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Nonuniform Wavelet Packets on Local Fields of Positive Characteristic

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Abstract. The concept of nonuniform multiresolution analysis on local field of positive characteristic was considered by Shah and Abdullah for which the translation set is a discrete set which is not a group. We construct the associated wavelet packets for such an MRA and investigate their properties by means of the Fourier transform.

1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. A multiresolution analysis is an increasing family of closed subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2\left(\mathbb{R}^d\right)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of multiresolution analysis to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. For more about wavelets and their applications, we refer the monograph [6].

In recent years there has been a considerable interest in the problem of constructing wavelet bases on various groups, namely, Cantor dyadic groups [10], locally compact Abelian groups [7], p-adic fields [9] and Vilenkin groups [11]. Recently, R. L. Benedetto and J. J. Benedetto [2] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Since local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local

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fields of characteristic zero include the p-adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, but their wavelet and multiresolution analysis theory are quite different. The concept of multiresolution analysis on a local field K of positive characteristic was introduced by Jiang et al.[8]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [21] using extension principles. More results in this direction can also be found in [15–18] and the references therein.

Recently, Shah and Abdullah [16] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace V_0 is no longer a group, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$, where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathfrak{D}$ in the locally compact Abelian group K^+. More precisely, this set is of the form $\Lambda = \{0, r/N\} + \mathcal{Z}$, where $N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. They call this a *nonuniform multiresolution analysis* on local fields of positive characteristic.

It is well known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman et al.[5] introduced the notion of orthogonal univariate wavelet packets. Well known Daubechies orthogonal wavelets are a special of wavelet packets. Chui and Li [4] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be employed to the spline wavelets and so on. Shen [22] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. The construction of wavelet packets and wavelet frame packets on local fields of positive characteristic were recently reported by Behera and Jahan in [1]. They proved lemma on the so-called splitting trick and several theorems concerning the Fourier transform of the wavelet packets and the construction of wavelet packets to show that their translates form an orthonormal basis of $L^2(K)$. Other notable generalizations are the vector-valued wavelet packets [3], wavelet packets and framelet packets related to the Walsh polynomials [13,14,19] and M-band framelet packets [20].

Motivated and inspired by the concept of nonuniform multiresolution analysis on local fields of positive characteristic, we construct the associated orthogonal wavelet packets for such an MRA. More precisely, we show that the collection of all dilations and translations of the wavelet packets is an overcomplete system in $L^2(K)$. Finally, we investigate certain properties of the nonuniform wavelet packets on local fields by virtue of the Fourier transform.

This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results which are required in the subsequent sections including the definitions of uniform and non-uniform multiresolution analysis on local fields of positive characteristic. In Section 3, we introduce the notion of nonuniform wavelet packets on local field K and prove that they generate an orthonormal basis for $L^2(K)$. In Section 4, we examine their properties by means of the Fourier transform.

2. Preliminaries on Local Fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [12, 23]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [23], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [23]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [23], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\left\{\zeta_j\right\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \;\; n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \;\; 0 \leq a_k < p, \;\; \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.2}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \ne 1. \end{cases} \tag{2.4}$$

A generalization of classical theory of multiresolution analysis on local fields of positive characteristic was considered by Jiang et al.[8]. Analogous to the Euclidean case, following is a definition of *uniform multiresolution analysis* on the local field K of positive characteristic.

Definition 2.1. Let K be a local field of positive characteristic $p > 0$ and $\mathfrak{p}$ be a prime element of K. A multiresolution analysis(MRA) of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There is a function $\varphi \in V_0$, called the *scaling function*, such that $\left\{\varphi\left(\,\cdot - u(k)\right) : k \in \mathbb{N}_0\right\}$ forms an orthonormal basis for V_0.

Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = K$, we can regard $\mathfrak{p}^{-1}$ as the dilation and since $\{u(n)\}_{n\in\mathbb{N}_0}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. Note that unlike the standard wavelet theory on the real line, the translation set is not a group.

Let $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$, where $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of $\mathfrak{D}$ in K^+. For an integer $N \ge 1$ and an odd integer r with $1 \le r \le qN - 1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{r}{N}\right\} + \mathcal{Z}. \tag{2.5}$$

It is easy to verify that Λ is not a group on local field K, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$. In this set up, Shah and Abdullah [16] formulated the notion of multiresolution analysis on local field of positive characteristic, which is called *nonuniform multiresolution analysis* (NUMRA) and is based on the theory of spectral pairs. We first recall the definition of a NUMRA on local fields of positive characteristic (as defined in [16]) and associated set of wavelets:

Definition 2.2. For an integer $N \ge 1$ and an odd integer r with $1 \le r \le qN - 1$ such that r and N are relatively prime, an associated nonuniform multiresolution analysis on local field K of positive characteristic is a sequence of closed subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ of $L^2(K)$ such that the following properties hold:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ if and only if $f\left(\mathfrak{p}^{-1}N\cdot\right) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There exists a function φ in V_0 such that $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$, is a complete orthonormal basis for V_0.

It is worth noticing that, when $N = 1$, one recovers from the definition above the definition of a multiresolution analysis on local fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathcal{Z} \subset \Lambda$.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if } \ell \neq \ell'. \tag{2.6}$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}\,, \tag{2.7}$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 2.2, this implies

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j, \tag{2.8}$$

a decomposition of $L^2(K)$ into mutually orthogonal subspaces.

