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Some Common Fixed Point Theorems for Tangential Generalized Weak Contractions in Metric-like Spaces

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Abstract. In this paper we define the tangential property in partial metric spaces and metric-like spaces to prove some common fixed point theorems for two pairs of generalized weakly contractions, some examples are given to illustrate ours results.

1. Introduction and Preliminaries

Jungck [14]introduced the compatible mappings concept for two self mappings in metric spaces to prove a common fixed point. The study of common fixed point for noncompatible mappings was initiated by pant [23], later Sastry and Murthy [26] introduced the notions of tangent point and tangential mappings in metric spaces to obtain some common fixed point results, after two years Aamri and El-Moutawakil [1] introduced the notion of property (E.A), which was generalized by Y. Liu et al.[20] to common (E.A) property.

In 2009, Pathak and Shahazad [24]introduced the concept of tangential mappings, they defined a weak tangent point in place of tangent point and pair-wise tangential property, in this paper we define the tangential property in the partial metric spaces and utilize it to prove some common fixed theorems for two weakly compatible pairs of self mappings under two different contractive condition.

On other hand, the concept of the partial metric spaces was introduced by Matthews [21], these spaces are a generalization of the usual metric spaces in such spaces the distance from an object to itself is not necessarily have a zero.

In 2012, Amini Harrandi [7] introduced a generalization to the partial metric spaces, which is called metric-like spaces and he proved some fixed point theorems in such spaces.

Firstly, we recall to some basic definitions and properties of partial metric spaces and metric-like spaces:

Definition 1.1. *[21] Let X be a nonempty set, a function:$p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if the following conditions satisfied:*

- *(P1)* $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$
- *(P2)* $p(x, x) \leq p(x, y)$

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- *(P3)* $p(x,y) = p(y,x)$
- *(P4)* $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$,

the space (X,p) *is called a partial metric space.*

Clearly that if $p(x,y) = 0$ then (P1) and (P2) imply $x = y$.
If p is a partial metric on X, then the function $p^s : X \times X \to R_+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

defines a metric on X.

Definition 1.2. *[21] Let* (X,p) *be a partial metric space.*

1. *A sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$*, with respect to* τ_p *, if and only if*

$$\lim_{n\to\infty} p(x,x_n) = p(x,x)$$

2. *A sequence* $\{x_n\}$ *in* X *is said to be a Cauchy sequence if* $\lim_{n\to\infty} p(x_n,x_m)$ *exists and is finite.*
3. (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *is convergent with respect to* τ_p *to a point* $x \in X$ *such that* $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.
   In this case, we say that the partial metric p is complete.

Definition 1.3. *[7] Let* X *be a nonempty set, a function:*$\sigma : X \times X \to \mathbb{R}_+$ *is said to be a metric-like on* X *if the following conditions satisfied:*

- $\sigma(x,y) = 0$ *implies that* $x = y$
- $\sigma(x,y) = \sigma(y,x)$
- $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$,

the space (X,σ) *is said to be a metric-like space.*

Each metric-like σ on X generates a topology τ_σ on X which has as a base the family of open σ-balls $\{B_\sigma(x,\varepsilon); x \in X, \varepsilon > 0\}$, where $B_\sigma(x,\varepsilon) = \{y \in X, |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.
A sequence $\{x_n\}$ in metric-like space (X,σ) is said to be convergent to a point $x \in X$, if and only if

$$\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x).$$

A sequence $\{x_n\}$ in X is said to be a σ-Cauchy sequence if $\lim_{n\to\infty} \sigma(x_n,x_m)$ exists and is finite.
(X,σ) is said to be σ-complete if every σ-Cauchy sequence $\{x_n\}$ in X is convergent to a point $x \in X$ such that $\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x)$.
Every partial metric is a metric-like.

Example 1.4. *[7] Let* $X = \{0,1\}$*, define* $\sigma : X \times X \to \mathbb{R}_+$ *as follows:*

$$\sigma(x,y) = \begin{cases} 2, & \text{if } x = y = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Then σ *is metric-like on* X*, since* $\sigma(0,0) > \sigma(0,1)$ *then* σ *is not partial metric.*

More recently, Nazir and Abbas[22] defined the (E.A) property in partial metric spaces as follows:

Definition 1.5. *[22] Let A, S be two self mappings of a partial metric space (X, p), the pair $\{A, S\}$ is said to be satisfies the (E.A) property if there exists a sequence $\{x_n\}$ in X such*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ and $p(z, z) = 0$.

