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Weak-2-Local Derivations on $\mathbb{M}_n$

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Abstract. We introduce the notion of weak-2-local derivation (respectively, *-derivation) on a C*-algebra A as a (non-necessarily linear) map $\Delta : A \to A$ satisfying that for every $a, b \in A$ and $\phi \in A^*$ there exists a derivation (respectively, a *-derivation) $D_{a,b,\phi} : A \to A$, depending on a, b and ϕ, such that $\phi\Delta(a) = \phi D_{a,b,\phi}(a)$ and $\phi\Delta(b) = \phi D_{a,b,\phi}(b)$. We prove that every weak-2-local *-derivation on M_n is a linear derivation. We also show that the same conclusion remains true for weak-2-local *-derivations on finite dimensional C*-algebras.

1. Introduction and Preliminaries

"Derivations appeared for the first time at a fairly early stage in the young field of C*-algebras, and their study continues to be one of the central branches in the field" (S. Sakai, 1991 [20, Preface]). We recall that *derivation* from an associative algebra A into an A-bimodule X is a linear mapping $D : A \to X$ satisfying

$$D(ab) = D(a)b + aD(b), \quad (a, b \in A).$$

If A is a C*-algebra and D is a derivation on A satisfying $D(a^*) = D(a)^*$ $(a \in A)$, we say that D is **-derivation* on A.

Some of the earliest, remarkable contributions on derivations are due to Sakai. For example, a celebrated result due to him shows that every derivation on a C*-algebra is continuous [18]. A subsequent contribution proves that every derivation on a von Neumann algebra M is inner, that is, for every derivation D on M there exists $a \in M$ satisfying $D(x) = [a, x] = ax - xa$, for every $x \in M$ (cf. [19, Theorem 4.1.6]).

We recall that, accordingly to the definition introduced by R.V. Kadison in [13], a linear mapping T from a Banach algebra A into a A-bimodule X is said to be a *local derivation* if for every a in A, there exists a derivation $D_a : A \to X$, depending on a, such that $T(a) = D_a(a)$. The contribution due to Kadison establishes that every continuous local derivation from a von Neumann algebra M into a dual M-bimodule

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X is a derivation. B.E. Johnson proves in [12] that every local derivation from a C*-algebra A into a Banach A-bimodule is a derivation.

A very recent contribution, due to A. Ben Ali Essaleh, M.I. Ramrez and the second author of this note, establishes a new characterization of derivations on a C*-algebra A, in weaker terms than those in the definition of local derivations given by Kadison (cf. [3]). A linear mapping $T : A \to A$ is a *weak-local derivation* if for every $a \in A$ and every $\phi \in A^*$, there exists a derivation $D_{a,\phi} : A \to A$, depending on a and ϕ, satisfying $\phi T(a) = \phi D_{a,\phi}(a)$ (cf. [3, Definition 1.1 and page 3]). Theorem 3.4 in [3] shows that every weak-local derivation on a C*-algebra is a derivation.

When in the definition of local derivation we relax the condition concerning linearity but we assume *locality* at two points, we find the notion of 2-local derivation introduced by P. Šemrl in [21]. Let A be a Banach algebra. A (non-necessarily linear) mapping $\Delta : A \to A$ is said to be a *2-local derivation* if for every $a, b \in A$ there exists a derivation $D_{a,b} : A \to A$, depending on a and b, satisfying $\Delta(a) = D_{a,b}(a)$ and $\Delta(b) = D_{a,b}(b)$. Šemrl proves in [21, Theorem 2] that for an infinite-dimensional separable Hilbert space H, every 2-local derivation on the algebra $B(H)$ of all linear bounded operators on H is linear and a derivation. S.O. Kim and J.S. Kim gave in [14] a short proof of the fact that every 2-local derivation on M_n, the algebra of $n \times n$ matrices over the complex numbers, is a derivation. In a recent contribution, S. Ayupov and K. Kudaybergenov prove that every 2-local derivation on an arbitrary von Neumann algebra is a derivation (see [1]).

In this note we introduce the following new class of mappings on C*-algebras:

Definition 1.1. *Let A be a C*-algebra, a (non-necessarily linear) mapping $\Delta : A \to A$ is said to be a weak-2-local derivation (respectively, a *-derivation) on A if for every $a, b \in A$ and $\phi \in A^*$ there exists a derivation (respectively, a *-derivation) $D_{a,b,\phi} : A \to A$, depending on a, b and ϕ, such that $\phi\Delta(a) = \phi D_{a,b,\phi}(a)$ and $\phi\Delta(b) = \phi D_{a,b,\phi}(b)$.*

The main result of this paper (Theorem 3.11) establishes that every (non-necessarily linear) weak-2-local *-derivation on M_n is a linear *-derivation. We subsequently prove that every weak-2-local *-derivation on a finite dimensional C*-algebra is a linear *-derivation. These results deepen on our knowledge about derivations on C*-algebras and the excellent behavior that these operators have in the set of all maps on a finite dimensional C*-algebra.

As in previous studies on 2-local derivations and *-homomorphisms (cf. [1, 5, 6, 15] and [2]), the techniques in this paper rely on the Bunce-Wright-Mackey-Gleason theorem [4], however, certain subtle circumstances and pathologies, which are intrinsical to the lattice $\mathcal{P}(M_n)$ of all projections in M_n, increase the difficulties with respect to previous contributions. More concretely, the just mentioned Bunce-Wright-Mackey-Gleason theorem asserts that every bounded, finitely additive (vector) measure on the set of projections of a von Neumann algebra M with no direct summand of Type I_2 extends (uniquely) to a bounded linear operator defined on M. Subsequent improvements due to S.V. Dorofeev and A.N. Sherstnev establish that every completely additive measure on the set of projections of a von Neumann algebra with no type I_n ($n < \infty$) direct summands is bounded ([8, 22]). In the case of M_n, there exist completely additive measures on $\mathcal{P}(M_n)$ which are unbounded (see Remark 3.6). We establish a new result on non-commutative measure theory by proving that every weak-2-local *-derivation on M_n ($n \in \mathbb{N}$) is bounded on the set $\mathcal{P}(M_n)$ (see Proposition 3.10). This result shows that under a weak algebraic hypothesis we obtain an analytic implication, which provides the necessary conditions to apply the Bunce-Wright-Mackey-Gleason theorem.

We have restricted our study to matrix algebras and finite dimensional C*-algebras. This is not a complete novelty, the results on weak-2-local derivations on matrix algebra are interesting by themself, and there exists an abundant literature on derivations and local and 2-local derivations on matrix algebras. As we have commented before, some of the papers about derivations for general C*-algebras were previously studied for matrix algebras, or subsequently revisited to find new and shorter proofs (compare, for example, [9, 14, 16] and [10]). On the other hand, the new concept of weak-2-local derivations is so weak and general that makes to fail all the techniques and arguments we can find in the studies of local and 2-local derivations. The results can be thought as algebric results at first look, but they are actually Analysis and non-commutative measure theory. The main contributions here can be thought as algebraic results at first look, but, as we have commented above, they are actually based on techniques of functional analysis and non-commutative measure theory.

In this paper we also prove that every weak-2-local derivation on M_2 is a linear derivation. Numerous topics remain to be studied after these first answers. Weak-2-local derivations on M_n and weak-2-local (*-)derivations on von Neumann algebras and C*-algebras should be examined.

2. General Properties of Weak-2-Local Derivations

Let A be a C*-algebra. Henceforth, the symbol A_{sa} will denote the self-adjoint part of A. It is clear, by the Hahn-Banach theorem, that every weak-2-local derivation Δ on A is 1-homogeneous, that is, $\Delta(\lambda a) = \lambda\Delta(a)$, for every $\lambda \in \mathbb{C}$, $a \in A$.

We observe that the set Der(A), of all derivations on A, is a closed subspace of the Banach space $B(A)$. This fact can be applied to show that a mapping $\Delta : A \to A$ is a weak-2-local derivation if and only if for any set $V \subseteq A^*$, whose linear span is A^*, the following property holds: for every $a, b \in A$ and $\phi \in V$ there exists a derivation $D_{a,b,\phi} : A \to A$, depending on a, b and ϕ, such that $\phi\Delta(a) = \phi D_{a,b,\phi}(a)$ and $\phi\Delta(b) = \phi D_{a,b,\phi}(b)$. This result guarantees that in Definition 1.1 the set A^* can be replaced, for example, with the set of positive functionals on A.

Let $\Delta : A \to A$ be a mapping on a C*-algebra. We define a new mapping $\Delta^\sharp : A \to A$ given by $\Delta^\sharp(a) := \Delta(a^*)^*$ $(a \in A)$. Clearly, $\Delta^{\sharp\sharp} = \Delta$. It is easy to see that Δ is linear (respectively a derivation) if and only if $\Delta^\sharp$ is linear (respectively, a derivation). We also know that $\Delta(A_{sa}) \subseteq A_{sa}$ whenever $\Delta^\sharp = \Delta$.

Let A be a C*-algebra. A mapping $\Delta : A \to A$ is said to be a *weak-2-local *-derivation* on A if for every $a, b \in A$ and $\phi \in A^*$ there exists a *-derivation $D_{a,b,\phi} : A \to A$, depending on a, b and ϕ, such that

$$\phi\Delta(a) = \phi D_{a,b,\phi}(a) \text{ and } \phi\Delta(b) = \phi D_{a,b,\phi}(b).$$

Clearly, every weak-2-local *-derivation Δ on A is a weak-2-local derivation and $\Delta^\sharp = \Delta$. However, we do not know if every weak-2-local derivation with $\Delta^\sharp = \Delta$ is a weak-2-local *-derivation. Anyway, for a weak-2-local derivation $\Delta : A \to A$ with $\Delta^\sharp = \Delta$, the mapping $\Delta|_{A_{sa}} : A_{sa} \to A_{sa}$ is a weak-2-local Jordan derivation, that is, for every $a, b \in A_{sa}$ and $\phi \in (A_{sa})^*$, there exists a Jordan *-derivation $D_{a,b,\phi} : A_{sa} \to A_{sa}$, depending on a, b and ϕ, such that

$$\phi\Delta(a) = \phi D_{a,b,\phi}(a) \text{ and } \phi\Delta(b) = \phi D_{a,b,\phi}(b).$$

To see this, let $a, b \in A_{sa}$ and $\phi \in (A_{sa})^*$, by assumptions, there exists a derivation $D_{a,b,\phi} : A \to A$, depending on a, b and ϕ, such that $\phi\Delta(a) = \phi D_{a,b,\phi}(a)$ and $\phi\Delta(b) = \phi D_{a,b,\phi}(b)$. Since $\phi\Delta(a) = \phi\Delta(a)^* = \phi D^\sharp_{a,b,\phi}(a)$ and $\phi\Delta(b) = \phi D^\sharp_{a,b,\phi}(b)$, we get

$$\phi\Delta(a) = \phi\frac{1}{2}\left(D_{a,b,\phi} - D^\sharp_{a,b,\phi}\right)(a), \text{ and } \phi\Delta(b) = \phi\frac{1}{2}\left(D_{a,b,\phi} - D^\sharp_{a,b,\phi}\right)(b),$$

where $\frac{1}{2}\left(D_{a,b,\phi} - D^\sharp_{a,b,\phi}\right)$ is a *-derivation on A.