As in the standard case, one expects the existence of $qN-1$ number of functions so that their translation by elements of λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2(K)$.

Definition 2.3. A set of functions $\left\{\psi_1, \psi_1, \ldots, \psi_{qN-1}\right\}$ in $L^2(K)$ is said to be a *set of basic wavelets* associated with the nonuniform multiresolution analysis $\left\{V_j : j \in \mathbb{Z}\right\}$ if the family of functions $\{\psi_\ell(\cdot - \lambda) : 1 \leq \ell \leq qN-1, \lambda \in \Lambda\}$ forms an orthonormal basis for W_0.

We denote $\psi_0 = \varphi$, the scaling function, and consider $qN-1$ functions $\psi_\ell, 1 \leq \ell \leq qN-1$, in W_0 as possible candidates for wavelets. Since $(1/qN)\,\psi_\ell(\mathfrak{p}/N\cdot) \in V_{-1} \subset V_0$, it follows from property (d) of Definition 2.2 that for each ℓ, $0 \leq \ell \leq qN-1$, there exists a sequence $\left\{h^\ell_\lambda : \lambda \in \Lambda\right\}$ with $\sum_{\lambda\in\Lambda} \left|h^\ell_\lambda\right|^2 < \infty$ such that

$$\frac{1}{qN}\,\psi_\ell\left(\frac{\mathfrak{p}x}{N}\right) = \sum_{\lambda\in\Lambda} h^\ell_\lambda\, \varphi(x-\lambda). \tag{2.9}$$

On taking Fourier transform, we obtain

$$\hat{\psi}_\ell\left(\mathfrak{p}^{-1}N\xi\right) = m_\ell(\xi)\,\hat{\varphi}(\xi), \tag{2.10}$$

where

$$m_\ell(\xi) = \sum_{\lambda\in\Lambda} h^\ell_\lambda\, \overline{\chi(\lambda, \xi)}, \tag{2.11}$$

are the integral periodic functions in $L^2(\mathfrak{D})$ and are called the *wavelet symbols*. In view of the specific form of Λ, we observe that

$$m_\ell(\xi) = m_\ell^1(\xi) + \overline{\chi\left(\frac{r}{N}, \xi\right)}\, m_\ell^2(\xi), \quad 0 \le \ell \le qN-1, \tag{2.12}$$

where m_ℓ^1 and m_ℓ^2 are locally L^2-periodic functions. Therefore, by the scaling property of the wavelet spaces W_j's and (2.8), it is clear that the family of functions

$$\left\{(qN)^{j/2}\psi_\ell\big((\mathfrak{p}^{-1}N)^j \cdot - \lambda\big) : 1 \le \ell \le qN-1, \lambda \in \Lambda\right\} \tag{2.13}$$

forms an orthonormal basis for $L^2(K)$. In fact, it was shown in [23, Lemma 3.3] that the orthonormality of the system $\left\{\psi_k\big(\,\cdot - \lambda\big) : 1 \le k \le qN-1, \lambda \in \Lambda\right\}$ is equivalent to the following two conditions:

$$\sum_{s=0}^{qN-1}\left\{m_k^1\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)\overline{m_\ell^1\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)} + m_k^2\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)\right.$$
$$\left.\times\, \overline{m_\ell^2\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)}\right\} = \delta_{k,\ell}, \tag{2.14}$$

$$\sum_{s=0}^{qN-1}\overline{\chi\left(\frac{r}{N}, \mathfrak{p}u(s)\right)}\left\{m_k^1\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)\overline{m_\ell^1\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)}\right.$$
$$\left. + \, m_k^2\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)\overline{m_\ell^2\left(\frac{\mathfrak{p}}{N}\big(\xi + \mathfrak{p}u(s)\big)\right)}\right\} = 0, \tag{2.15}$$

for $0 \le k, \ell \le qN-1$.

As we know that the classic technique involved for the construction of wavelet packets is through splitting the wavelet spaces W_j successively into finite number of orthogonal sub-spaces. This splitting is carried out by the following lemma, whose proof is similar to that of Lemma 3.3 in [16].

Lemma 2.4. *Let $\varphi \in L^2(K)$ be such that $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$ is an orthonormal system in $L^2(K)$ and let $V = \overline{span}\left\{(qN)^{1/2}\varphi\big((\mathfrak{p}^{-1}N)\cdot - \lambda\big) : \lambda \in \Lambda\right\}$. Let ψ_ℓ and $m_\ell(\xi)$ be the functions defined by (2.10) and (2.12), respectively. Then, $\{\psi_\ell(\cdot - \lambda) : 1 \le \ell \le qN-1, \lambda \in \Lambda\}$ is an orthonormal system if and only if m_ℓ^1 and m_ℓ^2 satisfy (2.14) and (2.15). Furthermore, this system is an orthonormal basis for V if and only if it is orthonormal.*

Corollary 2.5 *Let $\{E_\lambda : \lambda \in \Lambda\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$, and $m_\ell, 0 \le \ell \le qN-1$, be as in Lemma 2.4 satisfying (2.14) and (2.15). Define*

$$F_\sigma^\ell = (qN)^{1/2}\sum_{\lambda\in\Lambda} h_{\lambda - qN\sigma}\, E_\lambda, \quad 0 \le \ell \le qN-1, \sigma \in \Lambda.$$

Then, $\left\{F_\sigma^\ell : \sigma \in \Lambda\right\}$ is an orthonormal basis for its closed linear span $\mathcal{H}_\ell$ and $\mathcal{H} = \bigoplus_{\ell=0}^{qN-1}\mathcal{H}_\ell$.

