{\Gamma(q)} g(s)ds \Big) \\ &+ \frac{(T+a-2t)}{4} \Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} g(s)ds + \int_0^a \frac{(a-s)^{q-2}}{\Gamma(q-1)} g(s)ds \Big). \end{aligned} \tag{3}$$

Proof. For some constants $b_1, b_2 \in \mathbb{R}$, we know that the solution of the given equation can be written as

$$x(t) = I^q g(t) + b_1 + b_2 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds + b_1 + b_2 t. \tag{4}$$

Using the boundary conditions (2) in (4), it is found that

$$\begin{aligned} b_1 &= -\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s)ds - \frac{1}{2} \int_0^a \frac{(a-s)^{q-1}}{\Gamma(q)} g(s)ds \Big) + \frac{(T+a)}{4} \Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} g(s)ds + \int_0^a \frac{(a-s)^{q-2}}{\Gamma(q-1)} g(s)ds \Big) \\ b_2 &= -\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} g(s)ds + \int_0^a \frac{(a-s)^{q-2}}{\Gamma(q-1)} g(s)ds \Big). \end{aligned}$$

Substituting the values of b_1 and b_2 in (4) completes the solution (3).

Remark 2.3. *We note that the solution of the classical anti-periodic boundary value problem of fractional differential equations:*

$$
\begin{aligned}
&{}^cD^q x(t) = g(t), \;\; 0 < t < T, \;\; 1 < q \le 2, \\
&x(0) = -x(T), \qquad\qquad x'(0) = -x'(T)
\end{aligned}
$$

is [15]

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s)ds + \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} g(s)ds. \tag{5}$$

Comparing (3) and (5), it is found that the nonlocal anti-periodic boundary conditions give rise to two additional terms (third and fifth terms in (3) with coefficient function $(T-2t)$ replaced by $(T+a-2t)$).

2.2. Nonlocal fractional anti-periodic boundary conditions

As in Lemma 2.1, the solution of the equation ${}^cD^q x(t) = g(t)$ subject to the nonlocal fractional anti-periodic boundary conditions:

$$x(a) = -x(T), \;\; {}^cD^p x(a) = -{}^cD^p x(T), \;\; 0 < p < 1 \tag{6}$$

is

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - \frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s)ds + \int_0^a \frac{(a-s)^{q-1}}{\Gamma(q)} g(s)ds\Big) \\
&+ \frac{\Gamma(2-p)(T+a-2t)}{2T^{1-p}[1+(a/T)^{1-p}]}\Big(\int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} g(s)ds + \int_0^a \frac{(a-s)^{q-p-1}}{\Gamma(q-p)} g(s)ds\Big),
\end{aligned} \tag{7}
$$

where we have used

$${}^cD^p x(t) = \int_0^t \frac{(t-s)^{q-p-1}}{\Gamma(q-p)} y(s)ds - b_2 \frac{t^{1-p}}{\Gamma(2-p)}.$$

On the other hand, the solution of the given equation subject to the fractional anti-periodic boundary conditions $x(0) = -x(T)$, ${}^cD^p x(0) = -{}^cD^p x(T)$, $0 < p < 1$, is given by [16].

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s)ds + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}}\int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} g(s)ds. \tag{8}$$

Notice that the solution (7) contains five terms whereas the solution (2) has three terms with the coefficient function $(T-2t)$ modified by $(T+a-2t)/[1+(a/T)^{1-p}]$. Thus, two additional terms (with different coefficients) occur due to nonlocal anti-periodic boundary conditions in contrast to the classical case.

3. Uniqueness Results

This section deals with the existence and uniqueness of solutions for the problems (1)-(2) and (1)-(6).

Let $\mathcal{P} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ endowed with the usual norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$.

In the sequel, we need the following assumption:

(*H*) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the Lipschitz condition: $|f(t,u) - f(t,v)| \le L|u-v|$, $\forall t \in [0,T], L > 0,\; u,v \in \mathbb{R}$.

For computational convenience, we introduce the notations:

$$\mu = \max_{t\in[0,T]}\Big\{\frac{(2t+T^q+a^q)}{2\Gamma(q+1)} + \frac{|T-2t+a|(T^{q-1}+a^{q-1})}{4\Gamma(q)}\Big\}; \tag{9}$$

$$\sigma = \max_{t\in[0,T]}\Big\{\frac{(2t+T^q+a^q)}{2\Gamma(q+1)} + \frac{\Gamma(2-p)|T+a-2t|(T^{q-p}+a^{q-p})}{2T^{1-p}[1+(a/T)^{1-p}]\Gamma(q-p+1)}\Big\}. \tag{10}$$

Theorem 3.1. *Assume that the condition* (H) *holds and that* $\mu L < 1$*, where* μ *is given by (9). Then the boundary value problem (1)-(2) has a unique solution on* $[0,T]$.