In the framework we will need to the following definition due to Jungck[15]:

Definition 1.6. *[15] Let (X, d) be a metric space two mappings A and S are said to be weakly compatible if they commute at their coincidence point, i.e if $Au = Su$ for some $u \in X$, then $ASu = SAu$.*

Now, as an extension of the concept of tangential mappings due to Pathak and Shahazad [24] to the context of partial metric spaces and metric-like spaces, define:

Definition 1.7. *Let (X, σ) be a -metric-like space and let A, B, S and T be four self mappings on into itself, the pair (A, B) is said to be tangential with respect to (S, T) if there exist two sequences $\{x_n\}, \{y_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

and $\sigma(z, z) = 0$.

Example 1.8. *Let $X = \mathbb{R}_+$ with the metric-like σ such*

$$\sigma(x, y) =) = \max\{x, y\},$$

we define A, B, S and T as follows:

$$Ax = 2x, \quad Sx = \log(1 + x),$$

$$Bx = \begin{cases} 2 - x, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases} \quad Tx = \begin{cases} 1 - \frac{x}{2}, & 0 \le x \le 2 \\ x + 1,, & x > 2 \end{cases}$$

Consider two sequences $\{x_n\}, \{y_n\}$ which are defined for all $n \ge 1$ by:

$$x_n = \frac{1}{n}, \quad y_n = 2 - \frac{1}{n},$$

Clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0$ and $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0$, then (A, B) is tangential with respect to (S, T).

In paper[10], the authors improved the tangential property concept to strongly tangential in metric spaces, in next we will show this concept in -metric-like spaces:

Definition 1.9. *Let (X, σ) be a metric-like space and let A, B, S and T be self mappings on X, (A, B) is said to be strongly tangential w.r.t (S, T) if there exists two sequences $\{x_n\}, \{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ and $z \in AX \cap BX$ with $\sigma(z, z) = 0$.*

Example 1.10. *Let $X = [0, 2]$ with the metric-like:$\sigma(x, y) =) = \max\{x, y\}$, we define A, B, S and T as follows:*

$$Ax = \frac{x}{2}, \quad Sx = 1 - e^{-x},$$

$$Bx = \begin{cases} x, & 0 \le x \le 1 \\ 0,, & 1 < x \le 2 \end{cases} \quad Tx = \begin{cases} 2x & 0 \le x \le 1 \\ x - 1, & 1 < x \le 2 \end{cases}$$

$AX = [0,2]$, $BX = [0,1]$, so $0 \in AX \cap BX = [0,1]$.
Consider two sequences $\{x_n\}, \{y_n\}$ which are defined for all $n \geq 1$ by:

$$x_n = \frac{1}{n}, \quad y_n = e^{-n},$$

Clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0$ and $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0$ with $\sigma(0,0) = 0$, then (A,B) is strongly tangential with respect to (S,T).

Remark 1.11. *Since each partial metric is a metric-like, then the last two definitions are valid in partial metric space*

Let Ψ be a set of all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, such:

1. $\psi(x) = 0$ if and only if $x = 0$,
2. ψ is monotony non decreasing.

Let Φ be a set of all lower semicontinuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, such $\phi(x) = 0$ if and only if $x = 0$

2. Main Results

Theorem 2.1. *Let (X,σ) be a metric-like space and let A, B, S, T be four self mappings on X such for all $x, y \in X$*

$$\psi(\sigma(Sx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)), \tag{1}$$

where

$$M(x,y) = \max\{\sigma(Ax,By), \sigma(Ax,Sx), \sigma(By,Ty), \frac{1}{4}(\sigma(Ax,Ty) + \sigma(By,Sx))\}$$

and $\psi \in \Psi, \phi \in \Phi$, if the following conditions hold:

1. *AX and BX are closed,*
2. *the pair (A,B) is tangential w.r.t (S,T),*
3. *$(A,B),(S,T)$ are weakly compatible,*

then A, B, S and T have a unique common fixed point.