3. Nonuniform Wavelet Packets on Local Fields

In this Section, we construct nonuniform wavelet packets associated with nonuniform multiresolution analysis on local fields of positive characteristic.

Let $\{V_j : j \in \mathbb{Z}\}$ be an NUMRA with the scaling function φ. Then there exists a function m_0 such that $\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi/N)\hat{\varphi}(\mathfrak{p}\xi/N)$, where $m_0(\xi) = m_0^1(\xi) + \overline{\chi(r/N, \xi)}\, m_0^2(\xi)$. Applying the splitting Lemma 2.4 to the space V_1, we get functions Γ_ℓ, $\ell = 0, 1, \dots, qN-1$, where

$$\hat{\Gamma}_\ell(\xi) = m_\ell\left(\frac{\mathfrak{p}\xi}{N}\right)\hat{\varphi}\left(\frac{\mathfrak{p}\xi}{N}\right), \tag{3.1}$$

such that $\{\Gamma_\ell(\cdot - \lambda) : 1 \le \ell \le qN-1, \lambda \in \Lambda\}$ forms an orthonormal basis for V_1. For $\ell = 0$, we obtain the scaling function i.e., $\Gamma_0 = \varphi$ and for $\ell = 1, \dots, qN-1$, we have the basic wavelets $\Gamma_\ell = \psi_\ell$.

For $n \ge 0$, the *basic nonuniform wavelet packets* associated with a scaling function φ on a local fields of positive characteristic are defined recursively by

$$\Gamma_n(x) = \Gamma_{qN\gamma+\sigma}(x) = (qN)^{1/2}\sum_{\lambda\in\Lambda} h_\lambda^\sigma\, \Gamma_\gamma\Big(\mathfrak{p}^{-1}Nx - \lambda\Big), \quad 0 \le \sigma \le qN-1, \tag{3.2}$$

where $\gamma \in \mathbb{N}_0$ is the unique element such that $n = (qN)\gamma + \sigma$, $0 \le \sigma \le qN-1$ holds. By implementing the Fourier transform for the both sides of (3.2), we have

$$\Big(\Gamma_{qN\gamma+\sigma}\Big)^\wedge(\xi) = m_\sigma\left(\frac{\mathfrak{p}\xi}{N}\right)\hat{\Gamma}_\gamma\left(\frac{\mathfrak{p}\xi}{N}\right), \quad 0 \le \sigma \le qN-1. \tag{3.3}$$

Next, we obtain an expression for the Fourier transform of the nonuniform wavelet packets in terms of the wavelet masks m_ℓ as:

Proposition 3.1. *Let $\{\Gamma_n : n \ge 0\}$ be the basic nonuniform wavelet packets constructed above and*

$$n = \sum_{k=0}^{j-1} \varepsilon_k (qN)^k, \quad 0 \le \varepsilon_k \le qN-1,\ \varepsilon_j \ne 0 \tag{3.4}$$

be the unique expansion of the integer n in the base qN. Then

$$\hat{\Gamma}_n(\xi) = m_{\varepsilon_1}\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right) m_{\varepsilon_2}\left(\left(\frac{\mathfrak{p}}{N}\right)^2\xi\right)\dots m_{\varepsilon_j}\left(\left(\frac{\mathfrak{p}}{N}\right)^j\xi\right)\hat{\varphi}\left(\left(\frac{\mathfrak{p}}{N}\right)^j\xi\right). \tag{3.5}$$

Proof. If an integer n has an expansion of the form (3.4), then we say that it is of length j. We use induction on the length of n to prove the proposition. Since $\Gamma_0 = \varphi$ is the scaling function and $\Gamma_\sigma = \psi_\sigma, 1 \le \sigma \le qN-1$, are the basic nonuniform wavelets, it follows from (3.1) that the claim is true for all n of length 1. Assume that it holds for all integers of length j. Then an integer m of length $j+1$ is of the form $m = \sigma + (qN)n$, where $0 \le \sigma \le qN-1$, and n has length j. Therefore, we have

$$m = \sigma + (qN)n = \sigma + \sum_{k=1}^{j} \varepsilon_k (qN)^k.$$

Suppose n has the expansion of the form (3.4). Then, from (3.3) and (3.5), we have

$$\begin{aligned}\hat{\Gamma}_m(\xi) &= \left(\Gamma_{\sigma+(qN)n}\right)^{\wedge}(\xi)\\ &= m_\sigma\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right)\hat{\Gamma}_n\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right)\\ &= m_\sigma\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right)m_{\varepsilon_1}\left(\left(\frac{\mathfrak{p}}{N}\right)^2\xi\right)\ldots m_{\varepsilon j}\left(\left(\frac{\mathfrak{p}}{N}\right)^{j+1}\xi\right)\hat{\varphi}\left(\left(\frac{\mathfrak{p}}{N}\right)^{j+1}\xi\right).\end{aligned}$$

This completes the proof. $\square$

For the construction of the wavelet packets, it is necessary to show that their translates form an orthonormal basis for $L^2(K)$. This is evident from the following theorem.