The following properties can be also deduced from the fact stated in the second paragraph of this section.

Lemma 2.1. *Let A be a C*-algebra. The following statements hold:*

(*a*) *The linear combination of weak-2-local derivations on A is a weak-2-local derivation on A;*

(*b*) *A mapping $\Delta : A \to A$ is a weak-2-local derivation if and only if $\Delta^\sharp$ is a weak-2-local derivation;*

(*c*) *A mapping* $\Delta : A \to A$ *is a weak-2-local derivation if and only if* $\Delta_s = \frac{1}{2}(\Delta + \Delta^\sharp)$ *and* $\Delta_a = \frac{1}{2i}(\Delta - \Delta^\sharp)$ *are weak-2-local derivations. Clearly,* Δ *is linear if and only if both* Δ_s *and* Δ_a *are.*

Proof. (*a*) Suppose $\Delta_1, \ldots, \Delta_n : A \to A$ are weak-2-local derivations and $\lambda_1, \ldots, \lambda_n$ are complex numbers. Given $a, b \in A$ and $\phi \in A^*$, we can find derivations $D^j_{a,b,\phi} : A \to A$ satisfying $\phi\Delta_j(a) = \phi D^j_{a,b,\phi}(a)$ and $\phi\Delta_j(b) = \phi D^j_{a,b,\phi}(b)$, for every $j = 1, \ldots, n$. Then

$$\phi\left(\sum_{j=1}^{n} \lambda_j \Delta_j\right)(a) = \phi\left(\sum_{j=1}^{n} \lambda_j D^j_{a,b,\phi}\right)(a)$$

and

$$\phi\left(\sum_{j=1}^{n} \lambda_j \Delta_j\right)(b) = \phi\left(\sum_{j=1}^{n} \lambda_j D^j_{a,b,\phi}\right)(b),$$

which proves the statement.

(*b*) Suppose $\Delta : A \to A$ is a weak-2-local derivation. Given $a, b \in A$, $\phi \in A^*$, we consider the mapping $\phi^* \in A^*$ defined by $\phi^*(a) := \overline{\phi(a^*)}$ $(a \in A)$. By the assumptions on Δ there exists a derivation $D_{a,b,\phi} : A \to A$ such that $\phi^*\Delta(a^*) = \phi D_{a,b,\phi}(a^*)$ and $\phi\Delta(b^*) = \phi D_{a,b,\phi}(b^*)$. We deduce from the above that $\phi\Delta^\sharp(a) = \phi D^\sharp_{a,b,\phi}(a)$ and $\phi\Delta^\sharp(b) = \phi D^\sharp_{a,b,\phi}(b)$, which proves the statement concerning $\Delta^\sharp$. Since $\Delta^{\sharp\sharp} = \Delta$ the reciprocal implication is clear.

The statement in (*c*) follows from (*a*) and (*b*). □

Remark 2.2. A *-derivation on a C*-algebra A is a derivation D on A satisfying $D^\sharp = D$, equivalently, $D(a^*) = D(a)^*$, for every $a \in A$. It is easy to see that, for each *-derivation D on A, the mapping $D|_{A_{sa}} : A_{sa} \to A_{sa}$ is a Jordan derivation, that is, $D(a \circ b) = a \circ D(b) + b \circ D(a)$, for every $a, b \in A_{sa}$, where $a \circ b = \frac{1}{2}(ab + ba)$ (we should recall that A_{sa} is not, in general, an associative subalgebra of A, but it is always a Jordan subalgebra of A).

Conversely, if $\delta : A_{sa} \to A_{sa}$ is a Jordan derivation on A_{sa}, then the linear mapping $\widehat{\delta} : A \to A$, $\widehat{\delta}(a + ib) = \delta(a) + i\delta(b)$ is a Jordan *-derivation on A, and hence a *-derivation by [11, Theorem 6.3] and [17, Corollary 17]. When M is a von Neumann algebra, we can deduce, via Sakai's theorem (cf. [19, Theorem 4.1.6]) that for every Jordan derivation $\delta : M_{sa} \to M_{sa}$, there exists $z \in iM_{sa}$ satisfying $\delta(a) = [z, a]$, for every $a \in M$.

Lemma 2.3. *Let* Δ *be a weak-2-local* **-derivation on a* C**-algebra* A*. Then* $\Delta(a + ib) = \Delta(a) + i\Delta(b) = \Delta(a - ib)^*$*, for every* $a, b \in A_{sa}$*.*

Proof. Let us fix $a, b \in A_{sa}$. By assumptions, for each $\phi \in A^*$ with $\phi^* = \phi$ (that is, $\phi(a^*) = \overline{\phi(a)}$ $(a \in A)$. There exists a *-derivation $D_{a,a+ib,\phi}$ on A, depending on $a + ib$, a and ϕ, such that

$$\phi\Delta(a + ib) = \phi D_{a,a+ib,\phi}(a + ib) = \phi D_{a,a+ib,\phi}(a) + i\phi D_{a,a+ib,\phi}(b),$$

and

$$\phi\Delta(a) = \phi D_{a,a+ib,\phi}(a).$$

Then $\Re\mathrm{e}\phi\Delta(a+ib) = \phi D_{a,a+ib,\phi}(a)$, for every $\phi \in A^*$ with $\phi^* = \phi$, which proves that $\Delta(a+ib)+\Delta(a+ib)^* = 2\Delta(a)$. We can similarly check that $\Delta(a + ib) - \Delta(a + ib)^* = 2i\Delta(b)$. □

It is well known that every derivation D on a unital C*-algebra A satisfies that $D(1) = 0$. Since the elements in A^* separate the points in A, we also get:

Lemma 2.4. *Let* Δ *be a weak-2-local derivation on a unital* C**-algebra. Then* $\Delta(1) = 0$. □

Lemma 2.5. *Let Δ be a weak-2-local derivation on a unital C^*-algebra A. Then $\Delta(1-x)+\Delta(x)=0$, for every $x\in A$.*

Proof. Let $x \in A$. Given $\phi \in A^*$, there exists a derivation $D_{x,1-x,\phi} : A \to A$, such that $\phi\Delta(x) = \phi D_{x,1-x,\phi}(x)$ and $\phi\Delta(1-x) = \phi D_{x,1-x,\phi}(1-x)$. Therefore,

$$\phi(\Delta(1-x)+\Delta(x)) = \phi D_{x,1-x,\phi}(1-x+x) = 0.$$

We conclude by the Hahn-Banach theorem that $\Delta(1-x)+\Delta(x) = 0$. □

Lemma 2.6. *Let Δ be a weak-2-local derivation on a unital C^*-algebra, and let p be a projection in A. Then*

$$p\Delta(p)p = 0 \quad \textit{and} \quad (1-p)\Delta(p)(1-p) = 0.$$

Proof. Let ϕ be a functional in A^* satisfying $\phi = (1-p)\phi(1-p)$. Pick a derivation $D_{p,\phi} : A \to A$ satisfying $\phi\Delta(p) = \phi D_{p,\phi}(p)$. Then

$$\phi\Delta(p) = \phi\left(D_{p,\phi}(p)p + pD_{p,\phi}(p)\right) = 0,$$

where in the last equality we applied $\phi = (1-p)\phi(1-p)$. Lemma 3.5 in [3] implies that $(1-p)\Delta(p)(1-p) = 0$. Replacing p with $1-p$ and applying Lemma 2.5, we get $0 = p\Delta(1-p)p = -p\Delta(p)p$. □

The first statement in the following proposition is probably part of the folklore in the theory of derivations, however we do not know an explicit reference for it.

Proposition 2.7. *Let A be a C^*-algebra, $D : A \to A$ a derivation (respectively, a *-derivation), and let p be a projection in A. Then the operator $pDp|_{pAp} : pAp \to pAp$, $x \mapsto pD(x)p$ is a derivation (respectively, a *-derivation) on pAp. Consequently, if $\Delta : A \to A$ is a weak-2-local derivation (respectively, a weak-2-local *-derivation) on A, the mapping $p\Delta p|_{pAp} : pAp \to pAp$, $x \mapsto p\Delta(x)p$ is a weak-2-local derivation (respectively, a weak-2-local *-derivation) on pAp.*

Proof. Let T denote the linear mapping $pDp|_{pAp} : pAp \to pAp$, $x \mapsto pD(x)p$. We shall show that T is a derivation on pAp. Let $x, y \in pAp$. Since $px = xp = x$ and $py = yp = y$, we have

$$T(xy) = pD(xy)p = pD(x)yp + pxD(y)p = pD(x)py + xpD(y)p = T(x)y + xT(y).$$

□

3. Weak-2-Local Derivations on Matrix Algebras

In this section we shall study weak-2-local derivations on matrix algebras.

Lemma 3.1. *Let $\Delta : M_n \to M_n$ be a weak-2-local derivation on M_n. Let tr denote the unital trace on M_n. Then, $tr\Delta(x) = 0$, for every $x \in M_n$.*

Proof. Let x be an arbitrary element in M_n. By Sakai's theorem (cf. [19, Theorem 4.1.6]), every derivation on M_n is inner. We deduce from our hypothesis that there exists an element $z_{x,tr}$ in M_n, depending on tr and x, such that $tr\Delta(x) = tr[z_{x,\phi}, x] = tr(z_{x,\phi}x - xz_{x,\phi}) = 0$. □

The algebra M_2 of all 2 by 2 matrices must be treated with independent arguments.

We set some notation. Given two elements ξ, η in a Hilbert space H, the symbol $\xi \otimes \eta$ will denote the rank-one operator in $B(H)$ defined by $\xi \otimes \eta(\kappa) = (\kappa|\eta)\xi$. We can also regard $\phi = \xi \otimes \eta$ as an element in the trace class operators (that is, in the predual of $B(H)$) defined by $\xi \otimes \eta(a) = (a(\xi)|\eta)$ $(a \in B(H))$.

Theorem 3.2. *Every weak-2-local derivation on M_2 is linear and a derivation.*

Proof. Let Δ be a weak-2-local derivation on M_2. To simplify notation we set $e_{ij} = \xi_i \otimes \xi_j$ for $1 \leq i, j \leq 2$, where $\{\xi_1, \xi_2\}$ is a fixed orthonormal basis of $\mathbb{C}^2$. We also write $p_1 = e_{11}$ and $p_2 = e_{22}$. The proof is divided into several steps.

Lemma 3.1 shows that

$$tr\Delta(x) = 0, \tag{1}$$

for every $x \in M_2$.