Proof. Since (A,B) is tangential w.r.t (S,T), there are two sequences $\{x_n\},\{y_n\}$ such the four sequences $\{Ax_n\},\{By_n\},\{Sx_n\}$ and $\{Ty_n\}$ converge to the same point $z \in X$ with $\sigma(z,z) = 0$, also the closure of AX implies that $z \in AX$ as well as $z \in BX$, then there exist $u, v \in X$ such $z = Au = Bv$.
Firstly, we will prove $Su = z$, if not by using (1) we get:

$$\psi(Su,Ty_n) \leq \psi(\max\{\sigma(Au,By_n), \sigma(Au,Su), \sigma(By_n,Ty_n), \frac{1}{4}(\sigma(Au,Ty_n) + \sigma(By_n,Su))\})$$

$$-\phi(\max\{\sigma(Au,By_n), \sigma(Au,Su), \sigma(By_n,Ty_n), \frac{1}{4}(\sigma(Au,Ty_n) + \sigma(By_n,Su))\}),$$

letting $n \to \infty$, we get:

$$\psi(\sigma(Su,z)) \leq \psi(\sigma(Su,z)) - \phi(\sigma(Su,z) \leq \psi(\sigma(Su,z)),$$

which a contradiction, then $\sigma(Su,z) = 0$ and so $Su = z$.
Now, we claim $Tv = z$, if not by using (1) we get

$$\psi(\sigma(Sx_n,Tv)) \leq \psi(\max\{\sigma(Ax_n,Bv), \sigma(Ax_n,Sx_n), \sigma(Bv,Tv), \frac{1}{4}(\sigma(Ax_n,Tv) + \sigma(Bv,Sx_n))\})$$

$$-\phi(\max\{\sigma(Ax_n,Bv), \sigma(Ax_n,Sx_n), \sigma(Bv,Tv), \frac{1}{4}(\sigma(Ax_n,Tv) + \sigma(Bv,Sx_n))\}),$$

letting $n \to \infty$, we get

$$\psi(\sigma(z, Tv)) \leq \psi(\sigma(z, Tv)) - \phi(\sigma(z, Tv)) < \psi(\sigma(z, Tv)),$$

which a contradiction, then $Tv = z$.
On other hand, the pair (A, S) is weakly compatible implies that $Az = Sz$, similarly for the pair (B, T) we obtain $Bz = Tz$ and the point z is a coincidence point for the four mappings.
Nextly, we will prove $z = Az$, if not by using (1) we get

$$\psi(\sigma(Sz, Ty_n)) \leq \max\{\sigma(Az, By_n), \sigma(Az, Sz), \sigma(By_n, Ty_n), \frac{1}{4}(\sigma(Az, Ty_n) + \sigma(By_n, Sz))\},$$

letting $n \to \infty$, we get

$$\psi(\sigma(Az, z)) \leq \psi(\sigma(Az, z)) - \phi(\sigma(Az, z)) < \psi(\sigma(Az, z)),$$

which is a contradiction, then $Az = z$.
Similarly we can find that $Bz = Tz = z$.
For the uniqueness, suppose there is another common fixed point w, by using ((1) we get:

$$\psi(Sz, Tw) \leq \psi(M(z, w)) - \phi(\sigma(M(z, w))),$$

where $M(z, w) = \sigma(z, w)$ and $\psi(\sigma(Sz, Tw)) = \psi(\sigma(z, w))$, so

$$\psi(\sigma(z, w)) \leq \psi(\sigma(z, w)) - \phi(\sigma(z, w)) < \psi(\sigma(z, w)),$$

which is a contradiction, then z is unique. □

Corollary 2.2. *Let (X, p) be a partial metric space, $A, S : X \to X$ two self mappings satisfying for all $x, y \in X$:*

$$\psi(p(Sx, Ty) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where

$$M(x, y) = \max\{p(Ax, By), p(Ax, Sx), (p(By, Ty), \frac{1}{2}(p(Ax, Ty) + p(By, Sx))\},$$

if the following conditions satisfy:

1. *$A(X)$ and BX are closed,*
2. *(A, B) is tangential w.r.t (S, T),*
3. *the two pairs $(A, S), (B, T)$ are weakly compatible,*

then A, B, S and T have a unique common fixed point.

If we take $\psi(t) = t$ we obtain the following corollary:

Corollary 2.3. *Let (X, σ) be a metric-like space, A, B, S and T are self mappings of X into itself satisfying for all $x, y \in X$:*

$$\sigma(Sx, Ty) \leq M(x, y) - \phi(M(x, y)),$$

where $\phi \in \Phi$ and $M(x, y) = \max\{\sigma(Ax, By), \sigma(Ax, Sx), \sigma(By, Ty), \frac{1}{4}(\sigma(Ax, Ty) + \sigma(By, Sx))\}$, further assume that the following conditions satisfy:

1. *AX and BX are closed,*
2. *(A, B) is tangential w.r.t (S, T),*
3. *the pair (A, S) is weakly compatible as well as (B, T),*

then A, B, S and T have a unique common fixed point.