Theorem 3.2. *Let $\{\Gamma_n : n \geq 0\}$ be the basic nonuniform wavelet packets associated with the nonuniform multiresolution analysis $\left\{V_j : j \in \mathbb{Z}\right\}$. Then,*

(i) $\left\{\Gamma_n\left(\,\cdot - \lambda\right) : (qN)^j \leq n \leq (qN)^{j+1} - 1, \lambda \in \Lambda\right\}$ *is an orthonormal basis of W_j, $j \geq 0$.*

(ii) $\left\{\Gamma_n\left(\,\cdot - \lambda\right) : 0 \leq n \leq (qN)^j - 1, \lambda \in \Lambda\right\}$ *is an orthonormal basis of V_j, $j \geq 0$.*

(iii) $\left\{\Gamma_n\left(\,\cdot - \lambda\right) : n \geq 0,\ \lambda \in \Lambda\right\}$ *is an orthonormal basis of $L^2(K)$.*

Proof. We prove the theorem by induction on j. Since $\{\Gamma_n : 1 \leq n \leq qN - 1\}$ is the basic set of wavelets in W_0, so (i) is true for $j = 0$. Let us assume that it holds for j. We will prove it for $j + 1$. By our assumption, the family of functions $\left\{(qN)^{1/2}\Gamma_n\left((\mathfrak{p}^{-1}N)\cdot - \lambda\right) : (qN)^j \leq n \leq (qN)^{j+1} - 1,\ \lambda \in \Lambda\right\}$ is an orthonormal basis of W_{j+1}. Set

$$E_n = \overline{\operatorname{span}}\left\{(qN)^{1/2}\Gamma_n\left((\mathfrak{p}^{-1}N)\cdot - \lambda\right) : \lambda \in \Lambda\right\},$$

so that

$$W_{j+1} = \bigoplus_{n=(qN)^j}^{(qN)^{j+1}-1} E_n. \tag{3.6}$$

By applying the splitting Lemma 2.4 to E_n, we obtain

$$\left(g_\ell^n\right)^{\wedge}(\xi) = m_\ell\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right)\hat{\Gamma}_n\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right), \quad 0 \leq \ell \leq qN - 1, \tag{3.7}$$

such that $\left\{g_\ell^n(\cdot - \lambda) :\ 0 \leq \ell \leq qN - 1,\ \lambda \in \Lambda\right\}$ is an orthonormal basis of E_n. Now, if n has the expansion as in (3.4). Then, with the help of (3.5), we obtain

$$\left(g_\ell^n(\xi)\right)^{\wedge} = m_\ell\left(\left(\frac{\mathfrak{p}}{N}\right)\xi\right)m_{\varepsilon_1}\left(\left(\frac{\mathfrak{p}}{N}\right)^2\xi\right)\ldots m_{\varepsilon j}\left(\left(\frac{\mathfrak{p}}{N}\right)^{j+1}\xi\right)\hat{\varphi}\left(\left(\frac{\mathfrak{p}}{N}\right)^{j+1}\xi\right). \tag{3.8}$$

But the expression on the right-hand side of (3.8) is precisely $\hat{\Gamma}_m(\xi)$, where

$$m = \ell + (qN)\varepsilon_1 + (qN)^2\varepsilon_2 + \cdots + (qN)^j\varepsilon_j = \ell + (qN)n.$$

Hence, we get $g_\ell^n = \Gamma_{\ell+(qN)n}$. Application of this fact together with equation (3.6) shows that

$$\Big\{\Gamma_{\ell+(qN)n}(\cdot - \lambda) : 0 \le \ell \le qN-1, (qN)^j \le n \le (qN)^{j+1}-1, \lambda \in \Lambda\Big\}$$

$$= \Big\{\Gamma_n(\cdot - \lambda) : (qN)^{j+1} \le n \le (qN)^{j+2}-1,\ \lambda \in \Lambda\Big\},$$

is an orthonormal basis of W_{j+1}. Thus we have proved (i) for $j+1$ and the induction is complete. Part (ii) follows from the fact that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$, and (iii) from the decomposition (2.8). □

Definition 3.3. Let $\{\Gamma_n : n \ge 0\}$ be the basic wavelet packets associated with the nonuniform multiresolution analysis $\big\{V_j : j \in \mathbb{Z}\big\}$ of $L^2(K)$. The family of functions

$$\mathcal{F} = \Big\{(qN)^{j/2}\Gamma_n\Big(\big(\mathfrak{p}^{-1}N\big)^j \cdot - \lambda\Big) : n \ge 0,\ j \in \mathbb{Z},\ \lambda \in \Lambda\Big\}, \tag{3.9}$$

will be called the *general nonuniform wavelet packets* corresponding to the nonuniform multiresolution analysis $\big\{V_j : j \in \mathbb{Z}\big\}$ of $L^2(K)$.

Next, we prove several decompositions of the wavelet subspaces W_j by virtue of a series of subspaces of nonuniform wavelets packets on local fields of positive characteristic. For $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$, we define

$$U_j^n = \overline{\operatorname{span}}\Big\{(qN)^{j/2}\Gamma_n\Big(\big(\mathfrak{p}^{-1}N\big)^j \cdot - \lambda\Big) : \lambda \in \Lambda\Big\}, \tag{3.10}$$

Since Γ_0 is the scaling function and Γ_n, $1 \le n \le qN-1$ are the basic nonuniform wavelets, we observe that

$$U_j^0 = V_j, \quad \bigoplus_{\ell=1}^{qN-1} U_j^\ell = W_j, \quad j \in \mathbb{Z},$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = \bigoplus_{\ell=0}^{qN-1} U_j^\ell. \tag{3.11}$$

The following theorem decomposes U_{j+1}^n, into qN orthogonal subspaces.

