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On Statistically Sequentially Covering Maps

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Abstract. A mapping $f : X \to Y$ is statistically sequence covering map if whenever a sequence $\{y_n\}$ convergent to y in Y, there is a sequence $\{x_n\}$ statistically converges to x in X with each $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$. In this paper, we introduce the concept of statistically sequence covering map which is a generalization of sequence covering map and discuss the relation with covering maps by some examples. Using this concept, we prove that every closed and statistically sequence-covering image of a metric space is metrizable. Also, we give characterizations of statistically sequence covering compact images of spaces with a weaker metric topology.

1. Introduction

In 1971, Siwiec [12] introduced the concept of sequence covering maps which is closely related to the question about compact covering and s-images of metric spaces. In 1982, Chaber gave a characterization of perfect images and open and compact images of spaces that can be mapped onto metrizable spaces by a mapping with fibers having a given property P in [3]. After that characterizations of sequence covering compact images and sequentially quotient compact images of spaces with a weaker metric topology are studied. In this paper, we characterize statistically sequence covering compact images of spaces with a weaker metric topology. Also, we prove that every closed and sequence covering image of a metric space is metrizable. Also, we introduce *ssn*-cover and *scs*-cover which is a generalization of *sn*-cover and *cs*-cover, respectively, to characterize statistically sequence covering compact map.

Throughout this paper, all spaces are regular and T_1, all maps are continuous and onto, and $\mathbb{N}$ is the set of natural numbers. $x_n \to x$ denote a sequence $\{x_n\}$ converging to x. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n \mid n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X. Then $\cup\mathcal{P}$ and $\cap\mathcal{P}$ denote the union $\cup\{P \mid P \in \mathcal{P}\}$ and the intersection $\cap\{P \mid P \in \mathcal{P}\}$, respectively. Let A be a subset of a space X, $x \in X$, and $\mathcal{U}$ be a family of subsets of X. We write $st(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid x \in U\}$ and $st(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$.

Definition 1.1. Let X be a space, $P \subset X$ and $x \in P$. Then P is called a *sequential neighborhood* [5] of x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x_n\}$ is eventually in P.

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Definition 1.2. ([7]) Let X be a space, and let $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is a cs-cover of X, if for any convergent sequence S in X, there exists $P \in \mathcal{P}$ such that S is eventually in P.
(2) $\mathcal{P}$ is an sn-cover of X, if each element of $\mathcal{P}$ is a sequential neighborhood of some point of X and for each $x \in X$, there exists $P \in \mathcal{P}$ such that P is the sequential neighborhood of x.

Definition 1.3. A space X is *strongly Fréchet* [12] if whenever $\{A_n \mid n \in \mathbb{N}\}$ is a decreasing sequence of sets in X and x is a point which is in the closure of each A_n where $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$, there exists an $x_n \in A_n$ such that the sequence $x_n \to x$.

Definition 1.4. A space X is said to have *property* ωD [14] if every infinite closed discrete subset has an infinite subset A such that there exists a discrete open family $\{U_x \mid x \in A\}$ with $U_x \cap A = \{x\}$ for each $x \in A$.

Definition 1.5. A class of mappings is said to be *hereditary* [1] if whenever $f : X \to Y$ is in the class, then for each subspace H of Y, the restriction of f to $f^{-1}(H)$ is in the class.

Definition 1.6. Let $f : X \to Y$ be a mapping.

(a) f is a *sequence covering map* [7] if for every convergent sequence S in Y, there is a convergent sequence L in X such that $f(L) = S$. Equivalently, if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$ [12].
(b) f is a *sequentially quotient map* [7] if for every convergent sequence S in Y, there is a convergent subsequence L in X such that $f(L)$ is an infinite subsequence of S. Equivalently, if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$ [12].

Definition 1.7. If X is a space that can be mapped onto a metric space by a one-to-one mapping, then X is said to have a *weaker metric topology* [3].

Definition 1.8. [4, 11] If $K \subset \mathbb{N}$, then K_n will denote the set $\{k \in K \mid k \leq n\}$ and $|K_n|$ stands for the cardinality of K_n. The *natural density* of K is defined by $d(K) = \lim\limits_{n\to\infty} \frac{|K_n|}{n}$, if limit exists.

Definition 1.9. A subset K of the set $\mathbb{N}$ is called *statistically dense* [2] if $d(K) = 1$.

