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Existence of Non-subnormal Completely Semi-Weakly Hyponormal Weighted Shifts

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Abstract. In this paper, we introduce a new notion of completely semi-weakly hyponormal operator which is a special case of polynomially hyponormal operator. For an one-step backward extension of the Bergman weighted shift, we show that completely semi-weakly hyponormal weighted shifts need not be subnormal. In addition, we provide an example which can serve to distinguish the semi-weak m-hyponormality from the semi-weak m-hyponormality with positive determinant coefficients for such a shift. Finally we discuss flatness on semi-weakly m-hyponormal weighted shifts.

1. Preliminaries

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For bounded operators A and B, we denote $[A, B] := AB - BA$. A k-tuple $\mathbf{T} = (T_1, ..., T_k)$ of bounded operators on $\mathcal{H}$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^k$ is positive on the direct sum of $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ with k copies. Also an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (*strongly*) *k-hyponormal* if $(I, T, ..., T^k)$ is hyponormal ([3],[4],[5],[7],[8]). It is well known that an operator T is subnormal if and only if T is k-hyponormal for all $k \geq 1$ via Bram-Halmos criterion ([1]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for all complex polynomials p. For a positive integer k, an operator T is *weakly k-hyponormal* if for every polynomial p of degree k or less, $p(T)$ is hyponormal ([4],[7],[8]). It holds that every subnormal operator is a polynomially hyponormal operator and a k-hyponormal operator is a weakly k-hyponormal operator for each positive integer k. For $k = 1$, 1-hyponormality and weak 1-hyponormality of T are equivalent to the hyponormality of T.

Recently in [9], the classes of semi-weakly k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality. An operator T is called *semi-weakly k-hyponormal* if $T + sT^k$ is hyponormal for all $s \in \mathbb{C}$ ([9]). It is trivial that semi-weak 2-hyponormality is equivalent to weak 2-hyponormality. In particular, T is said to be *completely semi-weakly hyponormal* if T is semi-weakly k-hyponormal for all $k \geq 2$. We can easily show that every polynomially hyponormal operator is

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a completely semi-weakly hyponormal operator. Also it is obvious that weakly k-hyponormality implies semi-weakly k-hyponormality for each positive integer k. However it is known that converse implications are not always true ([9],[12]). Sometimes weak 2-, 3- and 4-hyponormality are referred to as quadratic, cubic and quartic hyponormality, respectively, and also semi-weak 3-hyponormality is referred to as semi-cubic hyponormality.

It is one of the old problems in operator theory to determine whether every polynomially hyponormal operator is subnormal. Curto-Putinar ([7]) proved that there exists an operator that is polynomially hyponormal but not 2-hyponormal. Although the existence of a weighted shift which is polynomially hyponormal but not subnormal was established in [7] and [8], concrete example of such weighted shifts has not been found yet.

Since Curto ([3]) began to study criteria for distinguishing weak n-hyponormality from n-hyponormality, the weighted shifts have played very important roles in various research areas containing these classes. Recall that $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ denotes a weight sequence in the set of positive real numbers $\mathbb{R}_+$. The *weighted shift* W_α acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_j = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It follows instantly from simple computations that W_α is hyponormal if and only if α is an increasing sequence.

The study of flatness for weighted shifts is a good approach to detect gaps between subnormality and hyponormality. Stampfli ([13]) showed that a subnormal W_α with $\alpha_k = \alpha_{k+1}$ for some $k \in \mathbb{N}_0$ is flat, i.e., $\alpha_1 = \alpha_2 = \cdots$. Stampfli's result has been used to attempt the construction of nonsubnormal polynomially hyponormal weighted shifts (cf. [2],[3],[9]). In [2], it is proved that every polynomially hyponormal weighted shift with any two equal weights has flatness. It is shown in [4] that flatness need not hold for quadratic hyponormality; for example, if $\alpha : \sqrt{2/3}, \sqrt{2/3}, \sqrt{(n+1)/(n+2)}$ $(n \geq 2)$, then W_α is quadratically hyponormal but not 2-hyponormal. Recently, authors in [11] proved that a cubically hyponormal weighted shift with first two equal weights has flatness. Also in [9], they proved that a semi-cubically hyponormal weighted shift with $\alpha_k = \alpha_{k+1}$ for some $k \geq 1$ is flat. Hence it is worthwhile to determine whether weakly m [or semi-weakly m]-hyponormal weighted shifts for $m \geq 4$ have flatness.

This paper consists of five sections. In Section 2 we recall some terminology and notations concerning semi-weakly m-hyponormal weighted shifts. We can explicitly obtain an interval I in x such that a weighted shift $W_{\alpha(x)}$ is completely semi-weakly hyponormal but not subnormal on I (see Theorem 2.3 below). In Section 3 we produce an interval on x in the positive real line for semi-weak m-hyponormality but not semi-weak m-hyponormality with positive determinant coefficients for such a shift. In Section 4, we show some properties of flatness for a completely semi-weakly hyponormal and semi weakly m-hyponormal weighted shifts. In Section 5, we give the rigorous proof for Theorem 2.1 which used some different methods from proofs in results [9].

Some of the calculations in this paper were aided by using the software tool *Mathematica* ([14]).

2. Characterizations

We recall some standard terminology and definitions about semi-weakly m-hyponormal weighted shifts ([9]). Throughout this paper we consider $m \geq 3$.

Let W_α be a weighted shift with a weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ and let P_n denote the orthogonal projection onto $\vee_{k=0}^{n}\{e_k\}$.

For $n \in \mathbb{N}_0$, define $D_n^{[m]}$ by

$$D_n^{[m]} := D_n^{[m]}(s) = P_n\left[(W_\alpha + sW_\alpha^m)^*, W_\alpha + sW_\alpha^m\right]P_n$$

$$= \begin{pmatrix} q_{m,0} & 0 & \cdots & 0 & z_{m,0} & 0 & \cdots & \\ 0 & q_{m,1} & \ddots & \ddots & 0 & z_{m,1} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & q_{m,m-2} & 0 & \ddots & \ddots & z_{m,n+1-m} \\ \bar{z}_{m,0} & 0 & \ddots & 0 & q_{m,m-1} & \ddots & \ddots & 0 \\ 0 & \bar{z}_{m,1} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & q_{m,n-1} & 0 \\ & & 0 & \bar{z}_{m,n+1-m} & 0 & \cdots & 0 & q_{m,n} \end{pmatrix} \tag{2.1}$$

for all $s \in \mathbb{C}$, where

$$\begin{aligned} &q_{m,n} := u_{m,n} + v_{m,n}|s|^2,\ z_{m,n} := \sqrt{w_{m,n}}\bar{s},\ u_{m,n} := \alpha_n^2 - \alpha_{n-1}^2, \\ &v_{m,n} := \alpha_n^2\alpha_{n+1}^2\cdots\alpha_{n+m-1}^2 - \alpha_{n-m}^2\alpha_{n-m+1}^2\cdots\alpha_{n-1}^2,\ w_{m,n} := \alpha_n^2\alpha_{n+1}^2\cdots\alpha_{n+m-2}^2(\alpha_{n+m-1}^2 - \alpha_{n-1}^2)^2, \end{aligned} \tag{2.2}$$