Step I. Let us write $\Delta(p_1) = \sum_{i,j=1}^{2} \lambda_{ij} e_{ij}$, where $\lambda_{ij} \in \mathbb{C}$. For $\phi = \xi_1 \otimes \xi_1 \in M_2^*$ there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ and p_1, such that $\phi\Delta(p_1) = \phi[z, p_1]$. Since

$$[z, p_1] = -z_{12}e_{12} + z_{21}e_{21}, \tag{2}$$

we deduce that $\lambda_{11} = \phi\Delta(p_1) = \phi[z, p_1] = 0$. Since $\lambda_{11} + \lambda_{22} = tr\Delta(p_1) = 0$, we also have $\lambda_{22} = 0$. Therefore,

$$\Delta(p_1) = \lambda_{12}e_{12} + \lambda_{21}e_{21}.$$

Defining $z_0 := \lambda_{21}e_{21} - \lambda_{12}e_{12}$, it follows that $\widetilde{\Delta} = \Delta - [z_0, .]$ is a weak-2-local derivation (cf. Lemma 2.1(a)) which vanishes at p_1. Applying Lemma 2.5, we deduce that

$$\widetilde{\Delta}(p_1) = \widetilde{\Delta}(p_2) = 0. \tag{3}$$

Step II. Let us write $\widetilde{\Delta}(e_{12}) = \sum_{i,j=1}^{2} \lambda_{ij} e_{ij}$, with $\lambda_{22} = -\lambda_{11}$ (cf. (1)). For $\phi = \xi_1 \otimes \xi_2 \in M_2^*$, there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ and e_{12}, such that $\phi\widetilde{\Delta}(e_{12}) = \phi[z, e_{12}]$. Since

$$[z, e_{12}] = -z_{21}p_1 + (z_{11} - z_{22})e_{12} + z_{21}p_2, \tag{4}$$

we see that $\lambda_{21} = 0$.

For $\phi = \xi_1 \otimes \xi_1 - \xi_1 \otimes \xi_2 \in M_2^*$, there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ, p_1 and e_{12}, such that $\phi\widetilde{\Delta}(p_1) = \phi[z, p_1]$ and $\phi\widetilde{\Delta}(e_{12}) = \phi[z, e_{12}]$. The identities (2) and (4) (and (3)) imply that $\lambda_{11} = -z_{21}$ and $0 = -z_{21}$, and hence $\lambda_{11} = 0$. Therefore, there exists a complex number δ satisfying

$$\widetilde{\Delta}(e_{12}) = \delta e_{12} = [z_1, e_{12}], \tag{5}$$

where $z_1 = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}$. We observe that $[z_1, \lambda p_1 + \mu p_2] = 0$, for every $\lambda, \mu \in \mathbb{C}$. Thus, the mapping $\widehat{\Delta} = \widetilde{\Delta} - [z_1, .] = \Delta - [z_0, .] - [z_1, .]$ is a weak-2-local derivation satisfying

$$\widehat{\Delta}(e_{12}) = \widehat{\Delta}(p_1) = \widehat{\Delta}(p_2) = 0. \tag{6}$$

Step III. Let us write $\widehat{\Delta}(e_{21}) = \sum_{i,j=1}^{2} \lambda_{ij} e_{ij}$, with $\lambda_{11} = -\lambda_{22}$ (see Lemma 3.1). For $\phi = \xi_2 \otimes \xi_1 \in M_2^*$, there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ and e_{21}, such that $\phi\widehat{\Delta}(e_{21}) = \phi[z, e_{21}]$. Since

$$[z, e_{21}] = z_{12}p_1 - (z_{11} - z_{22})e_{21} - z_{12}p_2, \tag{7}$$

we see that $\lambda_{12} = 0$.

Take now $\phi = \xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_1 \in M_2^*$. By hypothesis, there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ, p_1 and e_{21}, such that $\phi\widehat{\Delta}(p_1) = \phi[z, p_1]$ and $\phi\widehat{\Delta}(e_{21}) = \phi[z, e_{21}]$. We deduce from (2), (7) and (6) that $z_{12} = \lambda_{11}$ and $z_{12} = 0$, which gives $\lambda_{11} = 0$.

For $\phi = \xi_2 \otimes \xi_1 - \xi_1 \otimes \xi_2 \in M_2^*$, there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ, e_{12} and e_{21}, such that $\phi\widehat{\Delta}(e_{12}) = \phi[z, e_{12}]$ and $\phi\widehat{\Delta}(e_{21}) = \phi[z, e_{21}]$. We apply (4), (7) and (6) to obtain $-\lambda_{21} = z_{11} - z_{22}$ and $0 = \phi\widehat{\Delta}(e_{12}) = z_{11} - z_{22}$, which proves that $\lambda_{21} = 0$. Therefore

$$\widehat{\Delta}(e_{21}) = 0. \tag{8}$$

We shall finally prove that $\widehat{\Delta} \equiv 0$, and consequently $\Delta = [z_0, .] + [z_1, .]$ is a linear mapping and a derivation.

Step IV. Let us fix $\alpha, \beta \in \mathbb{C}$. We write $\widehat{\Delta}(\alpha e_{12} + \beta e_{21}) = \sum_{i,j=1}^{2} \lambda_{ij} e_{ij}$, where $\lambda_{11} = -\lambda_{22}$. For $\phi = \xi_2 \otimes \xi_1 \in M_2^*$, there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ, e_{12} and $\alpha e_{12} + \beta e_{21}$, such that $\phi\widehat{\Delta}(e_{12}) = \phi[z, e_{12}]$ and $\phi\widehat{\Delta}(\alpha e_{12} + \beta e_{21}) = \phi[z, \alpha e_{12} + \beta e_{21}]$. Since

$$[z, \alpha e_{12} + \beta e_{21}] = (\beta z_{12} - \alpha z_{21}) p_1 + \alpha(z_{11} - z_{22}) e_{12} + \beta(z_{22} - z_{11}) e_{21} + (\alpha z_{21} - \beta z_{12}) p_2, \tag{9}$$

we have $\lambda_{12} = \alpha(z_{11} - z_{22})$. Now, the identities (4) and (6) imply $z_{11} - z_{22} = 0$, and hence $\lambda_{12} = 0$.

For $\phi = \xi_1 \otimes \xi_2 \in M_2^*$ there exists an element $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ in M_2, depending on ϕ, e_{21} and $\alpha e_{12} + \beta e_{21}$, such that $\phi\widehat{\Delta}(e_{21}) = \phi[z, e_{21}]$ and $\phi\widehat{\Delta}(\alpha e_{12} + \beta e_{21}) = \phi[z, \alpha e_{12} + \beta e_{21}]$. We deduce from (7), (9) and (8), that $\lambda_{21} = \beta(z_{22} - z_{11})$ and $z_{22} - z_{11} = 0$, witnessing that $\lambda_{21} = 0$.

For $\phi = \xi_1 \otimes \xi_1 + \beta \xi_2 \otimes \xi_1 + \alpha \xi_1 \otimes \xi_2 \in M_2^*$ there exists an element z in M_2, depending on ϕ, p_1 and $\alpha e_{12} + \beta e_{21}$, such that $\phi\widehat{\Delta}(p_1) = \phi[z, p_1]$ and $\phi\widehat{\Delta}(\alpha e_{12} + \beta e_{21}) = \phi[z, \alpha e_{12} + \beta e_{21}]$. It follows from (9) and (2) that $\lambda_{11} + \beta\lambda_{12} + \alpha\lambda_{21} = \beta z_{12} - \alpha z_{21}$, and $-\beta z_{12} + \alpha z_{21} = 0$, which implies that $\lambda_{11} = 0$, and hence

$$\widehat{\Delta}(\alpha e_{12} + \beta e_{21}) = 0, \tag{10}$$

for every $\alpha, \beta \in \mathbb{C}$.

Step V. In this step we fix two complex numbers $t, \alpha \in \mathbb{C}$, and we write $\widehat{\Delta}(t p_1 + \alpha e_{12}) = \sum_{i,j=1}^{2} \lambda_{ij} e_{ij}$, with $\lambda_{11} = -\lambda_{22}$. Applying that $\widehat{\Delta}$ is a weak-2-local derivation with $\phi = \xi_1 \otimes \xi_1 \in M_2^*$, e_{12} and $t p_1 + \alpha e_{12}$, we deduce from the identity

$$[z, t p_1 + \alpha e_{12}] = -\alpha z_{21} p_1 + (\alpha z_{11} - t z_{12} - \alpha z_{22}) e_{12} + t z_{21} e_{21} + \alpha z_{21} p_2, \tag{11}$$

combined with (4) and (6), that $-\alpha z_{21} = \lambda_{11}$, and $z_{21} = 0$, and hence $\lambda_{11} = 0$.

Repeating the above arguments with $\phi = \xi_1 \otimes \xi_2 \in M_2^*$, p_1 and $t p_1 + \alpha e_{12}$, we deduce from (2), (11) and (6), that $\lambda_{21} = t z_{21}$ and $z_{21} = 0$, which proves that $\lambda_{21} = 0$.

A similar reasoning with $\phi = t \xi_1 \otimes \xi_1 - \alpha \xi_2 \otimes \xi_1 \in M_2^*$, $\alpha e_{12} + \alpha e_{21}$ and $t p_1 + \alpha e_{12}$, gives, via (9), (10), and (11), that $t\lambda_{11} - \alpha\lambda_{12} = t\alpha z_{12} - t\alpha z_{21} - \alpha^2 z_{11} + \alpha^2 z_{22}$ and $t\alpha z_{12} - t\alpha z_{21} - \alpha^2 z_{11} + \alpha^2 z_{22} = 0$. Therefore $\alpha\lambda_{12} = 0$ and

$$\widehat{\Delta}(t p_1 + \alpha e_{12}) = 0, \tag{12}$$

for every $t, \alpha \in \mathbb{C}$.

A similar argument shows that

$$\widehat{\Delta}(tp_1 + \beta e_{21}) = 0, \tag{13}$$

for every $t, \beta \in \mathbb{C}$.

Step VI. In this step we fix $t, \alpha, \beta \in \mathbb{C}$, and we write

$$\widehat{\Delta}(tp_1 + \alpha e_{12} + \beta e_{21}) = \sum_{i,j=1}^{2} \lambda_{ij} e_{ij},$$

with $\lambda_{11} = -\lambda_{22}$. Applying that $\widehat{\Delta}$ is a weak-2-local derivation with $\phi = \alpha\xi_1 \otimes \xi_2 + \beta\xi_2 \otimes \xi_1 \in M_2^*$, p_1 and $tp_1 + \alpha e_{12} + \beta e_{21}$, we deduce from the identity

$$[z, tp_1 + \alpha e_{12} + \beta e_{21}] = (\beta z_{12} - \alpha z_{21})p_1 + (\alpha z_{11} - \alpha z_{22} - tz_{12})e_{12} \tag{14}$$

$$+(\beta z_{22} - \beta z_{11} + tz_{21})e_{21} + (\alpha z_{21} - \beta z_{12})p_2,$$

combined with (2) and (6), that $\beta\lambda_{12} + \alpha\lambda_{21} = t(\alpha z_{21} - \beta z_{12})$ and $\alpha z_{21} - \beta z_{12} = 0$, which gives $\beta\lambda_{12} + \alpha\lambda_{21} = 0$.