Definition 1.10. A subsequence S of the sequence L is called *statistically dense in* L if the set of all indices of elements from S is statistically dense.

Definition 1.11. Let X be a space and $P \subset X$. P is called a *statistically sequential neighborhood* of $x \in P$, if every sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x is frequently statistically dense in P, that is, $d(\{n \in \mathbb{N} \mid x_n \notin P\}) = 0$.

Definition 1.12. Let X be a space and $\mathcal{P}$ be a cover of X.

(a) $\mathcal{P}$ is a *scs-cover* of X if for any convergent sequence S in X, there exists $P \in \mathcal{P}$ such that S is frequently statistically dense in P.
(b) $\mathcal{P}$ is a *ssn-cover* of X if each element of $\mathcal{P}$ is a statistically sequential neighborhood of some point of X and for each $x \in X$, there exists $P \in \mathcal{P}$ such that P is the statistically sequential neighborhood of x.

Definition 1.13. A sequence $\{x_n\}$ in a topological space X is said to *converge statistically* [8] to $x \in X$, if for every neighborhood U of x, $d(\{n \in \mathbb{N} \mid x_n \notin U\}) = 0$.

Lemma 1.14. ([8]) *Let X be a first countable space. If a sequence $\{x_n\}$ in X statistically converges to x, then there exists a statistically dense subsequence $\{x_{n_j}\}$ converges to x.*

Lemma 1.15. ([16]) *Let X be a space with a weaker metric topology. Then there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} st(K, \mathcal{P}_i) = K$ for each compact subset K of X.*

2. Statistically Sequence Covering Map

In this section, we introduce a map, namely, statistically sequence covering map and give their properties. A mapping $f : X \to Y$ is said to be a *statistically sequence covering map* if for given $y_n \to y$ in Y, there exists a sequence x_n statistically converges to x, $x \in f^{-1}(y)$ and $x_n \in f^{-1}(y_n)$

Proposition 2.1. *Let $f : X \to Y$ and $g : Y \to Z$ be any two maps. Then the following hold:*

(a) *If f and g are statistically sequence covering map and Y is a first countable space, then $g \circ f$ is statistically sequence cover.*

(b) *If $g \circ f$ is statistically sequence covering map, then g is statistically sequence cover.*

Proof. (a) Let $z \in Z$ and $z_n \to z$ be a sequence. Since g is a statistically sequence covering map, there exists a sequence statistically converges to y with $y_n \in g^{-1}(z_n)$ and $y \in g^{-1}(z)$. By Lemma 1.14, there exists a statistically dense subsequence y_{n_k} converges to y. Since f is a statistically sequence covering map, there exists a statistically convergent sequence $x_{n_k} \to x$, where $x_{n_k} \in f^{-1}(y_{n_k})$ and $x \in f^{-1}(y)$. Choose $x_n \in f^{-1}(y_n)$ for $n \neq n_k$.

To prove $\{x_n\}$ is statistically convergent to x, it is enough to prove $d(\{n \mid x_n \notin U_x\}) = 0$ for every open neighborhood U_x of x.

Since y_{n_k} is statistically dense in y_n, x_{n_k} is statistically dense in x_n and so $d(K_1) = 0$ where $K_1 = \{n \mid n \neq n_k\}$. Also, x_{n_k} statistically converges to x implies that for every open set U, $d(K_2) = 0$ where $K_2 = \{n_k \mid x_{n_k} \notin U\}$.

Now for U_x,

$$d(K) = \lim_{n\to\infty} \frac{|\{n_0 \le n \mid x_{n_0} \notin U_x\}|}{n}$$
$$= \lim_{n\to\infty} \frac{|\{n_0 \le n \mid x_{n_0} \notin U_x \text{ and } n_0 \neq n_k\} \cup \{n_0 \le n \mid x_{n_0} \notin U_x \text{ and } n_0 = n_k\}|}{n}$$
$$= \lim_{n\to\infty} \frac{|\{n_0 \le n \mid x_{n_0} \notin U_x \text{ and } n_0 \neq n_k\}| + |\{n_0 \le n \mid x_{n_0} \notin U_x \text{ and } n_0 = n_k\}|}{n}$$
$$= \lim_{n\to\infty} \frac{|\{n_0 \le n \mid x_{n_0} \notin U_x \text{ and } n_0 \neq n_k\}|}{n} + \lim_{n\to\infty} \frac{|\{n_0 \le n \mid x_{n_0} \notin U_x \text{ and } n_0 = n_k\}|}{n}$$
$$= d(K_1) + d(K_2)$$
$$= 0$$