Theorem 3.4. *For $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$, we have*

$$U_{j+1}^n = \bigoplus_{\ell=0}^{qN-1} U_j^{\ell+(qN)n}. \tag{3.12}$$

Proof. By definition

$$U_{j+1}^n = \overline{\operatorname{span}}\Big\{(qN)^{(j+1)/2}\,\Gamma_n\Big(\big(\mathfrak{p}^{-1}N\big)^{j+1} \cdot - \lambda\Big) : \lambda \in \Lambda\Big\}.$$

Let $b_\lambda(x) = (qN)^{(j+1)/2}\,\Gamma_n\Big((\mathfrak{p}^{-1}N)^{j+1}x - \lambda\Big)$, $\lambda \in \Lambda$. Then $\{b_\lambda : \lambda \in \Lambda\}$ forms an orthonormal basis for the Hilbert space U_{j+1}^n. For $0 \le \ell \le qN-1$, we define

$$F_\sigma^\ell(x) = (qN)^{1/2}\sum_{\lambda\in\Lambda} h^\ell_{\lambda-qN\sigma} b_\lambda(x), \quad \sigma\in\Lambda, \tag{3.13}$$

and

$$\mathcal{P}_\ell = \overline{\operatorname{span}}\left\{F_\sigma^\ell : \sigma\in\Lambda\right\}.$$

Then, by Corollary 2.5, we have

$$U^n_{j+1} = \bigoplus_{\ell=0}^{qN-1}\mathcal{P}_\ell.$$

Therefore, Eq. (3.13) becomes

$$\begin{aligned}
F_\sigma^\ell(x) &= \sum_{\lambda\in\Lambda}(qN)^{1/2} h^\ell_{\lambda-qN\sigma} b_\lambda(x)\\
&= \sum_{\lambda\in\Lambda}(qN)^{1/2} h^\ell_{\lambda}\, b_{\lambda+qN\sigma}(x)\\
&= \sum_{\lambda\in\Lambda}(qN)^{(j+2)/2} h^\ell_{\lambda}\, \Gamma_n\Big(\left(\mathfrak{p}^{-1}N\right)^{j+1} x - \lambda - \mathfrak{p}^{-1}N\sigma\Big)\\
&= (qN)^{j/2}\sum_{\lambda\in\Lambda}(qN) h^\ell_{\lambda}\, \Gamma_n\Big(\left(\mathfrak{p}^{-1}N\right)\Big(\left(\mathfrak{p}^{-1}N\right)^{j} x - \sigma\Big) - \lambda\Big)\\
&= (qN)^{j/2}\,\Gamma_{\ell+qNn}\Big(\left(\mathfrak{p}^{-1}N\right)^{j} x - \sigma\Big).
\end{aligned}$$

Hence

$$\mathcal{P}_\ell = U_j^{\ell+(qN)n} \quad \text{and} \quad U^n_{j+1} = \bigoplus_{\ell=0}^{qN-1} U_j^{\ell+(qN)n}.$$

The above decomposition can be used to obtain various decompositions of the wavelet subspaces W_j, $j\geq 0$ as follows:

$$W_j = \bigoplus_{\ell=1}^{qN-1} U_j^\ell = \bigoplus_{\ell=qN}^{(qN)^2-1} U_{j-1}^\ell = \cdots = \bigoplus_{\ell=(qN)^j}^{(qN)^{j+1}-1} U_0^\ell. \tag{3.14}$$

Note that one can construct various orthonormal basis of $L^2(K)$ by using (3.14). As we know $L^2(K) = V_0\oplus W_0\oplus W_1\oplus W_2\oplus\ldots$. Therefore, for each $j\geq 0$, we can choose any of the decomposition of W_j obtained above. For example, if we do not want to decompose any W_j, then we have the usual wavelet decomposition. On the other hand, if we prefer the last decomposition in (3.14) for each W_j, then we get the non-uniform wavelet packet decomposition.

Let $S\subset\mathbb{N}_0\times\mathbb{Z}$. We want to characterize the sets S such that the collection

$$\mathcal{F}_S = \left\{(qN)^{j/2}\,\Gamma_n\left(\left(\mathfrak{p}^{-1}N\right)^j\cdot - \lambda\right) : \lambda\in\Lambda, (n,j)\in S\right\},$$

will form an orthonormal basis of $L^2(K)$. In other words, we are searching those subsets S of $\mathbb{N}_0 \times \mathbb{Z}$ for which

$$\bigoplus_{(n,j)\in S} U_j^n = L^2(K). \tag{3.15}$$

Theorem 3.5. *Let $\{\Gamma_n : n \geq 0\}$ be the basic wavelet packets associated with the NUMRA $\left\{V_j : j \in \mathbb{Z}\right\}$ of $L^2(K)$ and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then $\mathcal{F}_S$ is an orthonormal basis of $L^2(K)$ if and only if $\left\{I_{n,j} : (n,j) \in S\right\}$ is a partition of $\mathbb{N}_0$, where $I_{n,j} = \left\{\ell \in \mathbb{N}_0 : (qN)^j n \leq \ell \leq (qN)^j(n+1) - 1\right\}$.*