Corollary 2.3 improves and generalizes theorem 2.7 of Amini Harandi [7].
In the next, we will prove a common fixed point for two tangential pairs of self mappings satisfying a generalized quasi contractive condition in metric-like space.

Theorem 2.4. *Let (X, σ) be a metric-like space, $A, B, S, T : X \to X$ are four self mappings such for all $x, y \in X$*

$$\sigma(Sx, Ty)) \leq \varphi(\max\{\sigma(Ax, By), \sigma(Ax, Sx), \sigma(By, Ty), \sigma(Ax, Ty), \sigma(By, Sx)\}), \tag{2}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a semicontinuous nondecreasing function such $\varphi(t) = 0$ if and only if $t = 0$ and for all $t > 0, \varphi(t) < t$. Suppose that the following conditions hold:

1. *$\overline{TX} \subseteq BX$ and $\overline{SX} \subseteq BX$,*
2. *the pair (A, B) is tangential w.r.t (S, T),*
3. *$(A, B), (S, T)$ are weakly compatible,*

then A, B, S and T have a unique common fixed point.

Proof. Since (A, B) is tangential w.r.t (S, T), there are two sequences $\{x_n\}, \{y_n\}$ satisfy:

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z,$$

with $p(z, z) = 0$ and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z$.
Since Ty_n is convergent to $z \in \overline{TX} \subset AX$ so there exists $u \in X$ such $z = Au$.
We will prove $Su = z$, if not by taking $x = u$ and $y = y_n$ in (2) we get:

$$\sigma(Su, Ty_n) \leq \varphi(\max\{\sigma(Au, By_n), \sigma(Au, Su), \sigma(By_n, Ty_n), \sigma(Au, Ty_n), \sigma(By_n, Su)\}),$$

letting $n \to \infty$, we get:

$$\sigma(Su, z)) \leq \varphi(\sigma(Su, z)) < (\sigma(Su, z),$$

which is a contradiction, then $p(Su, z) = 0$ and $Su = z$.
Since $Sx_n \to z \in \overline{SX} \subset BX$, there is $v \in X$ such that $z = Bv$, we claim $Tv = z$, if not by taking $x = x_n$ and $y = v$ in (2), we get

$$\sigma(Sx_n, Tv) \leq \varphi(\max\{\sigma(Ax_n, Bv), \sigma(Ax_n, Sx_n), \sigma(Bv, Tv), (Ax_n, Tv), p(Bv, Sx_n)\}),$$

letting $n \to \infty$, we get:

$$\sigma(z, Tv)) \leq \varphi(\sigma(z, Tv)) < \sigma(z, Tv),$$

which is a contradiction, then $Tv = z$.
The pair (A, S) is weakly compatible implies that $Az = Sz$, as well as (B, T) we obtain $Bz = Tz$, then z is a coincidence point for A and S and for B and T.
If $z \neq Az$, by using (2) we get:

$$\sigma(Sz, Ty_n) \leq \varphi(\max\{\sigma(Az, By_n), \sigma(Az, Sz), \sigma(By_n, Ty_n), \sigma(Az, Ty_n), \sigma(By_n, Sz)\}),$$

letting $n \to \infty$, we get:

$$\sigma(Az, z)) \leq \varphi(\sigma(Az, z)) < \varphi(\sigma(Az, z)),$$

which is a contradiction, so $\sigma(Az, z) = 0$ then $Az = z$.
Consequently z is a common fixed point for A, B, S and T.
for the uniqueness, it is similar as in Theorem 2.1. □

Theorem 2.4 improves and generalizes theorem 2.4 of Amini-Harandi[7] and theorem 1 in paper[29].

Corollary 2.5. *Let (X, p) be a partial metric space and let $A, S : X \to X$ be two self mappings satisfying:*

$$p(Sx, Ty)) \leq \varphi(\max\{p(Ax, By), p(Ax, Sx), p(By, Ty), p(Ax, Sy), p(By, Sx)\}),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a semicontinuous nondecreasing function such $\varphi(t) = 0$ if and only if $t = 0$ and for all $t > 0, \varphi(t) < t$. Further, if the following conditions hold:

1. *$\overline{TX} \subseteq BX$ and $\overline{SX} \subseteq BX$,*
2. *the pair (A, B) is tangential w.r.t (S, T),*
3. *$(A, B), (S, T)$ are weakly compatible,*

then A, B, S and T have a unique common fixed point.