(b) Since f is continuous and by Theorem 3 in [6], g is a statistically sequence covering map. □

Proposition 2.2. (a) *Finite product of statistically sequence covering mapping is a statistically sequence covering map.*

(b) *Statistically sequence covering mappings are hereditarily statistically sequence covering mappings.*

Proof. (a) Let $\prod_{i=1}^{N} f_i : \prod_{i=1}^{N} X_i \to \prod_{i=1}^{N} Y_i$ be a map where each $f_i : X_i \to Y_i$ is statistically sequence covering map for $i = 1, 2, 3, ...N$. Let $\{(y_{i,n})\}_{n\in\mathbb{N}}$ be converges to (y_i) in $\prod_{i=1}^{N} Y_i$. Then each $\{y_{i,n}\}$ is a sequence converges to y_i in Y_i. Since each f_i is a statistically sequence covering map, there exists a sequence $\{x_{i,n}\}$ statistically converges to x_i such that $f_i(x_{i,n}) = y_{i,n}$. Take a sequence $\{(x_{i,n})\}_{n\in\mathbb{N}}$ which is statistically converges to (x_i) by inductive application of Corollary 2.1 (a) and (b) in [2]. Therefore, $\prod_{i=1}^{N} f_i$ is a statistically sequence covering map.

(b) Let $f : X \to Y$ be a statistically sequence covering map and H be a subspace of Y. Take $g = f|_{f^{-1}(H)}$ such that $g : f^{-1}(H) \to H$ be a map.

Given a sequence $\{y_n\}$ convergence to y in H, there exists a sequence $x_n \in f^{-1}(y_n) \in f^{-1}(H)$ such that (x_n) statistically converges to $x \in f^{-1}(y) \in f^{-1}(H)$, since f is statistically sequence covering map and $\{y_n\}$ statistically converges to y in Y. Therefore, g is a statistically sequence covering map. □

We observe that every sequence covering map is a statistically sequence covering map. But the reverse implication need not be true as shown by the following Example 2.3. Also, Example 2.4 and Example 2.5 below shows that statistically sequence covering map and sequentially quotient map are independent.

Example 2.3. Let $\wedge = \{K \mid d(K) = 1, K$ is a subsequence of $\mathbb{N}$ obtained by deleting infinitely many elements$\}$ and S_α be a convergent sequence with its limit x_α where $\alpha = K \in \wedge$. That is, $S_\alpha = \{x_{\alpha,i}, x_\alpha \mid i \in K\}$. Let X be a disjoint union of S_α and Y be a sequence $\{x_n\} \to x$. Then $f : X \to Y$ defined by $f(x_{\alpha,i}) = f(x_i)$ and $f(x_\alpha) = f(x)$ is a statistically sequence covering map and a sequentially quotient map but not a sequence covering map.

Example 2.4. Let X be a topological sum of a collection $\{I, S_\alpha \mid \alpha \in I\}$, where I is the closed unit interval and each S_α is a sequence with its limit for each $\alpha \in I$ and Y be the space obtained from X by identifying the limit point of S_α with α. Let $f : X \to Y$ be the obvious map. Then Y is the quotient, finite to one image of a locally compact metric space X under f so that f is sequentially quotient. But f is not a statistically sequence covering map.

Example 2.5. Let $\wedge = \{K \mid d(K) = 1, K$ is a subsequence of $\mathbb{N}$ obtained by deleting infinitely many elements$\}$ and S_α be a statistically convergent sequence with its limit x_α where $\alpha = K \in \wedge$. That is, $S_\alpha = \{x_{\alpha,i}, x_\alpha \mid i \in K\}$. Let X be a disjoint union of S_α and Y be a sequence $\{x_n\} \to x$. Note that each S_α is not a sequential space, since singleton set $\{x_\alpha\}$ is vacuously sequentially open but not open, that is, there is no convergent sequence in S_α converges to x_α. Therefore, X is not a sequential space, since its open subspace must be. Then $f : X \to Y$ defined by $f(x_{\alpha,i}) = f(x_i)$ and $f(x_\alpha) = f(x)$ is a statistically sequence covering map but not a sequentially quotient map.