Repeating the above arguments with $\phi = t\xi_1 \otimes \xi_1 + \alpha\xi_1 \otimes \xi_2 \in M_2^*$, e_{21} and $tp_1 + \alpha e_{12} + \beta e_{21}$, we deduce from (7), (8) and (14), that $t\lambda_{11} + \alpha\lambda_{21} = \beta(tz_{12} + \alpha z_{22} - \alpha z_{11})$, and $tz_{12} + \alpha z_{22} - \alpha z_{11} = 0$ and hence $t\lambda_{11} + \alpha\lambda_{21} = 0$.

A similar reasoning with $\phi = t\xi_1 \otimes \xi_1 + \beta\xi_2 \otimes \xi_1 \in M_2^*$, e_{12} and $tp_1 + \alpha e_{12} + \beta e_{21}$, gives, via (4), (6) and (14), that $t\lambda_{11} + \beta\lambda_{12} = \alpha(-tz_{21} + \beta z_{11} - \beta z_{22})$ and $-tz_{21} + \beta z_{11} - \beta z_{22} = 0$. Therefore $t\lambda_{11} + \beta\lambda_{12} = 0$. The equations $\beta\lambda_{12} + \alpha\lambda_{21} = 0$, $t\lambda_{11} + \alpha\lambda_{21} = 0$, and $t\lambda_{11} + \beta\lambda_{12} = 0$ imply that $t\lambda_{11} = \beta\lambda_{12} = \alpha\lambda_{21} = 0$, which, combined with (10), (12) and (13), prove that

$$\widehat{\Delta}(tp_1 + \alpha e_{12} + \beta e_{21}) = 0, \tag{15}$$

for every $t, \alpha, \beta \in \mathbb{C}$.

Finally, since

$$[z, tp_1 + \alpha e_{12} + \beta e_{21} + sp_2] = [z, (t-s)p_1 + \alpha e_{12} + \beta e_{21}],$$

for every $z \in M_2$, it follows from the fact that $\widehat{\Delta}$ is a weak-2-local derivation, (15), and the Hahn-Banach theorem that

$$\widehat{\Delta}(tp_1 + \alpha e_{12} + \beta e_{21} + sp_2) = \widehat{\Delta}((t-s)p_1 + \alpha e_{12} + \beta e_{21}) = 0,$$

for every $t, s, \alpha, \beta \in \mathbb{C}$, which concludes the proof. □

The rest of this section is devoted to the study of weak-2-local derivations on M_n. For later purposes, we begin with a strengthened version of Lemma 2.6.

Lemma 3.3. *Let $\Delta : M \to M$ be a weak-2-local projection on a von Neumann algebra M. Suppose p, q are orthogonal projections in M, and a is an element in M satisfying $pa = ap = qa = aq = 0$. Then the identities:*

$$p\Delta(a + \lambda p + \mu q)q = p\Delta(\lambda p + \mu q)q, \text{ and, } p\Delta(a + \lambda p)p = \lambda p\Delta(p)p = 0,$$

hold for every $\lambda, \mu \in \mathbb{C}$. Furthermore, if b is another element in M, we also have

$$q\Delta(b + \lambda p)q = q\Delta(b)q, \text{ and } q\Delta(qbq + \lambda q)q = q\Delta(qbq)q.$$

Proof. Clearly, $p+q$ is a projection in M. Let ϕ any functional in M_* satisfying $\phi = (p+q)\phi(p+q)$. By hypothesis, there exists an element $z_{\phi,\lambda p+\mu q,a+\lambda p+\mu q} \in M$, depending on ϕ, $\lambda p+\mu q$, and $a+\lambda p+\mu q$, such that

$$\phi\Delta(a+\lambda p+\mu q) = \phi[z_{\phi,\lambda p+\mu q,a+\lambda p+\mu q}, a+\lambda p+\mu q],$$

and

$$\phi\Delta(\lambda p+\mu q) = \phi[z_{\phi,\lambda p+\mu q,a+\lambda p+\mu q}, \lambda p+\mu q].$$

Since

$$\phi[z_{\phi,\lambda p+\mu q,a+\lambda p+\mu q}, a+\lambda p+\mu q] = \phi[z_{\phi,\lambda p+\mu q,a+\lambda p+\mu q}, \lambda p+\mu q],$$

we deduce that $\phi(\Delta(a+\lambda p+\mu q) - \Delta(\lambda p+\mu q)) = 0$, for every $\phi \in M_*$ with $\phi = (p+q)\phi(p+q)$. Lemma 2.2 in [3] implies that

$$(p+q)\Delta(a+\lambda p+\mu q)(p+q) = (p+q)\Delta(\lambda p+\mu q)(p+q).$$

Multiplying on the left by p and on the right by q, we get $p\Delta(a+\lambda p+\mu q)q = p\Delta(\lambda p+\mu q)q$. The other statements follow in a similar way. □

Proposition 3.4. *Let $\Delta : M \to M$ be a weak-2-local derivation on a von Neumann algebra M. Then for every family $\{p_1, \ldots, p_n\}$ of mutually orthogonal projections in M, and every $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, we have*

$$\Delta\left(\sum_{j=1}^{n} \lambda_j p_j\right) = \sum_{j=1}^{n} \lambda_j \Delta(p_j).$$

Proof. Let $p_1, \ldots, p_n$ be mutually orthogonal projections in M. First, we observe that, by the last statement in Lemma 3.3, for any $1 \le i, k \le n,\ i \ne k$, we have

$$\begin{aligned} &(p_i+p_k)\Delta(\lambda_i p_i + \lambda_k p_k)(p_i+p_k) \\ &= (p_i+p_k)\Delta((\lambda_i-\lambda_k)p_i + \lambda_k(p_i+p_k))(p_i+p_k) = (p_i+p_k)\Delta((\lambda_i-\lambda_k)p_i)(p_i+p_k) \\ &= (p_i+p_k)\lambda_i\Delta(p_i)(p_i+p_k) - (p_i+p_k)\lambda_k\Delta(p_i)(p_i+p_k) \\ &= (p_i+p_k)\lambda_i\Delta(p_i)(p_i+p_k) - (p_i+p_k)\lambda_k\Delta(p_i+p_k-p_k)(p_i+p_k) \\ &= (p_i+p_k)\lambda_i\Delta(p_i)(p_i+p_k) + (p_i+p_k)\lambda_k\Delta(p_k)(p_i+p_k), \end{aligned}$$

where the last step is obtained by another application of Lemma 3.3. Multiplying on the left hand side by p_i and on the right hand side by p_k we obtain:

$$p_i\Delta(\lambda_i p_i + \lambda_k p_k)p_k = \lambda_i p_i\Delta(p_i)p_k + \lambda_k p_i\Delta(p_k)p_k,\ (1 \le i,k \le n,\ i \ne k). \tag{16}$$

Let us write $r = 1 - \sum_{j=1}^{n} p_j$ and

$$\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right) = r\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)r + \sum_{i=1}^{n}\left(p_i\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)r\right) + \sum_{k=1}^{n}\left(r\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)p_k\right) + \sum_{i,k=1}^{n}\left(p_i\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)p_k\right). \tag{17}$$

Applying Lemma 3.3 we get: $r\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)r = 0$. Given $1 \le i \le n$, the same Lemma 3.3 implies that

$$p_i\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)r = p_i\Delta\left(\sum_{j=1, j\ne i}^{n}\lambda_j p_j + \lambda_i p_i\right)r = \lambda_i p_i\Delta(p_i)r, \tag{18}$$

and similarly

$$r\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)p_i = \lambda_i r\Delta(p_i)p_i, \quad \text{and} \quad p_i\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)p_i = 0. \tag{19}$$

Given $1 \leq i, k \leq n$, $i \neq k$, Lemma 3.3 proves that

$$p_i\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)p_k = p_i\Delta\left(\sum_{j=1, j\neq i,k}^{n}\lambda_j p_j + \lambda_i p_i + \lambda_k p_k\right)p_k \tag{20}$$

$$= p_i\Delta(\lambda_i p_i + \lambda_k p_k)p_k = (\text{by } (16)) = \lambda_i p_i\Delta(p_i)p_k + \lambda_k p_i\Delta(p_k)p_k.$$

We also have:

$$\Delta(p_j) = p_j\Delta(p_j)r + r\Delta(p_j)p_j + \sum_{k=1}^{n} p_j\Delta(p_j)p_k + p_k\Delta(p_j)p_j. \tag{21}$$

Finally, the desired statement follows from (17), (18), (19), (20), and (21). □

Corollary 3.5. *Let $\Delta : M \to M$ be a weak-2-local derivation on a von Neumann algebra. Suppose a and b are elements in M which are written as finite linear complex linear combinations $a = \sum_{i=1}^{m_1}\lambda_i p_i$ and $b = \sum_{j=1}^{m_2}\mu_j q_j$, where $p_1, \ldots, p_{m_1}, q_1, \ldots, q_{m_2}$ are mutually orthogonal projections (these hypotheses hold, for example, when a and b are algebraic orthogonal self-adjoint elements in M). Then $\Delta(a + b) = \Delta(a) + \Delta(b)$.* □

Let $\Delta : M \to M$ be a weak-2-local derivation on a von Neumann algebra. Let $\mathcal{P}(M)$ denote the set of all projections in M. Proposition 3.4 asserts that the mapping $\mu : \mathcal{P}(M) \to M$, $p \mapsto \mu(p) := \Delta(p)$ is a finitely additive measure on $\mathcal{P}(M)$ in the usual terminology employed around the Mackey-Gleason theorem (cf. [4], [8], and [22]), i.e. $\mu(p + q) = \mu(p) + \mu(q)$, whenever p and q are mutually orthogonal projections in M. Unfortunately, we do not know if, the measure μ is, in general, bounded.

We recall some other definitions. Following the usual nomenclature in [1, 8, 22] or [15], a scalar or signed measure $\mu : \mathcal{P}(M) \to \mathbb{C}$ is said to be *completely additive* or a *charge* if

$$\mu\left(\sum_{i\in I} p_i\right) = \sum_{i\in I}\mu(p_i) \tag{22}$$

for every family $\{p_i\}_{i\in I}$ of mutually orthogonal projections in M, where $\sum_{i\in I} p_i$ is the sum of the family (p_i) with respect to the weak*-topology of M (cf. [19, Page 30]), and in the right hand side, the convergence of an uncountable family is understood as summability in the usual sense. The main results in [7] shows that if M is a von Neumann algebra of type I with no type I_n ($n < \infty$) direct summands and M acts on a separable Hilbert space, then any completely additive measure on $\mathcal{P}(M)$ is bounded. The conclusion remains true when M is a continuous von Neumann algebra (cf. [8], see also [22]). The next remark shows that is not always true when M is a type I_n factor with $2 \leq n < \infty$.