Corollary 2.5 generalizes and improves theorem 1 in paper[6].
If $\varphi(t) = \alpha t$, where $0 < \alpha \leq 1$ we obtain the following corollary:

Corollary 2.6. *Let (X, σ) be a metric-like space and let $A, S : X \to X$ be two self mappings satisfying:*

$$\sigma(Sx, Ty) \leq \alpha \max\{\sigma(Ax, By), \sigma(Ax, Sx), \sigma(By, Ty), \sigma(Ax, Sy), \sigma(By, Sx)\},$$

where $0 \leq \alpha < 1$, if $\overline{TX} \subset f(X)$, $\overline{SX} \subset BX$ and $\{A, B\}$ is tangential w.r.t $\{S, T\}$, then f and S have a coincidence point, further if the two pairs $\{A, S\}, \{B, T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Corollary 2.8 improves and generalizes corollary 2.10 of Amini Harandiah and corollary 1 in paper [6].

Now, we will present a common fixed point theorem for strongly tangential self-mappings in metric-like space:

Theorem 2.7. *Let (X, σ) be a metric-like space, A, B, S and T are self mappings on X satisfying for all $x, y \in X$:*

$$\psi(\sigma(Sx, Ty)) \leq \psi(N(x, y)) - \phi(N(x, y)), \tag{3}$$

where

$$N(x, y) = \max\{\sigma(Ax, By), \frac{1}{4}(\sigma(Ax, Sx) + \sigma(By, Ty)), \sigma(Ax, Ty), \sigma(By, Sx)\},$$

if (A, B) is strongly tangential w.r.t (S, T) and the two pairs (A, S),(B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Since the pair (A, B) is strongly tangential w.r.t (S, T) implies that there exists two sequences $\{x_n\}$, $\{y_n\}$ such:

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z,$$

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z,$$

and $z \in AX \cap BX$ with $\sigma(z, z) = 0$, so there are $u, v \in X$ such $z = Au = Bv$ Firstly, we will prove $Su = z$, if not by using (3) we get:

$$\psi(\sigma(Su, y_n)) \leq \psi(\max\{\sigma(Au, By_n), \frac{1}{4}(\sigma(Au, Su) + \sigma(By_n, Ty_n)), \sigma(Au, Ty_n, \sigma(By_n, Su)\}$$

$$-\psi(\max\{\sigma(Au, By_n), \frac{1}{4}(\sigma(Au, Su), \sigma(By_n, Ty_n)), \sigma(Au, Ty_n, \sigma(By_n, Su)\},$$

letting $n \to \infty$ we get:

$$\psi(\sigma(Su, z)) \leq \psi((\sigma(Su, z)) - \phi(\sigma(Su, z)) < \psi(\sigma(Su, z),$$

which is a contradiction and so $\sigma(Su, z) = 0$, then $Su = z$.
Now, we claim $Tv = z$, if not by taking $x = x_n$ and $y = v$ in the expression $N(x, y)$ we get

$$N(x_n, v) = \max\{\sigma(Ax_n, Bv), \frac{1}{4}(\sigma(Ax_n, Sx_n) + \sigma(Bv, Tv)), \sigma(Ax_n, Tv), \sigma(Bv, Sx_n\}$$

letting $n \to \infty$ we get $N(x_n, v) \to \sigma(z, Tv)$, by using (3) we obtain:

$$\psi(\sigma(z, Tv)) \leq \psi(\sigma(z, Tv)) - \phi(\sigma(z, Tv) < \psi(\sigma(z, Tv),$$

which is a contradiction, then $Tv = z$.
The weakly compatibility of the pair (A, S) implies that $Az = Sz$, similarly for the pair (B, T) we find $Bz = Tz$, consequently z is a coincidence point for A, B, S and T.
Now, we will prove $z = Az$, if not by using (3) we get

$$\psi(\sigma(Sz, Ty_n)) \leq \psi(\max\{\sigma(Az, By_n), \frac{1}{4}(\sigma(Az, Sz) + \sigma(By_n, Ty_n)), \sigma(Az, Ty_n), \sigma(By_n, Sz)\}),$$