with $\alpha_{-m} = \alpha_{-m+1} = \cdots = \alpha_{-1} = 0$ for our convenience. It is obvious that W_α is semi-weakly m-hyponormal if and only if $D_n^{[m]}(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$. By changing the basis of $\mathbb{C}^{m+1}$, we can see that $D_n^{[m]}(t)$ in (2.1) is unitarily equivalent to $\oplus_{j=0}^{m-2} D_{\ell,j}^{[m]}(t)$ for $t := |s|^2$ and $\ell := [\frac{n}{m-1}]$, where

$$D_{\ell,j}^{[m]}(t) = \begin{pmatrix} \check{q}_{0,j} & \check{z}_{0,j} & 0 & & & \\ \check{z}_{0,j} & \check{q}_{1,j} & \check{z}_{1,j} & 0 & & \\ 0 & \check{z}_{1,j} & \check{q}_{2,j} & \check{z}_{2,j} & \ddots & \\ & 0 & \check{z}_{2,j} & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \check{q}_{k-1,j} & \check{z}_{k-1,j} \\ & & & 0 & \check{z}_{k-1,j} & \check{q}_{k,j} \end{pmatrix} \tag{2.3}$$

and k is an integer ℓ or $\ell - 1$ satisfying $k(m-1) + j \leq n$ $(j = 0, 1, ..., m-2)$,

$$\begin{aligned} &\check{q}_{i,j} \equiv q_{m,i(m-1)+j} = u_{m,i(m-1)+j} + v_{m,i(m-1)+j}t \equiv \check{u}_{i,j} + \check{v}_{i,j}t, \\ &\check{z}_{i,j} \equiv z_{m,i(m-1)+j} = \sqrt{w_{m,i(m-1)+j}t} \equiv \sqrt{\check{w}_{i,j}t}\ \ (i = 0, 1, ..., k). \end{aligned} \tag{2.4}$$

It is clear that $D_{\ell,j}^{[m]}(t) \geq 0$ for every $0 \leq j \leq m-2$ and $n \geq 0$ is equivalent to $D_n(t) \geq 0$ for $n \geq 0$. To detect the positivity of each matrix $D_{\ell,j}^{[m]}(t)$ in (2.3), we will use Sylvester's Criterion (which is sometimes called the *Nested Determinants Test*, see [4]). Denote

$$d_{\ell,j}^{[m]}(t) := \det D_{\ell,j}^{[m]}(t) = \sum_{i=0}^{n+1} c_j^{[m]}(n,i)t^i.$$

If we follow the method in [3], then we can obtain that

$$\begin{aligned} &c_j^{[m]}(0,0) = \check{u}_{0,j},\ c_j^{[m]}(0,1) = \check{v}_{0,j}, \\ &c_j^{[m]}(1,0) = \check{u}_{0,j}\check{u}_{1,j},\ c_j^{[m]}(1,1) = \check{u}_{0,j}\check{v}_{1,j} + \check{u}_{1,j}\check{v}_{0,j} - \check{w}_{0,j},\ c_j^{[m]}(1,2) = \check{v}_{0,j}\check{v}_{1,j}, \\ &c_j^{[m]}(n,i) = \check{u}_{n,j}c_j^{[m]}(n-1,i) + \check{v}_{n,j}c_j^{[m]}(n-1,i-1) - \check{w}_{n-1,j}c_j^{[m]}(n-2,i-1), \\ &c_j^{[m]}(n,n+1) = \check{v}_{0,j}\check{v}_{1,j}\cdots\check{v}_{n,j},\ \text{ for all } n \geq 2 \text{ and } 0 \leq i \leq n, \end{aligned} \tag{2.5}$$

with $c_j^{[m]}(-n,-i) := 0$ for all $n, i \in \mathbb{N}$.

We recall that a hyponormal weighted shift W_α has *positive determinant coefficients* ($\equiv$ p.d.c.) of order m for some $m \geq 2$ if all coefficients in $d_{\ell,j}^{[m]}$ for all $j = 0, 1, ..., m-2$ are nonnegative and at least one (in each) is positive ([9]). It is obvious that for a weighted shift W_α, if W_α is semi-weakly m-hyponormal with p.d.c, then W_α is clearly semi-weakly m-hyponormal.

Now we consider an one-step backward extension of (Bergman) weighted shift $W_{\alpha(x)}$ with a weight sequence $\alpha(x)$,

$$\alpha(x):\ \sqrt{x},\ \sqrt{\frac{3}{4}},\ \sqrt{\frac{4}{5}},\cdots,\ \sqrt{\frac{k+2}{k+3}}\ (k \geq 1). \tag{2.6}$$

From simple computations via (2.2) and (2.4), we have

$$\check{u}_{n+1,j}\check{v}_{n,j} = u_{(n+1)(m-1)+j}v_{n(m-1)+j} = w_{n(m-1)+j} = \check{w}_{n,j} \quad (n \geq 2; 0 \leq j \leq m-2),$$

which induces the recurrence formula of coefficients $c_j^{[m]}(n,i)$ for $n \geq 3$:

$$c_j^{[m]}(n,i) = \begin{cases} \check{v}_{n,j}c_j^{[m]}(n-1,i-1), & \text{if } 3 \leq i \leq n+1, \\ \check{v}_{n,j}c_j^{[m]}(n-1,i-1) + \check{u}_{n,j}\cdots\check{u}_{3,j}h_{j,i}^{[m]}, & \text{if } i = 1,2, \\ \check{u}_{0,j}\cdots\check{u}_{n,j}, & \text{if } i = 0, \end{cases} \tag{2.7}$$

where $h_{j,i}^{[m]} := \check{u}_{2,j}c_j^{[m]}(1,i) - \check{w}_{1,j}c_j^{[m]}(0,i-1)$ for $i = 1,2$. In particular for the cases of $i = 2$ and $j \neq 0,1$, from definitions in (2.4), we have

$$\check{u}_{2,j}\check{v}_{1,j} = u_{m,2(m-1)+j}v_{m,(m-1)+j} = \frac{m^2}{(2m+j)(m+j+1)(2m+j+1)^2} = \check{w}_{1,j},$$

which forces that $h_{j,2}^{[m]} = 0$ for all $j = 2, ..., m-2$.

Now using (2.7), if we follow similar methods in [4] via a little monotonous computations, then we can have the following result which plays a crucial role in the proof of Theorem 2.3. (see Section 5 for the rigorous proof.)