Remark 3.6. In M_n (with $2 \leq n < \infty$) every family of non-zero pairwise orthogonal projections is necessarily finite so, every finitely additive measure μ on $\mathcal{P}(M_n)$ is completely additive. However, the existence of unbounded finitely additive measures on $\mathcal{P}(M_n)$ is well known in literature, see, for example, the following example inspired by [24]. By the arguments at the end of the proof of [24, Theorem 3.1], we can always find a countable infinite set of projections $\{p_n : n \in \mathbb{N}\}$ which is linearly independent over $\mathbb{Q}$, and we can extend it, via Zorn's lemma, to a Hamel base $\{z_j : j \in \Lambda\}$ for $(M_n)_{sa}$ over $\mathbb{Q}$. Clearly, every element in M_n can be written as a finite $\mathbb{Q} \oplus i\mathbb{Q}$-linear combination of elements in this base. If we define a $\mathbb{Q} \oplus i\mathbb{Q}$-linear mapping $\mu : M_n \to \mathbb{C}$ given by

$$\mu(z_j) := \begin{cases} (n+1), & \text{if } z_j = p_n \text{ for some natural number } n; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\mu|_{\mathcal{P}(M_n)} : \mathcal{P}(M_n) \to \mathbb{C}$ is an unbounded completely additive measure.

We shall show later that the pathology exhibited in the previous remark cannot happen for the measure μ determined by a weak-2-local *-derivation on M_n (cf. Proposition 3.10). The case $n = 2$ was fully treated in Theorem 3.2.

Proposition 3.7. *Let $\Delta : M_3 \to M_3$ be a weak-2-local *-derivation. Suppose p_1, p_2, p_3 are mutually orthogonal minimal projections in M_3, e_{k3} is the unique minimal partial isometry in M_3 satisfying $e_{k3}^* e_{k3} = p_3$ and $e_{k3}e_{k3}^* = p_k$ $(k = 1, 2)$. Let us assume that $\Delta(p_j) = \Delta(e_{k3}) = 0$, for every $j = 1, 2, 3$, $k = 1, 2$. Then*

$$\Delta\left(\sum_{j=1}^{3} \lambda_j p_j + \sum_{k=1}^{2} \mu_k e_{k3}\right) = 0,$$

for every $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2$ in $\mathbb{C}$.

Proof. Along this proof we write $M = M_3$. For each $i \neq j$ in $\{1, 2, 3\}$, we shall denote by e_{ij} the unique minimal partial isometry in M satisfying $e_{ij}^* e_{ij} = p_j$ and $e_{ij}e_{ij}^* = p_i$, while the symbol ϕ_{ij} will denote the unique norm-one functional in M^* satisfying $\phi_{ij}(e_{ij}) = 1$. In order to simplify the notation with a simple matricial notation, we shall assume that

$$p_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } p_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

however the arguments do not depend on this representation.

Step I. We claim that, under the hypothesis of the lemma,

$$\Delta(\lambda_2 p_2 + \mu_1 e_{13}) = 0 = \Delta(\lambda_1 p_1 + \mu_2 e_{23}), \tag{23}$$

for every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. We shall only prove the first equality, the second one follows similarly. Indeed, Corollary 3.5 implies that

$$\Delta(\lambda_2 p_2 + \mu_1 e_{13} \pm \overline{\mu_1} e_{31}) = \Delta(\lambda_2 p_2) + \Delta(\mu_1 e_{13} \pm \overline{\mu_1} e_{31}) = \Delta(\mu_1 e_{13} \pm \overline{\mu_1} e_{31}).$$

Having in mind that Δ is a weak-2-local *-derivation, we apply Lemma 2.3 to deduce that

$$\Delta(\mu_1 e_{13} \pm \overline{\mu_1} e_{31}) = \Delta(\mu_1 e_{13}) \pm \Delta(\mu_1 e_{13})^* = 0,$$

which proves that $\Delta(\lambda_2 p_2 + \mu_1 e_{13} \pm \overline{\mu_1} e_{31}) = 0$, for every $\mu_1, \lambda_2 \in \mathbb{C}$. Another application of Lemma 2.3 proves that

$$\Delta(\lambda_2 p_2 + \mu_1 e_{13}) = \Delta(\Re\text{e}(\lambda_2) p_2 + \frac{\mu_1}{2} e_{13} + \frac{\overline{\mu_1}}{2} e_{31}) + \Delta(i\Im\text{m}(\lambda_2) p_2 + \frac{\mu_1}{2} e_{13} - \frac{\overline{\mu_1}}{2} e_{31}) = 0.$$

Step II. We shall prove now that

$$\Delta(\lambda_2 p_2 + \mu_2 e_{23}) = 0 = \Delta(\lambda_1 p_1 + \mu_1 e_{13}), \tag{24}$$

for every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. Proposition 2.7 witnesses that

$$(p_2 + p_3)\Delta(p_2 + p_3)|_{(p_2+p_3)M(p_2+p_3)} : (p_2 + p_3)M(p_2 + p_3) \to (p_2 + p_3)M(p_2 + p_3)$$

is a weak-2-local *-derivation. Since $(p_2 + p_3)M(p_2 + p_3) \equiv M_2$, Theorem 3.2 implies that $(p_2 + p_3)\Delta(p_2 + p_3)|_{(p_2+p_3)M(p_2+p_3)}$ is a linear *-derivation. Therefore,

$$(p_2 + p_3)\Delta(\lambda_2 p_2 + \mu_2 e_{23})(p_2 + p_3) = \lambda_2(p_2 + p_3)\Delta(p_2)(p_2 + p_3) + \mu_2(p_2 + p_3)\Delta(e_{23})(p_2 + p_3) = 0,$$

by hypothesis. This shows that

$$\Delta(\lambda_2 p_2 + \mu_2 e_{23}) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ \omega_{31} & 0 & 0 \end{pmatrix},$$

where $\omega_{ij} \in \mathbb{C}$.

The identity

$$[z, \lambda_2 p_2 + \mu_2 e_{23}] = \begin{pmatrix} 0 & \lambda_2 z_{12} & \mu_2 z_{12} \\ -\lambda_2 z_{21} - \mu_2 z_{31} & -\mu_2 z_{32} & \mu_2(z_{22} - z_{33}) - \lambda_2 z_{23} \\ 0 & \lambda_2 z_{32} & \mu_2 z_{32} \end{pmatrix}, \tag{25}$$

holds for every matrix $z \in M$. Taking the functional ϕ_{11} (respectively ϕ_{31}) in M^*, we deduce, via the weak-2-local property of Δ at $\lambda_2 p_2 + \mu_2 e_{23}$, that $\omega_{11} = 0$ (respectively $\omega_{31} = 0$).

The weak-2-local behavior of Δ at the points $\lambda_2 p_2 + \mu_2 e_{23}$ and $\mu_2 e_{23}$ and the functional ϕ_{13}, combined with (25), and

$$[z, \mu_2 e_{23}] = \begin{pmatrix} 0 & 0 & \mu_2 z_{12} \\ -\mu_2 z_{31} & -\mu_2 z_{32} & \mu_2(z_{22} - z_{33}) \\ 0 & 0 & \mu_2 z_{32} \end{pmatrix}, \tag{26}$$

show that $\omega_{13} = 0$.

The identity

$$[z, -\lambda_2 p_1 + \mu_2 e_{23}] = \begin{pmatrix} 0 & \lambda_2 z_{12} & \mu_2 z_{12} + \lambda_2 z_{13} \\ -\lambda_2 z_{21} - \mu_2 z_{31} & -\mu_2 z_{32} & \mu_2(z_{22} - z_{33}) \\ -\lambda_2 z_{31} & 0 & \mu_2 z_{32} \end{pmatrix},$$

combined with (23), (25), and the weak-2-local property of Δ at $\lambda_2 p_2 + \mu_2 e_{23}$, $-\lambda_2 p_1 + \mu_2 e_{23}$ and the functional ϕ_{12} (respectively ϕ_{21}), we obtain $\omega_{12} = 0$ (respectively $\omega_{21} = 0$), which means that $\Delta(\lambda_2 p_2 + \mu_2 e_{23}) = 0$. The statement concerning $\Delta(\lambda_1 p_1 + \mu_1 e_{13})$ follows similarly.

Step III. We claim that

$$\Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_2 e_{23}) = 0 = \Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13}), \tag{27}$$

for every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. As before we shall only prove the first equality. Indeed, Corollary 3.5 assures that

$$\Delta(\lambda_1 p_1 + p_2 + \mu_2 e_{23} + \overline{\mu_2} e_{32}) = \lambda_1 \Delta(p_1) + \Delta(p_2 + \mu_2 e_{23} + \overline{\mu_2} e_{32}) = 0,$$

where in the last equality we apply the hypothesis, (24) and Lemma 2.3. Another application of Lemma 2.3 proves that $\Delta(\lambda_1 p_1 + p_2 + \mu_2 e_{23}) = 0$. The desired statement follows from the 1-homogeneity of Δ.

Step IV. In this step we show that

$$\Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}) = (1 - p_3)\Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}) p_3, \tag{28}$$

for every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$.

Since for any $z = (z_{ij}) \in M$, we have

$$[z, \mu_1 e_{13}] = \begin{pmatrix} -\mu_1 z_{31} & -\mu_1 z_{32} & \mu_1(z_{11} - z_{33}) \\ 0 & 0 & \mu_1 z_{21} \\ 0 & 0 & \mu_1 z_{31} \end{pmatrix},$$

using appropriate functionals in M^*, we deduce, via the weak-2-local property of Δ at $w_1 = \lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}$ and $w_2 = \lambda_1 p_1 + \lambda_2 p_2 + \mu_2 e_{23}$ ($w_1 - w_2 = \mu_1 e_{13}$), combined with (27), that

$$(p_2 + p_3)\Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23})(p_1 + p_2) = 0.$$

Considering the identity (26) and repeating the above arguments at the points $\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}$ and $\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13}$, we show that

$$p_1\Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23})(p_1 + p_2) = 0$$

The statement in the claim (28) follows from the fact that $\operatorname{tr}\Delta(\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}) = 0$.

Step V. We claim that,

$$\Delta(\mu_1 e_{13} + \mu_2 e_{23}) = 0, \tag{29}$$

for every μ_1, μ_2 in $\mathbb{C}$. By (28)

$$\Delta(\mu_1 e_{13} + \mu_2 e_{23}) = \begin{pmatrix} 0 & 0 & \delta_{13} \\ 0 & 0 & \delta_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

where $\delta_{ij} \in \mathbb{C}$.

Let $\phi = \phi_{12} + \phi_{13}$. It is not hard to see that

$$\phi[z, \mu_1 e_{13} + \mu_2 e_{23}] = \phi[z, \mu_1 e_{13} + \mu_2 p_2].$$

Considering this identity, the equality in (23), and the weak-2-local property of Δ at $\mu_1 e_{13} + \mu_2 e_{23}$ and $\mu_1 e_{13} + \mu_2 p_2$, we prove that $\delta_{13} = 0$. Repeating the same argument with $\phi = \phi_{21} + \phi_{23}$, $\mu_1 e_{13} + \mu_2 e_{23}$ and $\mu_1 p_1 + \mu_2 e_{23}$, we obtain $\delta_{23} = 0$.

Step VI. We claim that

$$\Delta(\lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}) = 0 = \Delta(\lambda_1 p_1 + \mu_1 e_{13} + \mu_2 e_{23}), \tag{30}$$

for every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$.