3. Statistically Sequence Covering and Compact Map

Theorem 3.1. *Let $f : X \to Y$ be a statistically sequence covering compact map. Then for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that if U is an open neighborhood of x, then $f(U)$ is a sequential neighborhood of y.*

Proof. Suppose not, that is, there exists $y \in Y$, for every $x \in f^{-1}(y)$, there exists an open neighborhood U_x of x such that $f(U_x)$ is not a sequential neighborhood of y. Since $f^{-1}(y) \subset \bigcup_{x \in f^{-1}(y)} U_x$ and f is a compact map, there exists a finite U_i such that $f^{-1}(y) \subset \bigcup_{i=1}^{n_0} U_i$. Since each $f(U_i)$ is not a sequential neighborhood of y, choose $\{y_{m,n}\}_{n=1}^{\infty} \to y$ such that $y_{m,n} \notin f(U_m)$ for all $m \in \{1, 2, ..., n_0\}$ and $n \in \mathbb{N}$. Now form a sequence $y_k = y_{m,n}$ where $k = (n-1)n_0 + m, 1 \le m \le n_0$ and $n \in \mathbb{N}$. Then $\{y_k\}$ is a sequence converging to y in Y. Since f is a statistically sequence covering map, there exists $x \in f^{-1}(y)$ and $x_k \in f^{-1}(y_k)$ such that $\{x_k\} \to x$ statistically. Since $x \in f^{-1}(y) \subset \bigcup_{i=1}^{n_0} U_i$, there exist U_{m_0} such that $x \in U_{m_0}$ so that $d(\{n \in \mathbb{N} \mid x_n \notin U_{m_0}\}) = 0$ and hence $d(\{n \in \mathbb{N} \mid y_n \notin f(U_{m_0})\}) = 0$ which is a contradiction. Since $y_{m_0,n} \notin f(U_{m_0})$, $d(\{n \in \mathbb{N} \mid y_n \notin f(U_{m_0})\}) \ge d(\{n \in \mathbb{N} \mid n = (n'-1)n_0 + m_0, n' \in \mathbb{N}\}) > \frac{1}{m_0}$. □

Theorem 3.2. *The following conditions are equivalent for a space Y:*

(a) Y is a statistically sequence covering compact image of a space with a weaker metric topology.
(b) Y has a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of point-finite ssn-covers such that $\bigcap_{i\in\mathbb{N}} st(y, \mathcal{F}_i) = \{y\}$ for each $y \in Y$.
(c) Y has a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of point-finite scs-covers such that $\bigcap_{i\in\mathbb{N}} st(y, \mathcal{F}_i) = \{y\}$ for each $y \in Y$.
(d) Y has a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of point-finite sn-covers such that $\bigcap_{i\in\mathbb{N}} st(y, \mathcal{F}_i) = \{y\}$ for each $y \in Y$.
(e) Y has a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of point-finite cs-covers such that $\bigcap_{i\in\mathbb{N}} st(y, \mathcal{F}_i) = \{y\}$ for each $y \in Y$.

Proof. It is clear that (b) ⇒ (c), (d) ⇒ (e), (d) ⇒ (b), (e) ⇒ (c).

(a) ⇒ (d) Suppose $f : X \to Y$ is a statistically sequence covering compact mapping. As X being a space with a weaker metric topology, there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} st(K, \mathcal{P}_i) = K$ for each compact subset $K \subset X$, by Lemma 1.15. For each $i \in \mathbb{N}$, put $\mathcal{F}_i = f(\mathcal{P}_i)$. Then $\mathcal{F}_i$ is a point finite cover of Y, since f is compact. By Theorem 3.1, for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that for every open neighborhood U_x of x, $f(U_x)$ is a sequential neighborhood of y. Since each $\mathcal{P}_i$ is an open cover of X, there exists $P \in \mathcal{P}_i$ such that $x \in P$, and so $F = f(P)$ is a sequential neighborhood of y. Choose $\mathcal{F}_i' \subset \mathcal{F}_i$ which are sequential neighborhoods of y. $\mathcal{F}_i'$ is a point finite *sn*-cover of Y. For each $y \in Y$, $f^{-1}(y)$ is a compact subset of X and $\bigcap_{i\in\mathbb{N}} st(f^{-1}(y), \mathcal{P}_i) = f^{-1}(y)$. Thus, $\bigcap_{i\in\mathbb{N}} st(y, \mathcal{F}_i) = \{y\}$.