Theorem 2.1. *Let $W_{\alpha(x)}$ be a weighted shift with $\alpha(x)$ in (2.6). Then $W_{\alpha(x)}$ is semi-weakly m- hyponormal with p.d.c. if and only if $0 < x \leq \min\{\frac{3}{4}, f(m)\}$, where*

$$f(m) := \frac{3(m^5 - m^4 + 4m^2 + 24m + 8)}{2(2m^5 - m^4 - 4m^3 + 3m^2 + 54m + 18)}.$$

Corollary 2.2. *Let $W_{\alpha(x)}$ be a weighted shift with $\alpha(x)$ in (2.6).*
(i) *If $W_{\alpha(x)}$ is m-hyponormal, then $W_{\alpha(x)}$ is semi-weakly m- hyponormal with p.d.c.*
(ii) *If $0 < x \leq \min\{\frac{3}{4}, f(m)\}$, then $W_{\alpha(x)}$ is semi-weakly m-hyponormal for $m \geq 3$.*
(iii) *$W_{\alpha(x)}$ is hyponormal if and only if $W_{\alpha(x)}$ is semi-weakly 3-hyponormal [or with p.d.c.] and also is equivalent to $W_{\alpha(x)}$ is semi-weakly 4-hyponormal [or with p.d.c.].*

Proof. (i) It follows from the result in [6] that $W_{\alpha(x)}$ is m-hyponormal is equivalent to the condition

$$0 < x \leq \frac{2(m+1)^2(m+2)^2}{3m(m+3)(m^2+3m+4)} \equiv H(m) \quad (m \geq 1).$$

From a computation, we have

$$f(m) - H(m) = \frac{(m-1)(m^8 + 2m^7 + m^6 + 68m^5 + 328m^4 + 848m^3 + 1200m^2 + 1152m + 288)}{6m(m+3)(m^2+3m+4)(2m^5 - m^4 - 4m^3 + 3m^2 + 54m + 18)} > 0,$$

for all $m \geq 3$, which induces the conclusion.

(ii) It is obvious from (i).

(iii) We note that $W_{\alpha(x)}$ is hyponormal $\Leftrightarrow 0 < x \leq \frac{3}{4}$. By a computation, we get $f(3) = \frac{139}{168} > f(4) = \frac{78}{101} > \frac{3}{4}$, which induces the results. □

Theorem 2.3. *Let $W_{\alpha(x)}$ be a weighted shift with $\alpha(x)$ in* (2.6). *If $0 < x \leq \frac{13\,259}{18\,228}$, then $W_{\alpha(x)}$ is completely semi-weakly hyponormal. Moreover, if $\frac{2}{3} < x \leq \frac{13\,259}{18\,228}$, then $W_{\alpha(x)}$ is not subnormal but completely semi-weakly hyponormal.*

Proof. We consider the function $f(m)$ on an interval $[3, \infty)$. It is easy to see that there is a unique $\delta_0(\approx 8.9645)$ such that $f(x)$ is decreasing on $[3, \delta_0]$, and $f(x)$ is increasing on $[\delta_0, \infty)$. Since $f(8) = \frac{21\,846}{30\,017} > f(9) = \frac{13259}{18228}$, $f(m) \geq f(9)$ for all $m \geq 3$. From the result in [6], $W_{\alpha(x)}$ is subnormal if and only if $0 < x \leq \frac{2}{3}$. Hence if $\frac{2}{3} < x \leq \frac{13\,259}{18\,228}$, by Theorem 2.1, $W_{\alpha(x)}$ is semi-weakly m-hyponormal with p.d.c. for all $m \geq 3$, so $W_{\alpha(x)}$ is completely semi-weakly hyponormal but not subnormal. □

3. Gaps Between Semi-Weak *m*-Hyponormality and Semi-Weak *m*-Hyponormality with p.d.c.

In this section we give an example of weighted shifts with Bergman tail, which separates semi-weak m-hyponormality from semi-weak m-hyponormality with p.d.c. for some $m \geq 5$ due to Theorem 2.1. First, we give the useful result in [9] as follows.

Lemma 3.1. ([9, Corollary 3.3]) *Let $\alpha(x, y) : \sqrt{y}, \sqrt{x}, \sqrt{(k+1)/(k+2)}$ $(k \geq 2)$ with $0 < y \leq x \leq 3/4$ and let $n \geq 4$. Then $W_{\alpha(x,y)}$ is semi-weakly n-hyponormal with p.d.c. if and only if it holds that*

$$0 < x \leq \min\{g(n), 3/4\} \;\; and \;\; 0 < y \leq \min\{x, f_1^{[n]}(x), f_2^{[n]}(x)\},$$

where $g(n) = \frac{3(n^5-n^4+4n^2+24n+8)}{4n^5-2n^4-8n^3+6n^2+108n+36}$, *and*

$$f_1^{[n]}(x) = \frac{4n+2+x\left(n^4-2n^2+1\right)}{(n+2)\left(n^3+4n^2+5n+2-x\left(12n^2+18n+6\right)+x^2\left(6n^2+15n+6\right)\right)},$$

$$f_2^{[n]}(x) = \frac{x\left(n^4-2n^3+2n^2+2n+9\right)}{n^4+4n^3+5n^2+2n-x\left(12n^3+18n^2+6n\right)+x^2\left(6n^3+15n^2+6n+27\right)}.$$

Remark 3.2. In Lemma 3.1, if we consider the cases $n \geq 5$, then the function g is exactly same to the function f on Theorem 2.1. In particular, for cases of $n \geq 5$, if we take $y = 0$ in Lemma 3.1, we obtain the same result in Theorem 2.1. However we note that two models, $\alpha(x, y)$ in Lemma 3.1 and $\alpha(x)$ in (2.6) show a little different sides, subnormality or semi-weak 3 [or semi-weak 4]-hyponormality of corresponding weight shifts $W_{\alpha(x,y)}$ and $W_{\alpha(x)}$. In fact, $W_{\alpha(x,y)}$ is subnormal if and only if $0 \leq y \leq \frac{1}{2}$ and $x = \frac{2}{3}$ (cf. [10]), but $W_{\alpha(x)}$ is subnormal if and only if $0 \leq x \leq \frac{2}{3}$. Also we can see from Corollary 2.2 that the hyponormality of $W_{\alpha(x)}$ with $\alpha(x)$ in (2.6) is equivalent to the semi-cubic [and semi-quartic] hyponormality.

From the method in Lemma 3.1, we have the following results.

Proposition 3.3. *Let $W_{\alpha(x,x)}$ be a weighted shift with $\alpha(x, x) : \sqrt{x}, \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$. Then the followings hold:*

(i) *$W_{\alpha(x,x)}$ is semi-weakly 5-hyponormal with p.d.c.* $\Longleftrightarrow \frac{165-\sqrt{433}}{197} \leq x \leq \frac{1023}{1372}$.

(ii) *$W_{\alpha(x,x)}$ is semi-weakly 6-hyponormal with p.d.c.* $\Longleftrightarrow 0 < x \leq \frac{1694}{2307}$.

Proof. Without loss of generality, we assume that $0 < x \leq \frac{3}{4}$. First, from a direct computation, it holds that $g(m) < \frac{3}{4}$ for $m \geq 5$, which can reduce the range of x, $0 < x \leq g(5)$ for (i) and $x \leq g(6)$ for (ii), respectively. In order to use Lemma 3.1, we show the inequality $f_i^{[m]}(x) \geq x$ for $m = 5, 6$ and $i = 1, 2$ on an interval of x.