As in the previous steps, we shall only prove the first equality. By (28)

$$\Delta(\lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}) = \begin{pmatrix} 0 & 0 & \xi_{13} \\ 0 & 0 & \xi_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

where $\xi_{ij} \in \mathbb{C}$.

Since for any matrix $z = (z_{ij}) \in M$ we have

$$\phi_{13}\left[z, \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}\right] = \phi_{13}\left[z, \mu_1 e_{13} + \mu_2 e_{23}\right],$$

the weak-2-local behavior of Δ at $\lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}$ and $\mu_1 e_{13} + \mu_2 e_{23}$, combined with (29), shows that $\xi_{13} = 0$. Let $\phi = \phi_{21} + \phi_{23}$. It is easy to see that

$$\phi\left[z, \lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}\right] = \phi\left[z, \mu_1 p_1 + \lambda_2 p_2 + \mu_2 e_{23}\right].$$

Thus, weak-2-local property of Δ at $\lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}$ and $\mu_1 p_1 + \lambda_2 p_2 + \mu_2 e_{23}$ and (27) show that $\xi_{23} = 0$, and hence $\Delta(\lambda_2 p_2 + \mu_1 e_{13} + \mu_2 e_{23}) = 0$.

Step VII. We shall prove that

$$\Delta\left(\sum_{j=1}^{2} \lambda_j p_j + \sum_{k=1}^{2} \mu_k e_{k3}\right) = 0, \tag{31}$$

for every $\lambda_1,\lambda_2,\mu_1,\mu_2$ in $\mathbb{C}$. By (28)

$$\Delta\left(\sum_{j=1}^{2}\lambda_j p_j+\sum_{k=1}^{2}\mu_k e_{k3}\right)=\begin{pmatrix}0&0&\gamma_{13}\\0&0&\gamma_{23}\\0&0&0\end{pmatrix},$$

where $\gamma_{ij}\in\mathbb{C}$.

Given $z=(z_{ij})\in M$ we have

$$\phi_{13}\left[z,\sum_{j=1}^{2}\lambda_j p_j+\sum_{k=1}^{2}\mu_k e_{k3}\right]=\phi_{13}\left[z,\lambda_1 p_1+\sum_{k=1}^{2}\mu_k e_{k3}\right],$$

and

$$\phi_{23}\left[z,\sum_{j=1}^{2}\lambda_j p_j+\sum_{k=1}^{2}\mu_k e_{k3}\right]=\phi_{23}\left[z,\lambda_2 p_2+\sum_{k=1}^{2}\mu_k e_{k3}\right].$$

Then the weak-2-local behavior of Δ at $\sum_{j=1}^{2}\lambda_j p_j+\sum_{k=1}^{2}\mu_k e_{k3}$ and $\lambda_1 p_1+\sum_{k=1}^{2}\mu_k e_{k3}$ (respectively, $\lambda_2 p_2+\sum_{k=1}^{2}\mu_k e_{k3}$), combined with (30), imply that $\gamma_{13}=0$ (respectively, $\gamma_{23}=0$).

Finally, for $\lambda_3\neq 0$, we have

$$\Delta\left(\sum_{j=1}^{3}\lambda_j p_j+\sum_{k=1}^{2}\mu_k e_{k3}\right)=\Delta\left(\lambda_3 1+\sum_{j=1}^{2}(\lambda_j-\lambda_3)p_j+\sum_{k=1}^{2}\mu_k e_{k3}\right)=\lambda_3\Delta\left(1+\lambda_3^{-1}\sum_{j=1}^{2}(\lambda_j-\lambda_3)p_j+\lambda_3^{-1}\sum_{k=1}^{2}\mu_k e_{k3}\right)$$

$$=\text{(by Lemma 2.5)}=\lambda_3\Delta\left(\lambda_3^{-1}\sum_{j=1}^{2}(\lambda_j-\lambda_3)p_j+\lambda_3^{-1}\sum_{k=1}^{2}\mu_k e_{k3}\right)=\text{(by (31))}=0,$$

for every $\lambda_1,\lambda_2,\mu_1,\mu_2$ in $\mathbb{C}$. □

Proposition 3.8. *Let $\Delta: M_n\to M_n$ be a weak-2-local *-derivation, where $n\in\mathbb{N}$, $2\leq n$. Suppose $p_1,\ldots,p_n$ are mutually orthogonal minimal projections in M_n, $q=p_1+\ldots+p_{n-1}$, $\lambda_1,\ldots,\lambda_n$ are complex numbers, and a is an element in M_n satisfying $a=qap_n$. Then*

$$\Delta\left(\sum_{j=1}^{n}\lambda_j p_j+a\right)=\Delta\left(\sum_{j=1}^{n}\lambda_j p_j\right)+\Delta(a)=\sum_{j=1}^{n}\lambda_j\Delta\left(p_j\right)+\Delta(a),$$

and the restriction of Δ to qM_np_n is linear. More concretely, there exists $w_0\in M_n$, depending on $p_1,\ldots,p_n$, satisfying $w_0^*=-w_0$ and

$$\Delta\left(\sum_{j=1}^{n}\lambda_j p_j+a\right)=\left[w_0,\sum_{j=1}^{n}\lambda_j p_j+a\right],$$

for every $\lambda_1,\ldots,\lambda_n$ and a as above.

Proof. We shall argue by induction on n. The statement for $n=1$ is clear, while the case $n=2$ follows from Theorem 3.2. We can therefore assume that $n\geq 3$. Let us suppose that the desired conclusion is true for every $k<n$.

As in the previous results, to simplify the notation, we write $M=M_n$. For each $i\neq j$ in $\{1,\ldots,n\}$, we shall denote by e_{ij} the unique minimal partial isometry in M satisfying $e_{ij}^*e_{ij}=p_j$ and $e_{ij}e_{ij}^*=p_i$. Henceforth, the symbol ϕ_{ij} will denote the unique norm-one functional in M^* satisfying $\phi_{ij}(e_{ij}) = 1$. We also note that every element $a \in M$ satisfying $a = qap_n$ writes in the form $a = \sum_{k=1}^{n-1} \mu_k e_{kn}$, for unique $\mu_1, \ldots, \mu_{n-1}$ in $\mathbb{C}$.

Fix $j \in \{1, \ldots, n\}$. We observe that, for each matrix $z = (z_{ij}) \in M_n$, we have

$$[z, p_j] = \sum_{k=1, k\neq j}^{n} z_{kj} e_{kj} - z_{jk} e_{jk}. \tag{32}$$

We deduce from the weak-2-local property of Δ that

$$\Delta(p_j) = \Delta(p_j)^* = \sum_{k=1, k\neq j}^{n} \overline{\lambda_k^{(j)}} e_{kj} + \lambda_k^{(j)} e_{jk}, \tag{33}$$

for suitable $\lambda_k^{(j)} \in \mathbb{C}$, $k \in \{1, \ldots, n\}\backslash\{j\}$. Given $i \neq j$, Lemma 2.6 and Proposition 3.4 imply that

$$0 = (p_i + p_j)\Delta(p_i + p_j)(p_i + p_j) = (p_i + p_j)(\Delta(p_i) + \Delta(p_j))(p_i + p_j),$$

which proves that

$$\lambda_i^{(j)} = -\overline{\lambda_j^{(i)}}, \quad \forall i \neq j.$$

These identities show that the matrix

$$z_0 = -z_0^* := \sum_{i>j} -\lambda_i^{(j)} e_{ji} + \sum_{i<j} \overline{\lambda_j^{(i)}} e_{ji},$$

is well defined, and $\Delta(p_i) = [z_0, p_i]$ for every $i \in \{1, \ldots, n\}$. The mapping $\widehat{\Delta} = \Delta - [z_0, .]$ is a weak-2-local *-derivation satisfying

$$\widehat{\Delta}\left(\sum_{j=1}^{n} \lambda_j p_j\right) = 0,$$

for every $\lambda_j \in \mathbb{C}$ (cf. Proposition 3.4).

Let us fix $i_0 \in \{1, \ldots, n-1\}$. It is not hard to check that the identity

$$[z, e_{i_0 n}] = (z_{i_0 i_0} - z_{nn}) e_{i_0 n} + \sum_{j=1, j\neq i_0}^{n} z_{j i_0} e_{jn} - \sum_{j=1}^{n-1} z_{nj} e_{i_0 j}, \tag{34}$$

holds for every $z \in M$. Combining this identity with (32) for $[z, p_n]$, and $[z, p_{i_0}]$, and the fact that $\widehat{\Delta}$ is a weak-2-local *-derivation, we deduce, after an appropriate choosing of functionals $\phi \in M^*$, that there exists $\gamma_{i_0 n} \in i\mathbb{R}$ satisfying

$$\widehat{\Delta}(e_{i_0 n}) = \gamma_{i_0 n} e_{i_0 n}, \ \ \forall i_0 \in \{1, \ldots, n-1\}.$$

If we set $z_1 := \sum_{k=1}^{n-1} \gamma_{kn} p_k$, then $z_1 = -z_1^*$,

$$\widehat{\Delta}(e_{i_0 n}) = [z_1, e_{i_0 n}],$$

for every $i_0 \in \{1, \ldots, n-1\}$, and further $\left[z_1, \sum_{j=1}^{n} \lambda_j p_j\right] = 0$, for every $\lambda_j \in \mathbb{C}$. Therefore, $\widetilde{\Delta} = \widehat{\Delta} - [z_1, .]$ is a weak-2-local *-derivation satisfying

$$\widetilde{\Delta}\left(\sum_{j=1}^{n} \lambda_j p_j\right) = \widetilde{\Delta}(e_{i_0 n}) = 0, \tag{35}$$

for every $i_0 \in \{1, \ldots, n-1\}$.

The rest of the proof is devoted to establish that

$$\widetilde{\Delta}\left(\sum_{j=1}^{n} \lambda_j p_j + \sum_{k=1}^{n-1} \mu_k e_{kn}\right) = 0,$$

for every $\mu_1, \ldots, \mu_{n-1}, \lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, which finishes the proof. The case $n = 3$ follows from Proposition 3.7. So, henceforth, we assume $n \geq 4$. We shall split the arguments in several steps.

Step I. We shall first show that, for each $1 \leq i_0 \leq n-1$,

$$p_{i_0}\widetilde{\Delta}\left(\sum_{i=1}^{n} \lambda_i p_i + \mu e_{i_0 n}\right) = 0, \tag{36}$$

for every $\lambda_1, \ldots, \lambda_n, \mu$ in $\mathbb{C}$.