Proof. Using decomposition (3.12) repeatedly, we obtain

$$\begin{aligned} U_j^n &= \bigoplus_{\ell=0}^{(qN)-1} U_{j-1}^{\ell+(qN)n} = \bigoplus_{\ell=(qN)n}^{(qN)(n+1)-1} U_{j-1}^{\ell} = \bigoplus_{\ell=(qN)n}^{(qN)(n+1)-1} \left[\bigoplus_{k=0}^{(qN)-1} U_{j-2}^{k+(qN)\ell}\right] \\ &= \bigoplus_{\ell=(qN)^2 n}^{(qN)^2(n+1)-1} U_{j-2}^{\ell} = \cdots = \bigoplus_{\ell=(qN)^j n}^{(qN)^j(n+1)-1} U_0^{\ell} = \bigoplus_{\ell\in I_{n,j}} U_0^{\ell}. \end{aligned}$$

Therefore, we have

$$\bigoplus_{(n,j)\in S} U_j^n = \bigoplus_{(n,j)\in S} \bigoplus_{\ell\in I_{(n,j)}} U_0^{\ell}.$$

By Theorem 3.2(iii), we get $L^2(K) = \bigoplus_{\ell\in\mathbb{N}_0} U_0^{\ell}$. Hence (3.15) holds if and only if $\left\{I_{n,j} : (n,j) \in S\right\}$ is a partition of $\mathbb{N}_0$. □

4. The Orthogonal Properties of Nonuniform Wavelet Packets

In this Section, we investigate certain orthonormal properties of the nonuniform wavelet packets on local fields of positive characteristic by virtue of the Fourier transform.

Lemma 4.1. *Let $f(x)$ be any function in $L^2(K)$. Then, the system $\left\{f(x-\lambda) : \lambda \in \Lambda\right\}$ is orthonormal if and only if*

$$\sum_{k\in\mathbb{N}_0} \left|\hat{f}\big(\xi + u(k)\big)\right|^2 = 1 \tag{4.1}$$

and

$$\sum_{k\in\mathbb{N}_0} \overline{\chi\left(\frac{r}{N}, u(k)\right)} \left|\hat{f}\big(\xi + u(k)\big)\right|^2 = 0. \tag{4.2}$$

Proof. We have

$$\begin{aligned} \Big\langle f(x), f(x-\lambda)\Big\rangle &= \int_K \hat{f}(\xi)\overline{\hat{f}(\xi)}\,\overline{\chi(\lambda,\xi)}\,d\xi \\ &= \sum_{k\in\mathbb{N}_0} \int_{\mathfrak{D}} \left|\hat{f}\big(\xi + u(k)\big)\right|^2 \overline{\chi\big(\lambda, \xi + u(k)\big)}\,d\xi. \end{aligned}$$

If $\lambda \in \mathbb{N}_0$, we have

$$\Big\langle f(x), f(x-\lambda)\Big\rangle = \int_{\mathfrak{D}} \left[\sum_{k\in\mathbb{N}_0} \left|\hat{f}\Big(\xi + u(k)\Big)\right|^2\right] \overline{\chi(\lambda,\xi)}\, d\xi.$$

On taking $\lambda = \dfrac{r}{N} + u(m), m \in \mathbb{N}_0$, we obtain

$$\begin{aligned}
\Big\langle f(x), f(x-\lambda)\Big\rangle &= \sum_{k\in\mathbb{N}_0} \int_{\mathfrak{D}} \left|\hat{f}\Big(\xi + u(k)\Big)\right|^2 \overline{\chi\left(\frac{r}{N} + u(m), \xi + u(k)\right)} d\xi \\
&= \int_{\mathfrak{D}} \left[\sum_{k\in\mathbb{N}_0} \overline{\chi\left(\frac{r}{N}, u(k)\right)} \left|\hat{f}\Big(\xi + u(k)\Big)\right|^2\right] \overline{\chi\left(\frac{r}{N}, \xi\right)}\, \overline{\chi\Big(u(m), \xi\Big)}\, d\xi.
\end{aligned}$$

Therefore, the system $\Big\{f(x-\lambda) : \lambda \in \Lambda\Big\}$ is orthonormal if and only if the equalities (4.1) and (4.2) hold. $\square$

Lemma 4.2. *If* $\{\psi_\ell(\cdot - \lambda) : 1 \le \ell \le qN-1, \lambda \in \Lambda\}$ *are the basic orthonormal wavelets associated with a NUMRA* $\Big\{V_j : j \in \mathbb{Z}\Big\}$. *Then*

$$\sum_{\lambda\in\Lambda} h_\lambda^k\, \overline{h_{\lambda qN-\sigma}^\ell} = \frac{1}{qN}\, \delta_{k,\ell}\, \delta_{0,\sigma}, \quad 1 \le k, \ell \le qN-1, \sigma \in \Lambda. \tag{4.3}$$

Proof. By wavelet equation (2.9), we have

$$\begin{aligned}
\Big\langle \psi_k(x), \psi_\ell(x-\sigma)\Big\rangle &= (qN)^2 \int_K \sum_{\lambda\in\Lambda} h_\lambda^k \varphi\left(\mathfrak{p}^{-1}Nx - \lambda\right) \sum_{\omega\in\Lambda} \overline{h_\omega^\ell}\, \overline{\varphi\left(\mathfrak{p}^{-1}Nx - \mathfrak{p}^{-1}N\sigma - \omega\right)}\, dx \\
&= (qN)^2 \int_K \sum_{\lambda\in\Lambda} h_\lambda^k \varphi\left(\mathfrak{p}^{-1}Nx - \lambda\right) \sum_{\omega\in\Lambda} \overline{h_{\omega-qN\sigma}^\ell}\, \overline{\varphi\left(\mathfrak{p}^{-1}Nx - \omega\right)}\, dx \\
&= (qN) \sum_{\lambda\in\Lambda} \sum_{\omega\in\Lambda} h_\lambda^k\, \overline{h_{\omega-qN\sigma}^\ell} \Big\langle \varphi(x-\lambda), \varphi(x-\omega)\Big\rangle \\
&= (qN) \sum_{\lambda\in\Lambda} h_\lambda^k\, \overline{h_{\lambda-qN\sigma}^\ell}.
\end{aligned}$$

which implies that (4.3) follows. $\square$

We are now in a position to investigate the orthogonal properties of the nonuniform wavelet packets on local fields of positive characteristic.