(i) It follows from some computations that for $0 < x \le 3/4$,

$$f_1^{[5]}(x) - f_2^{[5]}(x) = -\frac{2(3093x^3 - 9691x^2 + 8418x - 2310)}{21(77x^2 - 132x + 84)(197x^2 - 330x + 210)} > 0.$$

Also we have

$$f_2^{[5]}(x) - x = -\frac{x(197x^2 - 330x + 136)}{197x^2 - 330x + 210} \equiv \frac{-xp_1(x)}{q_1(x)}.$$

Since $q_1(x) > 0$ for all $x > 0$ and $p_1(x)$ has two roots $\frac{165 \mp \sqrt{433}}{197}$, we have $f_2^{[5]}(x) \ge x$ for $\frac{165-\sqrt{433}}{197} (\approx 0.7319) \le x \le \frac{3}{4}$. Using the first reduction of x, i.e. $0 < x \le g(5) = \frac{1023}{1372}$, we can see that $f_2^{[5]}(x) \ge x$ for $\frac{165-\sqrt{433}}{197} \le x \le \frac{1023}{1372}$, which induces our result.

(ii) To show (ii), we follow the previous method. From some calculations,

$$f_1^{[6]}(x) - f_2^{[6]}(x) = -\frac{20799x^3 - 72150x^2 + 68376x - 20384}{16(156x^2 - 273x + 196)(633x^2 - 1092x + 784)} > 0,$$

$$f_2^{[6]}(x) - x = -\frac{3x(211x^2 - 364x + 155)}{633x^2 - 1092x + 784} \equiv \frac{-3xp_2(x)}{q_2(x)}.$$

Since $q_2(x) > 0$ for $x > 0$ and $p_2(x) > 0$ for $0 < x \le \frac{3}{4}$, we have $f_2^{[6]}(x) < x$. From the range of x, $0 < x \le g(6) = \frac{1694}{2307}$, we proves this result. □

Corollary 3.4. *Let θ be any value in the interval $\left[\frac{165-\sqrt{433}}{197}, \frac{1023}{1372}\right]$ and let $W_{\alpha(x,\theta)}$ be a weighted shift with $\alpha(x,\theta)$: $\sqrt{x}$, $\sqrt{\theta}$, $\sqrt{3/4}$, $\sqrt{4/5}, \cdots$. Then the followings are equivalent:*
(i) *$W_{\alpha(x,\theta)}$ is semi-weakly 5-hyponormal with p.d.c.;*
(ii) *$W_{\alpha(x,\theta)}$ is semi-weakly 5-hyponormal;*
(iii) *$W_{\alpha(x,\theta)}$ is hyponormal;*
(iv) *$0 < x \le \theta$.*

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv): These implications are trivial.

Now we sufficiently to prove that (iv) ⇒ (i). Suppose $\frac{165-\sqrt{433}}{197} \le \theta \le \frac{1023}{1372}$. Using some computations in the proof of Proposition 3.3 (i), we have $\theta \le g(5) = \frac{1023}{1372}$ and $\theta \le f_2^{[5]}(\theta) \le f_1^{[5]}(\theta)$. It follows from Lemma 3.1 that $W_{\alpha(x,\theta)}$ is semi-weakly 5-hyponormal with p.d.c. ⇔ $0 < x \le \theta$. So our proof is completed. □

From Proposition 3.3 and Corollary 3.4, we can produce an interval of x with non-empty interior in the positive real line for semi-weak 6-hyponormality but not semi-weak 6-hyponormality with p.d.c. for such a shift.

Proposition 3.5. *Let $W_{\alpha(x)}$ be a weighted shift with $\alpha(x)$: $\sqrt{x}$, $\sqrt{\frac{183}{250}}$, $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{4}{5}}, \cdots$. Set*

$$s\text{-}\mathcal{H}_6 = \{x : W_{\alpha(x)} \text{ is semi-weakly 6-hyponormal}\},$$

$$s\text{-}\widehat{\mathcal{H}}_6 = \{x : W_{\alpha(x)} \text{ is semi-weakly 6-hyponormal with p.d.c.}\}.$$

Then it holds that $s\text{-}\mathcal{H}_6 \backslash s\text{-}\widehat{\mathcal{H}}_6 = \left(\frac{14594250}{20239537}, \frac{183}{250}\right]$.

Proof. For $0 < x \le \frac{183}{250}$, from Proposition 3.3 (ii), $W_{\alpha(x)}$ is semi-weakly 6-hyponormal. Since $f_1^{[6]}(\frac{183}{250}) = \frac{28834375}{39876272} > f_2^{[6]}(\frac{183}{250}) = \frac{14594250}{20239537}$ and $g(6) = \frac{1694}{2307}$, by Lemma 3.1, we obtain that $W_{\alpha(x)}$ is semi-weakly 6-hyponormal with p.d.c. ⇔ $0 < x \le \frac{14594250}{20239537}$. Thus the interval $\left(\frac{14594250}{20239537}, \frac{183}{250}\right]$ is a range in x for semi-weak 6-hyponormality but not semi-weak 6-hyponormality with p.d.c. of $W_{\alpha(x)}$. □

4. Flatness

In this section we consider the flatness of semi-weakly m-hyponormal weighted shifts for $m \geq 3$. First, we note two principal submatrices in (2.3) as followings:

$$D_1 = \begin{pmatrix} q_{m,0} & z_{m,0} & 0 \\ \bar{z}_{m,0} & q_{m,m-1} & z_{m,m-1} \\ 0 & \bar{z}_{m,m-1} & q_{m,2m-2} \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} q_{m,1} & z_{m,1} \\ \bar{z}_{m,1} & q_{m,m} \end{pmatrix}, \tag{4.1}$$

where $\{q_{m,i}\}$ and $\{z_{m,i}\}$ are given in (2.2).

Theorem 4.1. *Let W_α be a hyponormal weighted shift with $\alpha = \{\alpha_i\}_{i=0}^\infty$ and $\alpha_0 = \alpha_1 = 1$. If W_α is semi-weakly m-hyponormal, then $\left(2 - \alpha_{m-1}^2\right)\alpha_m^2 \geq 1$.*

Proof. Suppose that W_α is semi-weakly m-hyponormal. It follows from $D_1 \geq 0$ and $D_2 \geq 0$ in (4.1) that

$$q_{m,m-1}q_{m,0} - z_{m,0}^2 \geq 0 \quad\text{and}\quad q_{m,m}q_{m,1} - z_{m,1}^2 \geq 0.$$

From the assumption of hyponormality of W_α,

$$\alpha_m^2\alpha_{m+1}^2\cdots\alpha_{2m-2}^2 - \alpha_1^2\alpha_2^2\cdots\alpha_{m-3}^2\alpha_{m-2}^4 > 0,$$

for all $m \geq 3$, so we have

$$\frac{q_{m,m-1}q_{m,0} - z_{m,0}^2}{\alpha_0^2} = \alpha_{m-1}^2 - \alpha_{m-2}^2 + \alpha_1^2\alpha_2^2\cdots\alpha_{m-2}^2\alpha_{m-1}^4\alpha_m^2\cdots\alpha_{2m-2}^2\, t^2 + \alpha_{m-1}^2\left(\alpha_m^2\cdots\alpha_{2m-2}^2 - \alpha_1^2\alpha_2^2\cdots\alpha_{m-3}^2\alpha_{m-2}^4\right)t \geq 0,$$

for all $t > 0$. Moreover

$$\begin{aligned} \lim_{t\to 0+}\frac{q_{m,m}q_{m,1} - z_{m,1}^2}{\alpha_2^2\alpha_3^2\cdots\alpha_{m-1}^2 t} &= \lim_{t\to 0+}\left(\left(\alpha_m^2\alpha_{m+1}^2\cdots\alpha_{2m-1}^2 - \alpha_2^2\cdots\alpha_{m-2}^2\alpha_{m-1}^2\right)\alpha_m^2 t - \alpha_m^2\alpha_{m-1}^2 + 2\alpha_m^2 - 1\right) \\ &= \left(2 - \alpha_{m-1}^2\right)\alpha_m^2 - 1 \geq 0, \end{aligned}$$

which induces that $\left(2 - \alpha_{m-1}^2\right)\alpha_m^2 \geq 1$. □

Corollary 4.2. *Let W_α be a completely semi-weakly hyponormal weighted shift with $\alpha_0 = \alpha_1 = 1$. Then W_α is flat, i.e., $\alpha_n = 1$ for all $n \in \mathbb{N}$.*