$$-\phi(\max\{\sigma(Az, By_n), \frac{1}{4}(\sigma(Az, Sz) + \sigma(By_n, Ty_n)), \sigma(Az, Ty_n), \sigma(By_n, Sz)\}),$$

letting $n \to \infty$ we get

$$\psi(\sigma(Az, z)) \leq \psi(\sigma(Az, z)) - \phi(\sigma(Az, z)) < \psi(\sigma(Az, z)),$$

which is a contradiction, so $\sigma(Az, z) = 0$, then $Az = z$. Similarly we can find that $Bz = Tz = z$.
For the uniqueness, suppose there is another common fixed point w, by using (3) we get:

$$\psi(\sigma(z, w)) = \psi(\sigma(Sz, Tw)) \leq \psi(N(z, w)) - \phi(N(z, w)) < \psi(\sigma(z, w)),$$

which is a contradiction, then $\sigma(z, w) = 0$ and so z is unique. □

Corollary 2.8. *Let (X, p) be a partial metric space, A, B, S and T are self mappings on X satisfying for all $x, y \in X$:*

$$\psi(p(Sx, Ty)) \leq \psi(\max\{p(Ax, By), \frac{1}{2}(p(Ax, Sx) + p(By, Ty)), p(Ax, Ty), p(By, Sx)\})$$

$$-\phi(\max\{p(Ax, By), \frac{1}{2}(p(Ax, Sx) + p(By, Ty)), p(Ax, Ty), p(By, Sx)\}),$$

if $\{A, B\}$ is strongly tangential w.r.t (S, T) and the two pairs (A, S),(B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Corollary 2.9. *Let (X, p) be a partial metric space, A, B, S and T are self mappings on X satisfying for all $x, y \in X$:*

$$p(Sx, Ty)) \leq \max\{p(Ax, By), \frac{1}{2}(p(Ax, Sx) + p(By, Ty)), p(Ax, Ty), p(By, Sx)\})$$

$$-\phi(\max\{p(Ax, By), \frac{1}{2}(p(Ax, Sx) + p(By, Ty)), p(Ax, Ty), p(By, Sx)\}),$$

if (A, B) is strongly tangential w.r.t (S, T) and the two pairs (A, S),(B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

In the following example, we will apply Corollary 2.2 (with $\psi(t) = t$) to establish a common fixed point for four self mappings in partial metric space:

Example 2.10. *Let $X = \mathbb{R}_+$ be a set endowed with the partial metric: $p(x, y) = \max\{x, y\}$, define A, B, S and T by:*

$$Ax = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ 2, & x > 1 \end{cases} \qquad Bx = \begin{cases} e^x - 1, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$$Sx = \begin{cases} \frac{x}{6}, & 0 \leq x \leq 1 \\ \ln 2, & x > 1 \end{cases} \qquad Tx = \begin{cases} \ln(1 + x), & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

we have $AX = [0, \frac{1}{2}] \cup '\{2\}$ and $BX = [0, e^1 - 1]$, which are closed.
For all $n \geq 1$, consider the two sequences $\{x_n\}, \{y_n\}$ in X which are defined for all $n \geq 1$ by:

$$x_n = \frac{1}{n}, \quad y_n = e^{-n},$$

It is clear that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = 0,$$

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = 0$$

and $p(0,0) = 0$*, then the pair* (A, B) *is tangential w.r.t* (S, T).
On other hand, the point 0 *is the unique coincidence point for* A *and* S *as well as* B *and* T*, also we have* $AS0 = SA0 = 0$ *and* $TB0 = BT0 = 0$*, which implies that the two pairs* $(A, S), (B, T)$ *are weakly compatible.*
For the contractive condition 1 with $\psi(t) = t$ *and* $\phi(t) = \frac{1}{5}$*, we have:*

1. *For* $x, y \in [0, 1]$*, we have*

$$p(Sx, Ty) = \max\{\frac{x}{6}, \log(1 + x)\} \le \frac{2}{5}x = \frac{4}{5}p(Ax, Sx),$$

2. *For* $x \in [0, 1]$ *and* $y > 1$*, we get*

$$p(Sx, Ty) = \frac{x}{6} \le \frac{2}{5}x = \frac{4}{5}p(Ax, Sx),$$

3. *For* $x > 1$ *and* $y \in [0, 1]$ *we get*

$$p(Sx, Ty) = \ln 2 \le \frac{8}{5} = \frac{4}{5}p(Ax, Sx)$$

4. *For* $x, y \in (1, \infty)$*, we get*

$$p(Sx, Ty) = \ln 2 \le \frac{8}{5} = \frac{4}{5}p(Ax, By)$$

consequently all the conditions of Corollary 2.2 are satisfied (with $\psi(t) = t$ *and* $\phi(t) = \frac{1}{5}$*), therefore* 0 *is the unique fixed point for* A, B, S *and* T.