Let us pick $k \in \{1, \ldots, n-1\}$ with $k \neq i_0$. By the induction hypothesis

$$(1-p_k)\widetilde{\Delta}\left(\sum_{i=1, i\neq k}^{n} \lambda_i p_i + \mu e_{i_0 n}\right)(1-p_k) = \sum_{i=1, i\neq k}^{n} \lambda_i (1-p_k)\widetilde{\Delta}(p_i)(1-p_k) + \mu(1-p_k)\widetilde{\Delta}(e_{i_0 n})(1-p_k) = 0. \tag{37}$$

Since for any $z \in M$, the identity

$$(1-p_k)\left[z, \sum_{i=1}^{n} \lambda_i p_i + \mu e_{i_0 n}\right](1-p_k) = (1-p_k)\left[z, \sum_{i=1, i\neq k}^{n} \lambda_i p_i + \mu e_{i_0 n}\right](1-p_k),$$

holds, if we take $\phi = \phi_{i_0 j}$ with $j \neq k$, we get, applying (37) and the weak-2-local property of $\widetilde{\Delta}$, that

$$p_{i_0}\widetilde{\Delta}\left(\sum_{i=1}^{n} \lambda_i p_i + \mu e_{i_0 n}\right)p_j = 0. \quad (1 \leq j \leq n,\ j \neq k)$$

Since $4 \leq n$, we can take at least two different values for k to obtain (36).

Step II. In this step we prove that, for each $1 \leq i_0 \leq n-1$,

$$p_{i_0}\widetilde{\Delta}\left(\lambda p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right)p_n = 0, \tag{38}$$

for every λ and $\mu_1, \ldots, \mu_{n-1}$ in $\mathbb{C}$.

We fix $1 \leq i_0 \leq n-1$, and we pick $k \in \{1, \ldots, n-1\}$ with $k \neq i_0$. By the induction hypothesis, we have

$$(1-p_k)\widetilde{\Delta}\left(\lambda p_{i_0} + \sum_{i=1, i\neq k}^{n-1} \mu_i e_{in}\right)(1-p_k) = \lambda(1-p_k)\widetilde{\Delta}(p_{i_0})(1-p_k) + \sum_{i=1, i\neq k}^{n-1} \mu_i (1-p_k)\widetilde{\Delta}(e_{in})(1-p_k) = 0, \tag{39}$$

for every λ and $\mu_1, \ldots, \mu_{n-1}$ in $\mathbb{C}$.

Since for any $z \in M$, the equality

$$(1-p_k)\left[z, \lambda p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right](1-p_n) = (1-p_k)\left[z, \lambda p_{i_0} + \sum_{i=1, i\neq k}^{n-1} \mu_i e_{in}\right](1-p_n),$$

holds, we deduce from (39) and the weak-2-local property of $\widetilde{\Delta}$, applied to $\phi = \phi_{i_0 j}$ with $j \neq k, n$, that

$$p_{i_0}\widetilde{\Delta}\left(\lambda p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right) p_j = 0, \quad (\forall 1 \leq j \leq n-1,\ j \neq k).$$

By taking two different values for k, we see that

$$p_{i_0}\widetilde{\Delta}\left(\lambda p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right)(1 - p_n) = 0. \tag{40}$$

Let $\phi_0 = \sum_{j=1}^{n} \phi_{i_0 j}$. It is not hard to see that the equality

$$\phi_0\left[z, \sum_{i=1, i\neq i_0}^{n-1} \mu_i e_{in}\right] = \phi_0\left[z, \sum_{i=1, i\neq i_0}^{n-1} \mu_i p_i\right],$$

holds for every $z \in M$. Thus,

$$\phi_0\left[z, \lambda p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right] = \phi_0\left[z, \lambda p_{i_0} + \sum_{i=1, i\neq i_0}^{n-1} \mu_i p_i + \mu_{i_0} e_{i_0 n}\right],$$

for every $z \in M$. Therefore, the weak-2-local property of $\widetilde{\Delta}$ implies that

$$\phi_0\widetilde{\Delta}\left(\lambda p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right) = \phi_0\widetilde{\Delta}\left(\lambda p_{i_0} + \sum_{i=1, i\neq i_0}^{n-1} \mu_i p_i + \mu_{i_0} e_{i_0 n}\right) = 0,$$

where the last equality follows from (36). Combining this fact with (40), we get (38).

Step III. In this final step we shall show that

$$\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right) = 0, \tag{41}$$

for every $\mu_1, \ldots, \mu_{n-1}, \lambda_1, \ldots, \lambda_{n-1}$ in $\mathbb{C}$.

Let $k \in \{1, \ldots, n-1\}$. By the induction hypothesis

$$(1 - p_k)\widetilde{\Delta}\left(\sum_{i=1, i\neq k}^{n-1} \lambda_i p_i + \sum_{i=1, i\neq k}^{n-1} \mu_i e_{in}\right)(1 - p_k) = 0.$$

Since for any $z \in M$, we have

$$(1 - p_k)\left[z, \sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right](1 - p_k - p_n) = (1 - p_k)\left[z, \sum_{i=1, i\neq k}^{n-1} \lambda_i p_i + \sum_{i=1, i\neq k}^{n-1} \mu_i e_{in}\right](1 - p_k - p_n),$$

by taking $\phi = \phi_{lj}$, with $l \neq k$ and $j \neq k, n$, we deduce, via the weak-2-local behavior of $\widetilde{\Delta}$, that

$$p_l\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right) p_j = 0,$$

for every $l \neq k$ and $j \neq k, n$. Taking three different values for k, we show that

$$\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right)(1 - p_n) = 0. \tag{42}$$

Let us pick $i_0 \in \{1, \ldots, n-1\}$. It is easy to check that the identity

$$p_{i_0}\left[z, \sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right] p_n = p_{i_0}\left[z, \lambda_{i_0} p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right] p_n,$$

holds for every $z \in M$. So, taking $\phi = \phi_{i_0 n}$, we deduce from the weak-2-local property of $\widetilde{\Delta}$ that

$$p_{i_0}\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right) p_n = p_{i_0}\widetilde{\Delta}\left(\lambda_{i_0} p_{i_0} + \sum_{i=1}^{n-1} \mu_i e_{in}\right) p_n = 0,$$

where the last equality is obtained from (38). Since above identity holds for any $i_0 \in \{1, \ldots, n-1\}$, we conclude that

$$(1 - p_n)\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right) p_n = 0. \tag{43}$$

Now, Lemma 3.1 implies that $\operatorname{tr}\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right) = 0$, which combined with (42), shows that

$$p_n\widetilde{\Delta}\left(\sum_{i=1}^{n-1} \lambda_i p_i + \sum_{i=1}^{n-1} \mu_i e_{in}\right) p_n = 0. \tag{44}$$

Identities (42), (43) and (44) prove the statement in (41).

Finally, for $\lambda_n \neq 0$, we have

$$\widetilde{\Delta}\left(\sum_{j=1}^{n} \lambda_j p_j + \sum_{k=1}^{n-1} \mu_k e_{kn}\right) = \widetilde{\Delta}\left(\lambda_n 1 + \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) p_j + \sum_{k=1}^{n-1} \mu_k e_{kn}\right)$$

$$= \lambda_n \widetilde{\Delta}\left(1 + \lambda_n^{-1} \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) p_j + \lambda_n^{-1} \sum_{k=1}^{n-1} \mu_k e_{kn}\right) = \text{(by Lemma 2.5)}$$

$$= \lambda_n \widetilde{\Delta}\left(\lambda_n^{-1} \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) p_j + \lambda_n^{-1} \sum_{k=1}^{n-1} \mu_k e_{kn}\right) = \text{(by (41))} = 0,$$

for every $\mu_1, \ldots, \mu_{n-1}, \lambda_1, \ldots, \lambda_{n-1}$ in $\mathbb{C}$ $\square$

Our next result is a consequence of the above Proposition 3.8 and Lemma 2.3.

Corollary 3.9. *Let $\Delta : M_n \to M_n$ be a weak-2-local *-derivation, where $n \in \mathbb{N}$, $2 \leq n$. Suppose $p_1, \ldots, p_n$ are mutually orthogonal minimal projections in M_n, $q = p_1 + \ldots + p_{n-1}$, and $a \in M_n$ satisfies $a^* = a$ and $a = qap_n + p_n aq$. Then*

$$\Delta\left(\sum_{j=1}^{n} \lambda_j p_j + a\right) = \Delta\left(\sum_{j=1}^{n} \lambda_j p_j\right) + \Delta(a) = \sum_{j=1}^{n} \lambda_j \Delta\left(p_j\right) + \Delta(a),$$

for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and the restriction of Δ to $(M_n)_{sa} \cap (qM_n p_n + p_n M_n q)$ is linear.

Proof. Under the above hypothesis, Lemma 2.3 implies that

$$\Delta\left(\sum_{j=1}^{n}\lambda_j p_j + a\right) = \Delta\left(\frac{1}{2}\sum_{j=1}^{n}\lambda_j p_j + qap_n\right) + \Delta\left(\frac{1}{2}\sum_{j=1}^{n}\lambda_j p_j + qap_n\right)^* = \text{(by Proposition 3.8)}$$

$$= \sum_{j=1}^{n}\lambda_j\Delta\left(p_j\right) + \Delta\left(qap_n\right) + \Delta\left(qap_n\right)^* = \sum_{j=1}^{n}\lambda_j\Delta\left(p_j\right) + \Delta(qap_n + p_naq) = \sum_{j=1}^{n}\lambda_j\Delta\left(p_j\right) + \Delta(a).$$

□

We can prove now that the measure μ on $\mathcal{P}(M_n)$ determined by a weak-2-local *-derivation on M_n is always bounded.

Proposition 3.10. *Let $\Delta : M_n \to M_n$ be a weak-2-local *-derivation, where $n \in \mathbb{N}$. Then Δ is bounded on the set $\mathcal{P}(M_n)$ of all projections in M_n.*

Proof. We shall proceed by induction on n. The statement for $n = 1$ is clear, while the case $n = 2$ is a direct consequence of Theorem 3.2. We may, therefore, assume that $n \geq 3$. Suppose that the desired conclusion is true for every $k < n$. To simplify notation, we write $M = M_n$. We observe that, by hypothesis, $\Delta^\sharp = \Delta$.

Let $p_1, \ldots, p_n$ be (arbitrary) mutually orthogonal minimal projections in M. For each $i, j \in \{1, \ldots, n\}$, we shall denote by e_{ij} the unique minimal partial isometry in M satisfying $e_{ij}^* e_{ij} = p_j$ and $e_{ij}e_{ij}^* = p_i$. Henceforth, the symbol ϕ_{ij} will denote the unique norm-one functional in M^* satisfying $\phi_{ij}(e_{ij}) = 1$.

Let $q_n = p_1 + \ldots + p_{n-1}$. Proposition 2.7 implies that the mapping

$$q_n\Delta q_n|_{q_nMq_n} : q_nMq_n \to q_nMq_n$$

is a weak-2-local *-derivation on $q_nMq_n \equiv M_{n-1}(\mathbb{C})$. We know, by the induction hypothesis, that $q_n\Delta q_n|_{q_nMq_n}$ is bounded on the set $\mathcal{P}(q_nMq_n)$ of all projections in q_nMq_n. Proposition 3.4, assures that $\mu : \mathcal{P}(q_nMq_n) \to q_nMq_n$, $p \mapsto q_n\Delta(p)q_n$ is a bounded, finitely additive measure. An application of the Mackey-Gleason theorem (cf. [4]) proves the existence of a (bounded) linear operator $G : q_nMq_n \to q_nMq_n$ satisfying $G(p) = \mu(p) = q_n\Delta(p)q_n$, for every projection p in q_nMq_n. Another application of Proposition 3.4, combined with a simple spectral resolution, shows that $q_n\Delta(a)q_n = G(a)$, for every self-adjoint element in q_nMq_n. Therefore, $q_n\Delta(a + b)q_n = G(a + b) = G(a) + G(b) = q_n\Delta(a)q_n + q_n\Delta(b)q_n$, for every a, b in the self-adjoint part of q_nMq_n.