Proof. Put $a := \lim_{n\to\infty}\alpha_n$. Since W_α is semi-weakly m-hyponormal for all $m \geq 3$, $\left(2 - \alpha_{m-1}^2\right)\alpha_m^2 \geq 1$ for all m, which implies that $\left(2 - a^2\right)a^2 \geq 1$, i.e. $\left(a^2 - 1\right)^2 \leq 0$. Hence $a = 1$. Thus we have our conclusion. □

Example 4.3. Let W_α be a weighted shift with $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots$. Then W_α is a semi-cubically hyponormal but not semi-weakly m-hyponormal for any $m \geq 4$. In fact, this result is known in [9, Proposition 3.8]. In this example, we show simple method to check the result for $m \geq 5$. Denote a weight sequence $\beta = \{\beta_i\}_{i=0}^\infty$, where $\beta_n := \sqrt{\frac{3}{2}}\alpha_n$ $(n \geq 0)$. Then $\beta_0 = 1$, $\beta_n^2 = \frac{3(n+1)}{2(n+2)}$ $(n \geq 1)$. Since

$$\beta_m^2(2 - \beta_{m-1}^2) - 1 = \frac{4-m}{4(m+2)},$$

using Theorem 4.1, the corresponding weighted shift W_β is not semi-weakly m-hyponormal with $m > 4$, which induces our conclusion.

Example 4.4. Let W_α be a weighted shift with $\alpha : \sqrt{\frac{8}{9}}, 1, 1, \sqrt{\frac{4(n+2)}{3(n+3)}}$ $(n \geq 3)$. For $m \geq 3$ and $n \in \mathbb{N}_0$, denote $d_n^{[m]}(t)$ for the determinant of the matrix $D_n^{[m]}(t)$ in (2.1). For the cases of $m = 3$ and $m = 4$, by simple computations, we have

$$d_4^{[3]}(t) = \frac{320t(567 + 13812t + 143360t^2)(-2062071 + 35408688t + 256901120t^2)}{828805165333299},$$
$$d_5^{[4]}(t) = \frac{640t(217088t - 837)(297 + 17214t + 286720t^2)(11907 + 85392t + 286720t^2)}{604198965527974971}.$$

Then $d_4^{[3]}(t) < 0$ for $t < \delta$, where $\delta(\approx 0.0441)$ is the positive solution of the equation $256901120t^2 + 35408688t - 2062071 = 0$. So W_α is not semi-cubically hyponormal. And also $d_5^{[4]}(t) < 0$ for $t < \widetilde{\delta}$, where $\widetilde{\delta}(\approx 0.0039)$ is the solution of the equation $217088t - 837 = 0$. So W_α is not semi-quartically hyponormal.

Further for cases of $m \geq 5$, we use the similar methods above. Put

$$\Phi_{m+1}^{[m]}(t) := d_{m+1}^{[m]}(t)/q_{m,3} \cdots q_{m,m-3}q_{m,m-2}.$$

Then $\Phi_{m+1}^{[m]}(t) = \left(q_{m,0}q_{m,m-1} - z_{m,0}^2\right)\left(q_{m,1}q_{m,m} - z_{m,1}^2\right)\left(q_{m,2}q_{m,m+1} - z_{m,2}^2\right)$. Using the definitions in (2.2), each $q_{m,i}$ is strictly positive for all $i \geq 0$. From some computations containing with $\alpha_1 = 1 = \alpha_2$, we can see

$$\lim_{t \to 0} \frac{\Phi_{m+1}^{[m]}(t)}{t\alpha_0^2\alpha_3^2\alpha_4^2\alpha_5^2} = (\alpha_{m-1}^2 - \alpha_{m-2}^2)(1 - \alpha_0^2)(\alpha_m^2 - \alpha_{m-1}^2)\phi^{[m]},$$

where $\phi^{[m]} = \alpha_6^2(\alpha_{m+1}^2 - \alpha_m^2) - (\alpha_6^2 - 1)^2$. Since

$$\phi^{[m]} = \frac{852 - 175m - 25m^2}{729(3+m)(4+m)} < 0 \text{ for } m \geq 4,$$

from $\alpha_{n+1} \geq \alpha_n$ $(n \geq 2)$, $\Phi_{m+1}^{[m]}(t) < 0$ for some $t > 0$. Hence $d_{m+1}^{[m]}(t) \not\geq 0$ for all $t > 0$, which induces that W_α is not semi-weakly m-hyponormal for each $m \geq 5$.

Example 4.5. Consider a weighted shift W_α with $\alpha : \sqrt{\frac{8}{9}}, 1, 1, 1, \sqrt{\frac{4n+8}{3n+9}}$ $(n \geq 4)$. Then from simple computations,

$$d_5^{[4]}(t) = \det D_5^{[4]} = \frac{2048t^2(61 + 224t)(-837 + 235520t)(891 + 20078t + 172032t^2)}{1381341942222165}.$$

So we have $d_5^{[4]}(t) < 0$ for $0 < t < \delta$, where $\delta(\approx 0.00355)$ is the solution of $235520t - 837 = 0$. Hence W_α is not semi-weakly 4-hyponormal.