Theorem 4.3. *Let* $\{\Gamma_n : n \in \mathbb{N}_0\}$ *be the basic nonuniform wavelet packets associated with a NUMRA* $\Big\{V_j : j \in \mathbb{Z}\Big\}$ *of* $L^2(K)$. *Then, we have*

$$\Big\langle \Gamma_n(x), \Gamma_n(x-\lambda)\Big\rangle = \delta_{0,\lambda}, \quad \lambda \in \Lambda. \tag{4.4}$$

Proof. We prove this result by using induction on n. Since

$$\Big\langle \Gamma_0(x), \Gamma_0(x-\lambda) \Big\rangle = \Big\langle \varphi(x), \varphi(x-\lambda) \Big\rangle = \delta_{0,\lambda},$$

and hence, the claim is true for $n = 0$. Assume that (4.4) follows if $0 \le n \le (qN)^r$, r is a fixed positive integer. Then, for $(qN)^r \le n \le (qN)^{r+1}$, we have $(qN)^{r-1} \le [n/N] \le (qN)^r$. Let $n = N[n/N] + s$, $s = 0, 1, \ldots, qN - 1$. In view of induction hypothesis and Lemma 4.1, we have

$$\begin{aligned} \Big\langle \Gamma_{[n/N]}(x), \Gamma_{[n/N]}(x-\lambda) \Big\rangle = \delta_{0,\lambda} \Leftrightarrow \sum_{k \in \mathbb{N}_0} \Big| \hat{\Gamma}_{[n/N]}\big(\xi + u(k)\big) \Big|^2 &= 1, \\ \sum_{k \in \mathbb{N}_0} \overline{\chi\left(\frac{r}{N}, u(k)\right)} \Big| \hat{\Gamma}_{[n/N]}\big(\xi + u(k)\big) \Big|^2 &= 0. \end{aligned} \tag{4.5}$$

By virtue of (3.3), we obtain

$$\begin{aligned} &\Big\langle \Gamma_n(x), \Gamma_n(x-\lambda) \Big\rangle \\ &= \int_K \left| m_\sigma\left(\frac{\mathfrak{p}\xi}{N}\right) \right|^2 \left| \hat{\Gamma}_{[n/N]}\left(\frac{\mathfrak{p}\xi}{N}\right) \right|^2 \overline{\chi(\lambda,\xi)}\, d\xi \\ &= \sum_{k \in \mathbb{N}_0} \int_{qN(\mathfrak{D}+k)} \left| m_\sigma\left(\frac{\mathfrak{p}\xi}{N}\right) \right|^2 \left| \hat{\Gamma}_{[n/N]}\left(\frac{\mathfrak{p}\xi}{N}\right) \right|^2 \overline{\chi(\lambda,\xi)}\, d\xi \\ &= \sum_{k \in \mathbb{N}_0} \int_{qN\mathfrak{D}} \left| m_\sigma\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right) \right|^2 \left| \hat{\Gamma}_{[n/N]}\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right) \right|^2 \overline{\chi\left(\lambda, \xi + \frac{\mathfrak{p}}{N} u(k)\right)} d\xi. \end{aligned}$$

In view of the specific form of Λ, we can write

$$\begin{aligned} &\left| m_\sigma\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right) \right|^2 \\ &= \left\{ m_\sigma^1\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right) + \overline{\chi\left(\frac{r}{N}, \frac{\mathfrak{p}\xi}{N} + u(k)\right)} m_\sigma^2\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right) \right\} \\ &\quad \times \left\{ \overline{m_\sigma^1\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right)} + \chi\left(\frac{r}{N}, \frac{\mathfrak{p}\xi}{N} + u(k)\right) \overline{m_\sigma^2\left(\frac{\mathfrak{p}}{N}\big(\xi + u(k)\big)\right)} \right\} \\ &= \left| m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right) \right|^2 + \left| m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right) \right|^2 + \overline{m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right)} m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right) \overline{\chi\left(\frac{r}{N}, \frac{\mathfrak{p}\xi}{N}\right)}\, \overline{\chi\left(\frac{r}{N}, u(k)\right)} \\ &\quad + m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right) \overline{m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right)} \chi\left(\frac{r}{N}, \frac{\mathfrak{p}\xi}{N}\right) \chi\left(\frac{r}{N}, u(k)\right). \end{aligned}$$