Proof. As a first step, we transform the problem (1)-(2) to an equivalent fixed point problem via (3) as

$$x = \mathcal{U}_c x, \tag{11}$$

where $\mathcal{U}_c : \mathcal{P} \to \mathcal{P}$ is defined by

$$\begin{aligned}(\mathcal{U}_c x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &- \frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \int_0^a \frac{(a-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big) \\ &+ \frac{(T+a-2t)}{4}\Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds + \int_0^a \frac{(a-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big).\end{aligned}$$

Using the Banach contraction principle, we shall show that the operator $\mathcal{U}_c$ has a fixed point. Fixing $\max_{t\in[0,T]}|f(t,0)| = N < \infty$, and choosing $r \geq \frac{N\mu}{1-L\mu}$, we show that $\mathcal{U}_c B_r \subset B_r$, where $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \leq r\}$. For $x \in B_r$, it is straightforward to show that $\|\mathcal{U}_c\| \leq r$. Next, for $x, y \in C([0,T],\mathbb{R})$ and $t \in [0,T]$, we have

$$\begin{aligned}\|\mathcal{U}_c x - \mathcal{U}_c y\| &= \max_{t\in[0,T]}\Big|\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds \\ &- \frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds \\ &+ \int_0^a \frac{(a-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\Big) \\ &+ \frac{(T+a-2t)}{4}\Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s)) - f(s,y(s))|ds \\ &+ \int_0^a \frac{(a-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s)) - f(s,y(s))|ds\Big)\Big| \\ &\leq \max_{t\in[0,T]}\Big\{\frac{(2t+T^q+a^q)}{2\Gamma(q+1)} + \frac{|T-2t+a|(T^{q-1}+a^{q-1})}{4\Gamma(q)}\Big\}L\|x-y\|,\end{aligned}$$

which, in view of (9), can be written as $\|\mathcal{U}_c x - \mathcal{U}_c y\| \leq \mu L\|x-y\|$. Thus, by the assumption: $\mu L < 1$, it follows by Banach contraction principle that the operator $\mathcal{U}_c$ is a contraction. Thus there exists a unique solution for the problem (1)-(2) on $[0,T]$. This completes the proof.

Theorem 3.2. *Suppose that the condition* (H) *holds. Then there exists a unique solution on* $[0,T]$ *for the equation (1) supplemented with fractional boundary conditions (6) if* $\sigma L < 1$*, where* σ *is given by (10).*

Proof. Define an operator $\mathcal{U}_f : \mathcal{P} \to \mathcal{P}$ as

$$\begin{aligned}(\mathcal{U}_f x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \frac{1}{2}\Big(\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \int_0^a \frac{(a-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big)\\ &+ \frac{\Gamma(2-p)(T+a-2t)}{2T^{1-p}[1+(a/T)^{1-p}]}\Big(\int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} g(s)ds + \int_0^a \frac{(a-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\Big).\end{aligned}$$

Observe that the problem (1) and (6) has solutions if and only if the operator equation $x = \mathcal{U}_f x$ has fixed points. Following the method of proof for Theorem 3.1, it can be shown that the operator $\mathcal{U}_f$ has a fixed point which in fact is a solution of the problem (1) and (6). This completes the proof.

4. Sequential Fractional Differential Equations

We now consider a nonlocal anti-periodic boundary value problem of nonlinear sequential fractional differential equations given by

$$\begin{cases} ({}^cD^\alpha + k\,{}^cD^{\alpha-1})x(t) = f(t,x(t)),\ 1<\alpha\le 2,\ k\in\mathbb{R}^+,\ 0<t<T,\\ x(a) = -x(T), \qquad x'(a) = -x'(T),\ 0<a\ll T, \end{cases} \tag{12}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, and f is a given continuous function.

Our first result is concerned with the linear variant of (12).

Lemma 4.1. *Let* $0 < a \ll T$, *and* $h \in C([0,T],\mathbb{R})$. *Then the unique solution of the problem:*

$$\begin{cases} ({}^cD^\alpha + k\,{}^cD^{\alpha-1})x(t) = h(t),\ 1<\alpha\le 2,\ 0<t<T,\\ x(a) = -x(T), \qquad x'(a) = -x'(T),\ 0<a\ll T, \end{cases} \tag{13}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\Big)ds + \nu_1(t)\Big[\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}h(s)ds + \int_0^a \frac{(a-s)^{\alpha-2}}{\Gamma(\alpha-1)}h(s)ds\Big]\\ &+\nu_2(t)\Big[\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\Big)ds + \int_0^a e^{-k(a-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\Big)ds\Big], \end{aligned} \tag{14}$$

where

$$\nu_1(t) = \frac{2e^{-kt} - (e^{-kT}+e^{-ka})}{2k(e^{-kT}+e^{-ka})},\ \nu_2(t) = \frac{-e^{-kt}}{e^{-kT}+e^{-ka}}. \tag{15}$$