Theorem 4.6. *Let W_α be a semi-weakly m-hyponormal weighted shift with $\alpha = \{\alpha_i\}_{i=0}^\infty$. If $\alpha_n = \alpha_{n+1} = \cdots = \alpha_{n+2m-5}$ for some $n \in \mathbb{N}$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

Proof. By the following Lemma 4.7 and Lemma 4.8, we prove it. □

Lemma 4.7. (Outer propagation) *Let W_α be a semi-weakly m-hyponormal. If $\alpha_n = \alpha_{n+1} = \cdots = \alpha_{n+2m-5}$ for some $n \in \mathbb{N}$, then $\alpha_{n+k} = \alpha_n$, for all $k \geq 1$.*

Proof. Since the restriction of a semi-weakly m-hyponormal operator $(m \geq 3)$ to an invariant subspace is also semi-weakly m-hyponormal, we are sufficient to prove the result for the case $n = 1$. Suppose that $\alpha_1 = \alpha_2 = \cdots = \alpha_{2m-4} = 1$. From the hypothesis of semi-weak m-hyponormality of W_α, we note that the first matrix D_1 in (4.1) is positive, so $\det D_1 \geq 0$ for any $t > 0$. By a computation, we have

$$\lim_{t \to 0^+} \frac{\det D_1}{t} = -\alpha_0^2\left(\alpha_{2m-3}^2 - 1\right)^2 \alpha_{2m-2}^2 \geq 0,$$

which induces that $\alpha_{2m-3} = 1$, so $\alpha_1 = \cdots = \alpha_{2m-4} = \alpha_{2m-3} = 1$. Continuing the above methods, we obtain the result via mathematical induction. □

Lemma 4.8. (Inner propagation) *Let W_α be a semi-weakly m-hyponormal. If $\alpha_n = \alpha_{n+1} = \cdots = \alpha_{n+2m-5}$ for some $n \in \mathbb{N}$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.*

Proof. Without loss of generality, we assume that $n = 2$, i.e., $\alpha_2 = \alpha_3 = \cdots = \alpha_{2m-3} = 1$. By Lemma 4.7, we can have $\alpha_n = 1$ for all $n \geq 2$. Now we are sufficient to show that $\alpha_1 = 1$. From the hypothesis of semi-weak m-hyponormality of W_α, we note that the second matrix D_2 in (4.1) is positive, so $\det D_2 \geq 0$ for any $t \geq 0$. By a computation, we have

$$\lim_{t \to 0^+} \frac{\det D_2}{t} = -\alpha_0^2 \left(\alpha_1^2 - 1\right)^2 \geq 0,$$

which implies that $\alpha_1 = 1$. □

Corollary 4.9. *Assume that W_α is semi-cubically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \in \mathbb{N}$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

Corollary 4.10. *Assume that W_α is semi-weakly 4-hyponormal. If $\alpha_n = \alpha_{n+1} = \alpha_{n+2} = \alpha_{n+3}$ for some $n \in \mathbb{N}$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

5. Proof of Theorem 2.1

Proof of Theorem 2.1. From the definitions, we will find equivalent conditions to $c_j^{[m]}(n,i) \geq 0$ for all $n \geq 0$, $0 \leq i \leq n+1$ and $0 \leq j \leq m-2$. First, we note that by (2.5), $c_j^{[m]}(n,0) = \check{u}_{0,j} \cdots \check{u}_{n,j} > 0$ and $c_j^{[m]}(n,n+1) = \check{v}_{0,j} \cdots \check{v}_{n,j} > 0$ for all $n \geq 0$ and $0 \leq j \leq m-2$. So we only consider cases of $n \geq 1$ and $1 \leq i \leq n$ for $j = 0, 1, ..., m-2$. For our convenience, we may omit coding j ($j = 0, 1, ..., m-2$) of $\check{u}_{n,j}$, $\check{v}_{n,j}$ and $\check{w}_{n,j}$ in the expression of coefficients $c_j^{[m]}(n,i)$.

Now we consider to check the positivity of $c_j^{[m]}(n,i)$ for $j = 2, ..., m-2$ (i.e. $j \neq 0, 1$). From easy computations,

$$c_j^{[m]}(1,1) = \frac{m^3 - (j+4)m^2 + (j+3)^2 m + j^3 + 6j^2 + 11j + 6}{(j+2)(j+3)(m+j+1)(m+j+2)(2m+j+1)},$$

using the positivity of numerator in $c_j^{[m]}(1,1)$ for $m \geq 3$, $c_j^{[m]}(1,1) > 0$. It follows from a direct computation that

$$\check{v}_2 \check{u}_1 - \check{w}_1 = \frac{m^2(m-1)^2}{(2m+j)(3m+j)(m+j+1)(m+j+2)(2m+j+1)^2} > 0,$$

which induces that $c_j^{[m]}(2,1) = \check{u}_2 c_j^{[m]}(1,1) + \check{u}_0(\check{v}_2 \check{u}_1 - \check{w}_1) > 0$. Since $c_j^{[m]}(2,2) = h_{j,2}^{[m]} + \check{v}_2 c_j^{[m]}(1,1)$ in (2.5), using the facts $h_{j,2}^{[m]} = 0$ ($j \neq 0, 1$) in (2.7) and $c_j^{[m]}(1,1) > 0$, we have $c_j^{[m]}(2,2) > 0$. For all $n \geq 3$ and $2 \leq j \leq m-2$, using (2.7), we have

$$c_j^{[m]}(n,1) = \check{v}_n c_j^{[m]}(n-1,0) = \check{v}_n \check{u}_{n-1} \cdots \check{u}_1 \check{u}_0 > 0,$$

which implies that $c_j^{[m]}(n,2) = \check{v}_n c_j^{[m]}(n-1,1) > 0$ for all $n \geq 3$.
For the case $3 \leq i \leq n$ ($n \geq 3$), from the recurrence form (2.7),

$$c_j^{[m]}(n,i) = \check{v}_n c_j^{[m]}(n-1,i-1) = \cdots = \check{v}_n \check{v}_{n-1} \cdots \check{v}_{n-i+3} c_j^{[m]}(n-i+2,2).$$

Since $n - i + 2 \geq 2$, $c_j^{[m]}(n-i+2,2) > 0$. Using the mathematical induction, $c_j^{[m]}(n,i) > 0$ for all $n \geq 2$ with $2 \leq i \leq n$ and $j = 2, 3, \cdots, m-2$.

Now we sufficiently show that $W_{\alpha(x)}$ has positive determinant coefficients(p.d.c.) of order $m \Leftrightarrow c_0^{[m]}(n,i) \geq 0$ and $c_1^{[m]}(n,i) \geq 0$ for all $n \geq 1$ with $1 \leq i \leq n$.

Claim 1°. $c_0^{[m]}(n,i) \geq 0$ for all $n \geq 1$ and $1 \leq i \leq n$.
(1°-i) $i = 1$: It follows from a direct computation via (2.5) that

$$c_0^{[m]}(1,1) = \frac{(m^3 - 2m^2 + 2m + 2)x}{(m+1)(m+2)(2m+1)} > 0,$$

$$c_0^{[m]}(2,1) = \check{u}_2 c_0^{[m]}(1,1) + \frac{\check{u}_0(m-1)^2}{6(m+1)(m+2)(2m+1)^2} > 0.$$

For $n \geq 3$, from (2.5), (2.7), and the definition of $h_{0,1}^{[m]}$ we have

$$\begin{aligned} c_0^{[m]}(n,1) &= \check{v}_n c_0^{[m]}(n-1,0) + \check{u}_n \cdots \check{u}_3 h_{0,1}^{[m]} \\ &= \check{v}_n \check{u}_0 \cdots \check{u}_{n-1} + \check{u}_n \cdots \check{u}_3 \left[\check{u}_2 c_0^{[m]}(1,1) - \check{w}_1 c_0^{[m]}(0,0) \right] \\ &= \check{u}_2 \check{u}_3 \cdots \check{u}_n c_0^{[m]}(1,1) + \check{u}_0 \check{u}_3 \cdots \check{u}_{n-1} \left(\check{u}_1 \check{u}_2 \check{v}_n - \check{w}_1 \check{u}_n \right). \end{aligned}$$