(c) ⇒ (d) First we collect the set of all sequential neighborhoods $\mathcal{F}_i'$ from $\mathcal{F}_i$. Suppose there is no such sequential neighborhood in $\mathcal{F}_i$. For each $y \in Y$, put $(\mathcal{F}_i)_y = \{F \mid y \in F, F \in \mathcal{F}_i\} = \{F_m \mid m \le k\}$. Since each F_m is not a sequential neighborhood for each F_m, there exists a convergent sequence $y_{m,n} \to y$ such that $y_{m,n} \notin F_m$, for all $n \in \mathbb{N}$. Now form a new sequence $\{z_{n'}\}$ by taking $z_{n'} = y_{m,n}$, where $n' = (n-1)k + m$, $1 \leq m \leq k$ and $n \in \mathbb{N}$. Then the sequence $\{z_{n'}\}$ is also converging to y, but $d(\{n' \mid z_{n'} \notin F_m\}) \geq \frac{1}{k} > 0$, for all m. That is, $\mathcal{F}_i$ is not a *scs*-covers, which is a contradiction.

(b) ⇒ (a) For each $i \in \mathbb{N}$, take $\mathcal{F}_i = \{F_\alpha \mid \alpha \in X_i\}$ and each X_i is endowed with discrete topology. Let $M = \{(\alpha_i) \in \prod_{i\in\mathbb{N}} X_i \mid$ there is $y \in Y$ such that $\bigcap_{i\in\mathbb{N}} F_{\alpha_i} = \{y\}\}$ and give M the subspace topology induced from the usual product topology. Then M is a metric space. Let $X = \{(y, (\alpha_i)) \in Y \times M \mid y \in \bigcap_{i\in\mathbb{N}} F_{\alpha_i}\}$. Let $f : X \to Y$ and $P : X \to M$ be the onto projection map.

(1) X is a space with weaker metric topology.
Since $P : X \to M$ is the onto projection map, for each $(\alpha_i) \in M$, there is $y \in Y$ such that $\bigcap_{i\in\mathbb{N}} F_{\alpha_i} = \{y\}$ which implies $P^{-1}((\alpha_i)) = (y, (\alpha_i))$ and hence P is a one-to-one mapping. Thus, X is a space with a weaker metric topology.

(2) f is a compact map.
$f^{-1}(y) = \{(y, (\alpha_i)) \in Y \times M | \bigcap F_{\alpha_i} = \{y\}\}$.
Let $\{U_\beta\}_{\beta\in\wedge'}$ be a cover of $f^{-1}(y)$. Since M being a subspace topology induced from the usual product topology, for $\beta' \in \wedge'$, $P(U_{\beta'}) = V_M = (.., V_{i_1}, ..., V_{i_2}, .., V_{i_n}, ..)$, that is, $\prod_i(V_M) = X_i \bigcap \prod_i(M)$ except some finite place $i = \{i_1, i_2, ..., i_n\}$ where $\prod_i : M \to X_i$ is a projection map. Since $\mathcal{F}_i$ is a point finite cover of Y for each $i \in \mathbb{N}$, $\prod_{i_j}(P(f^{-1}(y)))$ is finite for each $j \in \{i_1, i_2, ..., i_n\}$. Therefore, we can choose a finite subcover of $\{U_\beta\}$ to cover the element of $\prod_{i_j}(P(f^{-1}(y)))$ where $j \in \{i_1, i_2, ..., i_n\}$. In addition, $U_{\beta'}$ will cover $f^{-1}(y)$. Therefore, f is a compact map.