Proof. As argued in [20], the general solution of the equation $({}^cD^\alpha + k\,{}^cD^{\alpha-1})x(t) = h(t)$ can be written as

$$u(t) = c_1 e^{-kt} + \int_0^t e^{-k(t-s)}(I^{\alpha-1}h(s))ds + c_2, \tag{16}$$

where c_1 and c_2 are arbitrary constants and

$$I^{\alpha-1}h(t) = \int_0^t \frac{(t-x)^{\alpha-2}}{\Gamma(\alpha-1)}h(x)dx.$$

Using the given boundary conditions in (16), we find that

$$c_1 = \frac{1}{k(e^{-kT}+e^{-ka})}\Big[I^{\alpha-1}h(T) + I^{\alpha-1}h(a) - k\int_0^T e^{-k(T-s)}(I^{\alpha-1}h(s))ds - k\int_0^a e^{-k(a-s)}(I^{\alpha-1}h(s))ds\Big],$$

$$c_2 = -\frac{1}{2k}\Big[I^{\alpha-1}h(T) + I^{\alpha-1}h(a)\Big].$$

Substituting the values of c_1 and c_2 in (16) yields the solution (17). This completes the proof. □

Using Lemma 4.1, we can transform the problem (12) into a fixed point problem: $x = \mathcal{U}_s x$, where the operator $\mathcal{U}_s : \mathcal{P} \to \mathcal{P}$ is given by

$$\begin{aligned}(\mathcal{U}_s x)(t) &= \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p,x(p))dp\Big)ds \\ &+\nu_1(t)\Big[\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds + \int_0^a \frac{(a-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds\Big] \\ &+\nu_2(t)\Big[\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p,x(p))dp\Big)ds \\ &+\int_0^a e^{-k(a-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p,x(p))dp\Big)ds\Big],\end{aligned} \tag{17}$$

Notice that the problem (12) has solutions if and only if the operator equation $x = \mathcal{U}_s x$ has fixed points. In the sequel, we use the following notations:

$$\bar{\nu}_1 = \max_{t\in[0,T]}\Big|\frac{2e^{-kt} - (e^{-kT} + e^{-ka})}{2k(e^{-kT} + e^{-ka})}\Big|, \ \bar{\nu}_2 = \max_{t\in[0,T]}\Big|\frac{-e^{-kt}}{e^{-kT} + e^{-ka}}\Big|, \tag{18}$$

$$\delta = \frac{1}{k\Gamma(\alpha)}\Big[T^{\alpha-1}(1-e^{-kT})(1+\bar{\nu}_2) + k\bar{\nu}_1(T^{\alpha-1} + a^{\alpha-1}) + \bar{\nu}_2 a^{\alpha-1}(1-e^{-ka})\Big]. \tag{19}$$

Theorem 4.2. *Let the assumption* (H) *and the condition* $\delta L < 1$ *hold, where* δ *is given by (19). Then there exists a unique solution for the problem (12) on* $[0, T]$.

Proof. As in the proof of Theorem 3.1, it can easily be shown that $\mathcal{U}_s B_{\Re} \subset B_{\Re}$, where $\mathcal{U}_s$ is given by (17), $B_{\Re} = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \Re\}$, $\Re \ge \dfrac{N\delta}{1-L\delta}$, $\max_{t\in[0,T]} |f(t,0)| = N < \infty$ and δ is given by (19). Next we show that the operator $\mathcal{U}_s$ is a contraction. for $x, y \in C([0,T],\mathbb{R})$ and $t \in [0,T]$, we have

$$\begin{aligned}\|\mathcal{U}_s x - \mathcal{U}_s y\| &= \max_{t\in[0,T]}\Big|\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}|f(p,x(p)) - f(p,y(p))|dp\Big)ds \\ &+ \nu_1(t)\Big[\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,x(s)) - f(s,y(s))|ds \\ &+ \int_0^a \frac{(a-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,x(s)) - f(s,y(s))|ds\Big] \\ &+ \nu_2(t)\Big[\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}|f(p,x(p)) - f(p,y(p))|dp\Big)ds \\ &+ \int_0^a e^{-k(a-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}|f(p,x(p)) - f(p,y(p))|dp\Big)ds\Big]\Big| \\ &\le \frac{1}{k\Gamma(\alpha)}\Big[T^{\alpha-1}(1-e^{-kT})(1+\bar{\nu}_2) + k\bar{\nu}_1(T^{\alpha-1} + a^{\alpha-1}) + \bar{\nu}_2 a^{\alpha-1}(1-e^{-ka})\Big]L\|x-y\|,\end{aligned}$$

which, by (19), can be expressed as $\|\mathcal{U}_s x - \mathcal{U}_s y\| \le \delta L\|x-y\|$. In view of the assumption: $\delta L < 1$, it is clear that the operator $\mathcal{U}_s$ is a contraction by Banach's fixed point theorem. In consequence, the problem (12) has a unique solution on $[0, T]$. This completes the proof. □