By a simple computation, we have

$$\check{u}_1 \check{u}_2 \check{v}_n - \check{w}_1 \check{u}_n = \frac{(m-1)^2 m (n-1)}{2(m+1)(m+2)(2m+1)^2 (2-n+mn)(3-n+mn)(2+m-n+mn)},$$

so $c_0^{[m]}(n,1) > 0$ for all $n \geq 3$. Hence $c_0^{[m]}(n,1) > 0$ for all $n \geq 1$.
(1°-ii) $i = 2$: From $h_{0,2}^{[m]} = (\check{u}_2 \check{v}_1 - \check{w}_1)\check{v}_0 = \check{v}_0/(2m(m+1)(2m+1))$, we have

$$c_0^{[m]}(2,2) = \check{v}_2 c_0^{[m]}(1,1) + h_{0,2}^{[m]} > 0.$$

Now for $n \geq 3$, using the recurrence form (2.7), we can obtain that

$$\begin{aligned} c_0^{[m]}(n,2) &= \check{v}_n c_0^{[m]}(n-1,1) + \check{u}_n \cdots \check{u}_3 h_{0,2}^{[m]} \\ &= \check{v}_n \left[\check{v}_{n-1} c_0^{[m]}(n-2,0) + \check{u}_{n-1} \cdots \check{u}_3 h_{0,1}^{[m]} \right] + \check{u}_n \cdots \check{u}_3 h_{0,2}^{[m]} \\ &= \check{u}_3 \cdots \check{u}_{n-2} \check{v}_n \left[\check{u}_0 \check{u}_1 \check{u}_2 \check{v}_{n-1} + \check{u}_{n-1} h_{0,1}^{[m]} \right] + \check{u}_3 \cdots \check{u}_n h_{0,2}^{[m]}. \end{aligned}$$

Put $\beta_n^{[m]} := \check{u}_0 \check{u}_1 \check{u}_2 \check{v}_{n-1} + \check{u}_{n-1} h_{0,1}^{[m]}$ $(n \geq 3)$. Then

$$\beta_n^{[m]} = \frac{x\left(n(m^3 - 3m^2 + 4m - 2) - m^3 + 4m^2 - 6m + 6\right)}{2m(m+1)(m+2)(2m+1)(mn-n+3)(mn-m-n+3)(mn-m-n+4)}.$$

Since $x > 0$ and $n \geq 3$, $\beta_n^{[m]} > 0$. Hence $c_0^{[m]}(n,2) > 0$ for all $n \geq 1$. Finally we consider $3 \leq i \leq n$ for $n \geq 3$. Also, using (2.7), we have

$$c_0^{[m]}(n,i) = \check{v}_n c_0^{[m]}(n-1,i-1) = \cdots = \check{v}_n \check{v}_{n-1} \cdots \check{v}_{n-i+3} c_0^{[m]}(n-i+2,2).$$

Since $n - i + 2 \geq 1$ and $c_0^{[m]}(n,2) > 0$ $(n \geq 1)$, $c_0^{[m]}(n-i+2,2) > 0$ for $3 \leq i \leq n$, which induces that $c_0^{[m]}(n,i) > 0$ for all $n \geq 1$ and $3 \leq i \leq n$.

Claim 2°. $c_1^{[m]}(n,i) > 0$ $(n \geq 1, 1 \leq i \leq n) \Leftrightarrow 0 < x \leq \min\{\frac{3}{4}, f(m)\}$.
(2°-i) $i = 1$: For the cases $n = 1, 2$, using (2.5), we can obtain two solutions, $g_1(m)$ and $g_2(m)$ of the linear equations $c_1^{[m]}(1,1) = 0$ and $c_1^{[m]}(2,1) = 0$, respectively, where

$$c_1^{[m]}(1,1) = \frac{3(m^3 - m^2 + 4) - 2(2m+3)(m-1)^2 x}{8(m+1)(m+2)(m+3)},$$

$$c_1^{[m]}(2,1) = \frac{6(m^3-2m^2+2m+1)-(8m+3)(m-1)^2x}{8\,(m+1)\,(m+2)\,(m+3)\,(2m+1)\,(3m+1)}.$$

Then $c_1^{[m]}(1,1) \geq 0 \Leftrightarrow x \leq g_1(m)$ and $c_1^{[m]}(2,1) \geq 0 \Leftrightarrow x \leq g_2(m)$, respectively.
For $n \geq 3$ and $i = 1$, using (2.5) and (2.7), we have

$$\begin{aligned} c_1^{[m]}(n,1) &= \check{v}_n c_1^{[m]}(n-1,0) + \check{u}_n \cdots \check{u}_3 h_{1,1}^{[m]} \\ &= \check{u}_3 \cdots \check{u}_n \left[\check{u}_0\check{u}_1\check{u}_2\check{v}_n/\check{u}_n + h_{1,1}^{[m]}\right] \equiv \check{u}_3 \cdots \check{u}_n \Theta_n^{[m]}(x). \end{aligned}$$

Denote $\check{\eta}_n$ for $\frac{\check{v}_n}{\check{u}_n}$ $(n \geq 3)$. From definitions in (2.4), $\{\check{\eta}_n\}$ is increasing. In particular, for each j, $\check{\eta}_n = \frac{\check{v}_{n,j}}{\check{u}_{n,j}} \left(= \frac{v_{n(m-1)+j}}{u_{n(m-1)+j}}\right) \nearrow m^2$ $(n \to \infty)$. From a direct computation,

$$\Theta_3^{[m]}(x) = \check{u}_0\check{u}_1\check{u}_2\check{\eta}_3 + h_{1,1}^{[m]} = \frac{3m^2-7m+8-4(m-1)^2x}{32(m+1)(m+2)(m+3)(2m+1)},$$

using $\check{\eta}_{n+1} \geq \check{\eta}_n$ $(n \geq 3)$, $\check{u}_3 \cdots \check{u}_n > 0$ and $0 < x \leq \frac{3}{4}$, we see

$$c_1^{[m]}(n,1) \geq 0 \ (n \geq 3) \iff \Theta_3^{[m]}(x) \geq 0 \iff 0 < x \leq \min\{3/4, g_3(m)\},$$

where $g_3(m)$ is the solution of the equation $\Theta_3^{[m]}(x) = 0$. Moreover from simple calculations, it holds that $g_i(m) > \frac{3}{4}$ for $m = 3, 4$ and $g_i(m) \leq \frac{3}{4}$ for $m \geq 5$ $(i = 1, 2, 3)$. Further, we get the followings:

$$g_1(m) - g_2(m) = \frac{3m^2(5-m)}{2(2m+3)(8m+3)(m-1)^3}, \quad g_3(m) - g_1(m) = \frac{m(m-5)}{4(2m+3)(m-1)^2},$$

which induce $g_1(m) \leq g_2(m)$ and $g_1(m) \leq g_3(m)$ for all $m \geq 5$.
Hence $c_1^{[m]}(n,1) \geq 0$ for all $n \geq 1 \Leftrightarrow 0 < x \leq \min\{\frac{3}{4}, g_1(m)\}$.
(2°-ii) $i = 2$: It is obvious that $c_1^{[m]}(1,2) = \check{v}_1\check{v}_0 > 0$. Write $\varphi^{[m]}(x) \equiv c_1^{[m]}(2,2)$ for convenience. By a direct computation via (2.5),

$$\varphi^{[m]}(x) = \frac{3(m^5-m^4+4m^2+24m+8)-2(2m^5-m^4-4m^3+3m^2+54m+18)x}{8(m+1)(m+2)(m+3)(2m+1)(3m+1)}.$$