(3) f is a statistically sequence covering map.
Take $y_0 \in Y$ and then choose $\beta_0 \in f^{-1}(y_0) \subset Y\times M$ such that for each $i \in \mathbb{N}$, choose $\alpha_i \in X_i$ such that F_{α_i} is a statistically sequential neighborhood of y_0. Let $\beta_0 = (y_0, (\alpha_i)) \in Y \times \prod_{i\in\mathbb{N}} X_i$. Then $\beta_0 \in f^{-1}(y_0) \subset Y\times M$. Now for given convergent sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y converging to y_0, we choose a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X as follows: Since F_{α_i} is a *ssn*-neighborhood of y_0, $\{y_n\}_{n\in\mathbb{N}}$ is frequently statistically dense in F_{α_i} for each $i \in \mathbb{N}$.
Choose $\alpha_{i_n} = \alpha_i$ if $y_n \in F_{\alpha_i}$, otherwise choose $\beta_i \in X_i$ such that $y_n \in F_{\beta_i}$ so that $\alpha_{i_n} = \beta_i$. Then $\{\alpha_{i_n}\}$ statistically converges to α_i in X_i and hence $\{(\alpha_{i_n})\}$ statistically converges to (α_i) in M. Put $\beta_n = (y_n, (\alpha_{i_n}))$ for each $n \in \mathbb{N}$. Then $f(\beta_n) = y_n$ and the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ is statistically converges to β_0 in X. Therefore, f is a statistically sequence covering mapping.

□

Theorem 3.3. *Let X be a strongly Fréchet space with property ωD. If $f : X \to Y$ is a closed and statistically sequence covering map, then Y is strongly Fréchet.*

Proof. Clearly, Y is a Fréchet space, since it is a closed image of a strongly Fréchet, in particular, Fréchet. Suppose Y is not strongly Fréchet. Then Y contains a homeomorphic copy of the sequential fan S_ω [13], and the copy can be closed in Y [10]. Hence let $S_\omega \subset Y$ as a closed set. Let it be $S_\omega = \{y\} \cup \{y_{m,n} \mid m, n \in \omega\}$ where each $S_m = \{y_{m,n}\}_{n\in\omega}$ is a convergent sequence converges to y.
For each $m \in \mathbb{N}$, choose

$$y_{m_k} = \begin{cases} y_{0,\frac{k+1}{2}}, & \text{if } k \text{ is odd} \\ y_{m,\frac{k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

Then the sequence $\{y_{m_k}\}$ converges to y. Since f is statistically sequence cover, there exist $x_m \in f^{-1}(y)$ and a sequence Q_m statistically converges to x_m such that $f(Q_m) = \{y_{m_k}\}$. For each $k \in \omega$, let $T_k = \cup\{f^{-1}(S_m) \mid m \geq k\}$ Suppose that there exists $z \in X$ such that for every open neighborhood U of z, $\{n \in \mathbb{N} \mid x_n \in U\}$ is infinite. Then $z \in \cap_{k\in\mathbb{N}}\overline{T_k}$. Since X is strongly Fréchet, there exists a convergent sequence $\{z_k\}_{k\in\mathbb{N}}$ converges to z, where $z_k \in T_k$. But $\{f(z_k)\}_{k\in\mathbb{N}}$ does not converge to y, which is a contradiction. Suppose the set $\{x_n\}_{n\in\mathbb{N}}$ is finite. Let it be $z = x_n$, $n \in N'$, N' is a infinite subset of $\mathbb{N}$. Then for every open neighborhood U of z, $\{n \in \mathbb{N} | x_n \in U\}$ is infinite which is a contradiction. Therefore, the set $\{x_n\}_{n\in\mathbb{N}}$ is infinite, closed and discrete

in X.
Since X has the property ωD, there exist an infinite subset $\{x_{n_j}\}_{n\in\mathbb{N}}$ and a discrete open family $\{U_j\}_{j\in\omega}$ such that $U_j \cap \{x_{n_j}\}_{j\in\omega} = \{x_{n_j}\}$. Recall that Q_{n_j} statistically converges to x_{n_j} and $f(Q_{n_j}) = \{y_{n_j}\}$. Therefore, we can take $u_j \in U_j \cap Q_{n_j}$ such that $\{f(u_j)\}_{j\in\omega}$ is infinite and contained in $\{y_{0,n} \mid n \in \mathbb{N}\}$. Since $\{u_j\}_{j\in\omega}$ is closed in X, $\{f(u_j)\}_{j\in\omega}$ is closed in S_ω, which is a contradiction. Thus, Y is strongly Fréchet. □

Corollary 3.4. *Every closed and statistically sequence covering image of a metric space is metrizable.*

Proof. Since every metrizable space has ωD property [14], Y is strongly Fréchet by Theorem 3.3. Also, m every strongly Fréchet space which is a closed image of a metric space is metrizable [9]. Hence Y is metrizable. □

Corollary 3.5. ([15]) *Every closed and sequence covering image of a metric space is metrizable.*

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