Now, Lemma 2.3 implies that $q_n\Delta q_n|_{q_nMq_n}$ is a *-derivation on q_nMq_n (compare also [3, Theorem 3.4]). Therefore there exists $z_0 = -z_0^* \in q_nMq_n$ such that

$$q_n\Delta(q_naq_n)q_n = [z_0, q_naq_n], \tag{45}$$

for every $a \in M$.

Now, it is not hard to see that the identities:

$$q_n\left[z, e_{1n}\right]q_n = -z_{n1}p_1 - \sum_{j=2}^{n-1} z_{nj}e_{1j} = -\sum_{j=1}^{n-1} z_{nj}e_{1j}, \tag{46}$$

and

$$q_n\left[z, e_{kn}\right]q_n = -\sum_{j=1}^{n-1} z_{nj}e_{kj},\ q_n\left[z, e_{nk}\right]q_n = \sum_{j=1}^{n-1} z_{jn}e_{jk}, \tag{47}$$

hold for every $z \in M$, and $1 \leq k \leq n-1$ (cf. (34)). The weak-2-local property of Δ, combined with (46) and (47), implies that

$$\phi_{kl}\left(\Delta(e_{kn})\right) = \phi_{1l}\left(\Delta(e_{1n})\right),$$

for every $1 \leq k \leq n-1$ and every $1 \leq l \leq n-1$. Furthermore, for $2 \leq i \leq n-1$, $1 \leq j \leq n-1$ there exits $z \in M$, depending on e_{1n} and ϕ_{ij}, such that $\phi_{ij}\Delta(e_{1n}) = \phi_{ij}[z, e_{1n}] = \phi_{ij}(q_n[z, e_{1n}]q_n) = (\text{by } (46)) = 0$. Therefore

$$q_n\Delta(e_{1n})q_n = \sum_{j=1}^{n-1} \lambda_{nj}e_{1j}, \tag{48}$$

for suitable (unique) λ_{nj}'s in $\mathbb{C}$ $(1 \leq j \leq n-1)$, and consequently,

$$q_n\Delta(e_{n1})q_n = q_n\Delta(e_{1n})^*q_n = (q_n\Delta(e_{1n})q_n)^* = \sum_{j=1}^{n-1} \overline{\lambda_{nj}}e_{j1}. \tag{49}$$

We similarly obtain

$$q_n\Delta(e_{kn})q_n = \sum_{j=1}^{n-1} \lambda_{nj}e_{kj},$$

for every $1 \leq k \leq n-1$.

Let us define

$$z_1 = -z_1^* := \sum_{j=1}^{n-1} \overline{\lambda_{nj}}e_{jn} - \lambda_{nj}e_{nj} \in p_nMq_n + q_nMp_n.$$

It is easy to check that

$$q_n\Delta(e_{kn})q_n = q_n[z_1, e_{kn}]q_n, \ q_n\Delta(e_{nk})q_n = q_n[z_1, e_{nk}]q_n, \quad \forall 1 \leq k \leq n-1,$$

$$q_n[z_1, q_naq_n]q_n = 0, \text{ and, } q_n[z_0, q_nap_n + p_naq_n]q_n = 0,$$

for every $a \in M$. Therefore

$$\begin{aligned} q_n\Delta(q_naq_n)q_n &= q_n[z_0 + z_1, q_naq_n]q_n = q_n[z_0, q_naq_n]q_n, \\ q_n\Delta(e_{kn})q_n &= q_n[z_0 + z_1, e_{kn}]q_n = q_n[z_1, e_{kn}]q_n, \end{aligned} \tag{50}$$

and

$$q_n\Delta(e_{nk})q_n = q_n[z_0 + z_1, e_{nk}]q_n = q_n[z_1, e_{nk}]q_n,$$

for every $a \in M$, $1 \leq k \leq n-1$.

We claim that the set

$$\left\{q_n\Delta(b)q_n : b \in M, b^* = b, \|b\| \leq 1\right\} \tag{51}$$

is bounded. Indeed, let us take $b = b^* \in M$ with $\|b\| \leq 1$. The last statement in Lemma 3.3 shows that

$$q_n\Delta(b)q_n = q_n\Delta(q_nbq_n + q_nbp_n + p_nbq_n + p_nbp_n)q_n = q_n\Delta(q_nbq_n + q_nbp_n + p_nbq_n)q_n. \tag{52}$$

The element q_nbq_n is self-adjoint in q_nMq_n, so, there exist mutually orthogonal minimal projections $r_1, \ldots, r_{n-1}$ in q_nMq_n and real numbers $\lambda_1, \ldots, \lambda_{n-1}$ such that $q_nbq_n = \sum_{j=1}^{n-1} \lambda_jr_j$ and $r_1 + \ldots + r_{n-1} = q_n$. We also observe that $p_nbq_n + q_nbp_n$ is self-adjoint in $q_nMp_n + p_nMq_n$, thus, Corollary 3.9 implies that

$$\begin{aligned} q_n\Delta(b)q_n &= q_n\Delta(q_nbq_n + q_nbp_n + p_nbq_n)q_n = q_n\Delta(q_nbq_n)q_n + q_n\Delta(q_nbp_n + p_nbq_n)q_n \\ &= (\text{by } (50)) = q_n[z_0, q_nbq_n]q_n + q_n[z_1, q_nbp_n + p_nbq_n]q_n, \end{aligned}$$

and hence

$$\|q_n\Delta(b)q_n\| \leq 2\|z_0\| + 4\|z_1\|,$$

which proves the claim in (51).

Following a similar reasoning to that given in the proof of (51) we can obtain that the sets

$$\left\{q_1\Delta(b)q_1 : b \in M, b^* = b, \|b\| \leq 1\right\} \tag{53}$$

and

$$\left\{q_2\Delta(b)q_2 : b \in M, b^* = b, \|b\| \leq 1\right\} \tag{54}$$

are bounded, where $q_2 = 1 - p_2$ and $q_1 = 1 - p_1$.

The boundedness of Δ on the set $\mathcal{P}(M_n)$ of all projections in M_n is a direct consequence of (51), (53), and (54). □

We can establish now the main result of this paper.

Theorem 3.11. *Every (non-necessarily linear nor continuous) weak-2-local *-derivation on M_n is linear and a derivation.*

Proof. Let $\Delta : M_n \to M_n$ be a weak-2-local *-derivation. Propositions 3.4 and 3.10 assure that the mapping $\mu : \mathcal{P}(M_n) \to M_n$, $p \mapsto \mu(p) := \Delta(p)$ is a bounded completely additive measure on $\mathcal{P}(M_n)$. By the Mackey-Gleason theorem (cf. [4]) there exists a bounded linear operator G on M_n such that $G(p) = \mu(p) = \Delta(p)$ for every $p \in \mathcal{P}(M_n)$.

We deduce from the spectral resolution of self-adjoint matrices and Proposition 3.4 that $\Delta(a) = G(a)$, for every $a \in (M_n)_{sa}$. Thus, given two self-adjoint elements a, b in M_n, we have

$$\Delta(a + b) = G(a + b) = G(a) + G(b) = \Delta(a) + \Delta(b).$$

This shows that $\Delta|_{(M_n)_{sa}}$ is a linear mapping. The linearity of Δ follows from Lemma 2.3, and the final conclusion from [3, Theorem 3.4]. □

Corollary 3.12. *Every weak-2-local *-derivation on a finite dimensional C*-algebra is a derivation.*

Proof. Let A be a finite dimensional C*-algebra. It is known that A is unital and there exists a finite sequence of mutually orthogonal central projections $q_1, \cdots, q_m$ in A such that $A = \bigoplus_{i=1}^{m} Aq_i$ and $Aq_i \cong M_{n_i}(\mathbb{C})$ for some $n_i \in \mathbb{N}$ $(1 \leq i \leq m)$ (cf. [23, Page 50]).

Let Δ be a weak-2-local *-derivation on A. Fix $1 \leq i \leq m$. By Proposition 2.7 the restriction $q_i\Delta q_i|_{Aq_i} = \Delta q_i|_{Aq_i} : q_iAq_i = Aq_i \to Aq_i$ is a weak-2-local *-derivation. Since $Aq_i \cong M_{n_i}(\mathbb{C})$, Theorem 3.11 asserts that $\Delta q_i|_{Aq_i}$ is a derivation.

Let a be a self-adjoint element in Aq_i. Then a writes in the form $a = \sum_{j=1}^{k_i} \lambda_j p_j$, where $p_1, \cdots, p_{k_i}$ are mutually orthogonal projections in Aq_i and $\lambda_1, \cdots, \lambda_{k_i}$ are real numbers. Proposition 3.4 implies that

$$\Delta(a) = \sum_{j=1}^{k_i} \lambda_j \Delta(p_j).$$

Multiplying on the right by the central projection $1 - q_i$ we get:

$$\Delta(a)(1 - q_i) = \sum_{j=1}^{k_i} \lambda_j \Delta(p_j)(1 - q_i). \tag{55}$$

However, Lemma 2.6 implies that $(1 - p_j)\Delta(p_j)(1 - p_j) = 0$, for every $1 \leq j \leq k_i$. Since $p_j \leq q_i$ for every j, we have $1 - q_i \leq 1 - p_j$, which implies that $0 = (1 - q_i)\Delta(p_j)(1 - q_i) = \Delta(p_j)(1 - q_i)$, for every $1 \leq j \leq k_i$. We deduce from (55) that $\Delta(a) = \Delta(a)q_i = q_i\Delta(a)q_i$ for every self-adjoint element $a \in Aq_i$. Lemma 2.3 shows that the same equality holds for every $a \in Aq_i$. That is, $\Delta(Aq_i) \subseteq Aq_i$ and $\Delta|_{Aq_i}$ is linear for every $1 \leq i \leq m$.

Let (a_i) be a self-adjoint element in A, where $a_i \in Aq_i$. Having in mind that every a_i admits a finite spectral resolution in terms of minimal projections and $Aq_i \perp Aq_j$, for every $i \neq j$, it follows from Corollary 3.5 (or from Proposition 3.4) that $\Delta((a_i)) = (\Delta(a_i))$. Having in mind that $\Delta|_{Aq_i}$ is linear for every $1 \leq i \leq m$, we deduce that Δ is additive in the self-adjoint part of A. Lemma 2.3 shows that Δ is actually additive on the whole A. Theorem 3.4 in [3] gives the desired conclusion. □

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