5. Examples

Example 5.1. *Consider the following anti-periodic boundary value problem*

$$\begin{aligned} {}^cD^{3/2}x(t) &= m\Big(x + \ln(x + \sqrt{x^2+1})\Big) + (t+1)^2, \ \ t \in [0,1], &(20)\\ x(0.01) &= -x(1), \qquad x'(0.01) = -x'(1). &(21) \end{aligned}$$

where the positive real number m will be fixed later. Here $q = 3/2,\ f(t,x) = m(x + \ln(x + \sqrt{x^2+1})) + (t+1)^2,\ a = 0.01,\ T = 1$. *Evidently,* $|f(t,x) - f(t,y)| \leq 2m|x-y|$. *With* $L_1 = 2m$, *the condition* $\mu L < 1$ *holds with*

$$\mu = \max_{t\in[0,T]}\Big\{\frac{(2t + T^q + a^q)}{2\Gamma(q+1)} + \frac{|T - 2t + a|(T^{q-1} + a^{q-1})}{4\Gamma(q)}\Big\}$$

when $m < 3000\sqrt{\pi}/15337$. *Thus, by Theorem 3.1, the problem (20) has a unique solution on* $[0,1]$.

Example 5.2. *Consider the fractional boundary value problem:*

$$\begin{aligned} {}^cD^{3/2}x(t) &= \frac{1}{(1+t)^2}\tan^{-1}x + \sin(t+2), \ \ t\in[0,1], &(22)\\ & x(0.01) = -x(1),\ {}^cD^{1/2}x(0.01) = -{}^cD^{1/2}x(1). &(23) \end{aligned}$$

Here $q = 3/2, p = 1/2,\ a = 0.01,\ T = 1$ *and* $f(t,x) = \frac{1}{(2+t)^2}\tan^{-1}x + \sin(t+2)$. *Clearly* $|f(t,x) - f(t,y)| \leq \frac{1}{4}|x-y|$. *With* $L = 1/4$, *and*

$$\sigma = \max_{t\in[0,1]}\Big\{\frac{(2t + T^q + a^q)}{2\Gamma(q+1)} + \frac{\Gamma(2-p)|T + a - 2t|(T^{q-p} + a^{q-p})}{2T^{1-p}[1 + (a/T)^{1-p}]\Gamma(q-p+1)}\Big\} = 1.539682,$$

we find that $\sigma L = 0.384921 < 1$. *Thus, all the conditions for Theorem 3.2 are satisfied and hence, by its conclusion, there exists a unique solution for the problem (20) on* $[0,1]$.

Example 5.3. *Consider the problem:*

$$\begin{aligned} &({}^cD^{3/2} + 5\,{}^cD^{1/2})x(t) = \frac{|x|}{(t+6)(|x|+1)} + e^{-t},\ 0 < t < 1, &(24)\\ &x(0.01) = -x(1),\ x'(0.01) = -x'(1). &(25) \end{aligned}$$

Here $\alpha = 3/2, k = 5,\ a = 0.01,\ T = 1$ *and* $f(t,x) = \frac{|x|}{(t+2)(|x|+1)} + e^{-t}$. *Clearly* $f(t,x)$ *satisfies the condition* (H) *with* $L = 1/2$. *With the given data, it is found that* $\bar{v}_1 = 0.108775,\ \bar{v}_2 = 1.043877,\ \delta = 0.594308$. *In consequence, we have* $\delta L = 0.297154 < 1$. *Since all the conditions of Theorem 4.2 are satisfied, therefore, its conclusion applies to the problem (20).*

6. Conclusions

In this paper, we have discussed a new kind of nonlocal anti-periodic boundary value problems of fractional-order. It has been found that the consideration of nonlocal anti-periodic boundary conditions gives rise to some additional terms in the integral solutions of fractional-order problems at hand. Further, the results obtained in this paper are flexible and correspond to the situation when a shift in the position of the anti-periodic phenomena occurs at the left-end of the interval $[0,T]$ by fixing $a \ll T$. It is found that the results for classical anti-periodic boundary conditions [15, 16] follow from the results of this paper in the limit $a \to 0^+$. It is worth-mentioning that the results of Section 4 dealing with sequential fractional differential equations in the limit $a \to 0^+$ are new. In the nutshell, the nonlocal nature of anti-periodic classical/fractional boundary conditions allows the antiperiodic phenomena to occur at any intermediate position of the interval under consideration.

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