From the assumption of $0 < x \leq \frac{3}{4}$, we have $\varphi^{[m]}(x) \geq 0 \Leftrightarrow 0 < x \leq \min\{\frac{3}{4}, f(m)\}$, where $f(m)$ is the solution of $\varphi^{[m]}(x) = 0$. In fact, $f(m) > \frac{3}{4}$ for $m = 3, 4$ and $f(m) \leq \frac{3}{4}$ otherwise. Further, elementary computations induce that for $m \geq 5$,

$$g_1(m) - f(m) = \frac{3(3m+1)p(m)}{(m-1)^2(2m+3)q(m)},$$

where $p(m) = m^3-5m^2+16m+24$ and $q(m) = 2m^5-m^4-4m^3+3m^2+54m+18$. Indeed, $p'(m) > 0$ and $q'(m) > 0$ $(m \geq 5)$. Then $p(m)$ and $q(m)$ are strictly positive increasing functions, which implies that $g_1(m) > f(m)$ for $m \geq 5$. Hence the condition of $0 < x \leq \min\{\frac{3}{4}, f(m)\}$ guarantees $c_1^{[m]}(2,2) \geq 0$ and $c_1^{[m]}(n,1) \geq 0$ for all $n \geq 1$.
Next we consider $n \geq 3$. Using (2.7), we can obtain that

$$\begin{aligned} c_1^{[m]}(n,2) &= \check{v}_n c_1^{[m]}(n-1,1) + \check{u}_n \cdots \check{u}_3 h_{1,2}^{[m]} \\ &= \check{v}_n\check{v}_{n-1} c_1^{[m]}(n-2,0) + \check{v}_n\check{u}_{n-1} \cdots \check{u}_3 h_{1,1}^{[m]} + \check{u}_n \cdots \check{u}_3 h_{1,2}^{[m]} \\ &= \check{u}_3 \cdots \check{u}_n \left[\frac{\check{u}_0\check{u}_1\check{u}_2\check{v}_{n-1}\check{v}_n}{\check{u}_{n-1}\check{u}_n} + \frac{\check{v}_n}{\check{u}_n} h_{1,1}^{[m]} + h_{1,2}^{[m]}\right]. \end{aligned}$$

Put $F^{[m]}(\check{\eta}_{n-1},\check{\eta}_n)=\check{u}_0\check{u}_1\check{u}_2\check{\eta}_{n-1}\check{\eta}_n+\check{\eta}_n h_{1,1}^{[m]}+h_{1,2}^{[m]}$ with $\check{\eta}_n=\frac{\check{v}_n}{\check{u}_n}$ for $n\geq 3$. Then

$$F^{[m]}(\check{\eta}_n,\check{\eta}_{n+1})-F^{[m]}(\check{\eta}_{n-1},\check{\eta}_n)=(\check{\eta}_{n+1}-\check{\eta}_n)(\xi_1\phi_n+\xi_2),$$

where $\xi_1:=\check{u}_0\check{u}_1\check{u}_2$, $\xi_2:=h_{1,1}^{[m]}$ and $\phi_n:=\check{\eta}_{n+1}\left(\frac{\check{\eta}_n-\check{\eta}_{n-1}}{\check{\eta}_{n+1}-\check{\eta}_n}\right)+\check{\eta}_{n-1}$.
If $\xi_1\phi_n+\xi_2\geq 0$, then $F^{[m]}(\check{\eta}_{n-1},\check{\eta}_n)$ is increasing for $n\geq 3$. So

$$F^{[m]}(\check{\eta}_2,\check{\eta}_3)\leq F^{[m]}(\check{\eta}_3,\check{\eta}_4)\leq\cdots\leq F^{[m]}(\check{\eta}_{n-1},\check{\eta}_n)\leq\cdots.$$

Since

$$F^{[m]}(\check{\eta}_2,\check{\eta}_3)=\frac{6(m^3-2m^2+2m+1)-(m-1)^2(8m+3)x}{32(m+1)(m+2)(m+3)(2m+1)},$$

$c_1^{[m]}(n,2)\geq 0$ $(n\geq 3)$ $\Leftrightarrow$ $F^{[m]}(\check{\eta}_2,\check{\eta}_3)\geq 0$ $\Leftrightarrow$ $0<x\leq\varphi_1(m)$, where $\varphi_1(m)$ is the solution of the equation $F^{[m]}(\check{\eta}_2,\check{\eta}_3)=0$.
If $\xi_1\phi_n+\xi_2<0$, then $F^{[m]}(\check{\eta}_{n-1},\check{\eta}_n)$ is decreasing for $n\geq 3$. Since $\lim_{n\to\infty}\check{\eta}_n=m^2$,

$$F^{[m]}(\check{\eta}_2,\check{\eta}_3)\geq\cdots\geq F^{[m]}(\check{\eta}_{n-1},\check{\eta}_n)\geq\cdots\geq F^{[m]}(m^2,m^2).$$

From a simple computation,

$$F^{[m]}(m^2,m^2)=\frac{m^2(3(m^2-2m+2)-(4m^2-7m+3)x)}{8(m+1)(m+2)(m+3)(2m+1)},$$

we know that $c_1^{[m]}(n,2)\geq 0$ for all $n\geq 3\Leftrightarrow F^{[m]}(m^2,m^2)\geq 0\Leftrightarrow 0<x\leq\varphi_2(m)$, where $\varphi_2(m)$ is the solution of $F^{[m]}(m^2,m^2)=0$. From a direct computation, we have $\varphi_i(3)>\frac{3}{4}$ and $\varphi_i(4)>\frac{3}{4}$ for $i=1,2$. Moreover, from the similar methods the above, we can obtain that $\varphi_i(m)>f(m)$ $(i=1,2)$ for all $m\geq 5$. Hence by the assumption of $0<x\leq\frac{3}{4}$, $c_1^{[m]}(n,2)\geq 0$ for all $n\geq 3\Leftrightarrow 0<x\leq\min\{\frac{3}{4},f(m)\}$.
For the final cases of $3\leq i\leq n$ and $n\geq 3$, using (2.7), we have

$$c_1^{[m]}(n,i)=\check{v}_n c_1^{[m]}(n-1,i-1)=\cdots=\check{v}_n\check{v}_{n-1}\cdots\check{v}_{n-i+3}c_1^{[m]}(n-i+2,2).$$

Since $n-i+2\geq 2$, using the above equivalence formula for $c_1^{[m]}(n,2)\geq 0$ for all $n\geq 1$, we can obtain $c_1^{[m]}(n,i)\geq 0\Leftrightarrow 0<x\leq\min\{\frac{3}{4},f(m)\}$ for all $3\leq i\leq n$ $(n\geq 3)$. Therefore we have proved completely. □

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