If $\lambda \in \mathbb{N}_0$, then by using (4.5), we obtain

$$\Big\langle \Gamma_n(x), \Gamma_n(x-\lambda) \Big\rangle$$

$$
\begin{aligned}
&= \int_{qN\mathfrak{D}} \left\{ \left| m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right)\right|^2 + \left| m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right)\right|^2 \right\} \overline{\chi(\lambda,\xi)} \sum_{k\in\mathbb{N}_0} \left| \hat{\Gamma}_{[n/N]}\left(\frac{\mathfrak{p}}{N}\big(\xi+u(k)\big)\right)\right|^2 d\xi \\
&+ \int_{qN\mathfrak{D}} \left[\overline{m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right)} m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right) \overline{\chi\left(\frac{r}{N},\frac{\mathfrak{p}\xi}{N}\right)}\, \overline{\chi(\lambda,\xi)} \sum_{k\in\mathbb{N}_0} \overline{\chi\left(\frac{r}{N},u(k)\right)} \left| \hat{\Gamma}_{[n/N]}\left(\frac{\mathfrak{p}}{N}\big(\xi+u(k)\big)\right)\right|^2 \right] d\xi \\
&+ \int_{qN\mathfrak{D}} \left[m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right) \overline{m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right)} \chi\left(\frac{r}{N},\frac{\mathfrak{p}\xi}{N}\right) \overline{\chi(\lambda,\xi)} \sum_{k\in\mathbb{N}_0} \chi\left(\frac{r}{N},u(k)\right) \left| \hat{\Gamma}_{[n/N]}\left(\frac{\mathfrak{p}}{N}\big(\xi+u(k)\big)\right)\right|^2 \right] d\xi \\
&= \sum_{s=0}^{qN-1} \int_{s\mathfrak{D}} \left\{ \left| m_\sigma^1\left(\frac{\mathfrak{p}\xi}{N}\right)\right|^2 + \left| m_\sigma^2\left(\frac{\mathfrak{p}\xi}{N}\right)\right|^2 \right\} \overline{\chi(\lambda,\xi)} d\xi \\
&= \int_{\mathfrak{D}} \sum_{s=0}^{qN-1} \left\{ \left| m_\sigma^1\left(\frac{\mathfrak{p}}{N}\big(\xi+u(s)\big)\right)\right|^2 + \left| m_\sigma^2\left(\frac{\mathfrak{p}}{N}\big(\xi+u(s)\big)\right)\right|^2 \right\} \overline{\chi(\lambda,\xi)} d\xi \\
&= \int_{\mathfrak{D}} \overline{\chi(\lambda,\xi)} d\xi \\
&= \delta_{0,\lambda}.
\end{aligned}
$$

Similarly, let $\lambda = \dfrac{r}{N} + u(m), m \in \mathbb{N}_0$, we obtain

$$
\begin{aligned}
&\Big\langle \Gamma_n(x), \Gamma_n(x-\lambda) \Big\rangle \\
&\quad = \int_{\mathfrak{D}} \sum_{s=0}^{qN-1} \overline{\chi\left(\frac{r}{N},u(s)\right)} \left\{ \left| m_\sigma^1\left(\frac{\mathfrak{p}}{N}\big(\xi+u(s)\big)\right)\right|^2 + \left| m_\sigma^2\left(\frac{\mathfrak{p}}{N}\big(\xi+u(s)\big)\right)\right|^2 \right\} \\
&\qquad\qquad \overline{\chi\left(\frac{r}{N},\xi\right)}\, \overline{\chi\big(u(s),\xi\big)} d\xi \\
&\quad = 0.
\end{aligned}
$$

This completes the proof. □

Theorem 4.4. *Let $\{\Gamma_n : n \in \mathbb{N}_0\}$ be the basic nonuniform wavelet packets associated with a NUMRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. Then for every $\gamma \in \mathbb{N}_0$, $0 \le k, \ell \le qN-1$, we have*

$$
\Big\langle \Gamma_{qN\gamma+k}(x), \Gamma_{qN\gamma+\ell}(x-\lambda) \Big\rangle = \delta_{k,\ell}\, \delta_{0,\lambda}, \quad \lambda \in \Lambda. \tag{4.6}
$$

Proof. By Lemma 4.2 and Theorem 4.3, we have

$$
\begin{aligned}
&\left\langle \Gamma_{qN\gamma+k}(x), \Gamma_{qN\gamma+\ell}(x-\lambda) \right\rangle \\
&\quad = (qN)^2 \int_K \sum_{\lambda\in\Lambda} h_\lambda^k\, \Gamma_\gamma\left(\mathfrak{p}^{-1}Nx-\lambda\right) \sum_{\omega\in\Lambda} \overline{h_\omega^\ell}\; \overline{\Gamma_\gamma\left(\mathfrak{p}^{-1}Nx-\mathfrak{p}^{-1}N\lambda-\omega\right)}\, dx \\
&\quad = (qN) \sum_{\lambda\in\Lambda} \sum_{\omega\in\Lambda} h_\lambda^k\, \overline{h_{\omega-qN\lambda}^\ell} \int_K \Gamma_\gamma(x-\lambda)\, \overline{\Gamma_\gamma(x-\omega)}\, dx \\
&\quad = \sum_{\lambda\in\Lambda} \sum_{\omega\in\Lambda} h_\lambda^k\, \overline{h_{\omega-qN\lambda}^\ell} \left\langle \Gamma_\gamma(x-\lambda), \overline{\Gamma_\gamma(x-\omega)} \right\rangle \\
&\quad = \delta_{k,\ell}\, \delta_{0,\lambda}.
\end{aligned}
$$

Thus, the system $\left\{\Gamma_n(\cdot-\lambda) : n\in\mathbb{N}_0\right\}$ forms an orthogonal system in $L^2(K)$. □

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