Example 2.11. *Let* $X[0, 2]$ *with the metric-like:* $\sigma(x, y) = \max\{x, y\}$ *and let* A, B, S *and* T *be four mappings defined by:*

$$Ax = \begin{cases} 2x, & 0 \le x \le 1 \\ x, & 1 < x \le 2 \end{cases} \quad Bx = \begin{cases} \frac{3}{2}, & 0 \le x \le 1 \\ 2, & 1 < x \le 2 \end{cases}$$

$$Sx = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{4}, & 1 < x \le 2 \end{cases} \quad Tx = \begin{cases} 0, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 < x \le 2 \end{cases}$$

In this example we will utilize Corollary 2.8 with $\varphi(t) = \frac{3}{4}t$*, firstly we have:*

$$\overline{SX} = [0, \frac{1}{2}] \subset [0, \frac{3}{4}] \cup \{2\} = BX,$$

$$\overline{TX} = \{0, \frac{1}{4}\} \subset [0, 2] = AX.$$

For all $n \ge 1$*, consider the two sequences* $\{x_n\}, \{y_n\}$ *in* X *such for all* $n \le 1$*:*

$$x_n = \frac{1}{n}, \quad y_n = \ln(1 + \frac{1}{n}),$$

It is clear that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = 0,$$

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = 0$$

and $\sigma(0,0) = 0$*, then the pair* (A, B) *is tangential w.r.t* (S, T).
The mappings A *and* S *have a unique coincidence point* $x = 0$ *and satisfying* $AS0 = SA0 = 0$*, as well as* B *and* T *we have ,* $TB0 = BT0 = 0$*, which imply that the two pairs* $(A, S), (B, T)$ *are weakly compatible.*
For the inequality 2, by taking $\varphi(t) = \frac{3}{4}t$*, we get the following cases:*

1. *For $x, y \in [0,1]$, we have*

$$\sigma(Sx, Ty) = \frac{x}{2} \leq \frac{3}{2}x = \frac{3}{4}\sigma(Ax, Sx),$$

2. *For $x \in [0,1]$ and $1 < y \leq 2$, we have*

$$\sigma(Sx, Ty) = \frac{1}{2} \leq \frac{3}{2} = \frac{3}{4}\sigma(Ax, By),$$

3. *For $1 < x \leq 2$ and $y \in [0,1]$ we have*

$$\sigma(Sx, Ty) = \frac{1}{4} \leq \frac{3}{4}x = \frac{3}{4}\sigma(Ax, Sx)$$

4. *For $x, y \in (1,2]$, we have*

$$\sigma(Sx, Ty) = \frac{1}{2} \leq \frac{3}{2} = \frac{3}{4}\sigma(By, Ty)$$

consequently all the conditions of Corollary 2.8 are satisfied with $\varphi(t) = \frac{3}{4}t$, moreover 0 is the unique fixed point for A, B, S and T.

Example 2.12. *Let $X = \{0,1,2\}$ endowed with the metric-like σ which defined as:*
$\sigma(0,0) = 0$, $\sigma(1,1) = 2$, $\sigma(2,2) = 3$, $\sigma(0,1) = \sigma(1,0) = 1$, $\sigma(0,2) = \sigma(2,0) = 2$, $\sigma(1,2) = \sigma(2,1) = 4$, *it is clear that σ is not partial metric because $\sigma(1,1) = 2 > \sigma(0,1)$, define A, B, S and T as follows:*

x	Ax	Bx	Sx	Tx
0	0	0	0	0
1	2	2	1	1
2	2	0	1	0

$$Ax = BX = \{0,2\}, SX = TX = \{0,1\},$$

clearly that $0 \in AX \cap BX$ and the two pairs (A,S),(B,T) are weakly compatible.
Define for each $n \geq 0$ the two sequences $x_n = 0$ and $y_n = 2$, we have

$$\lim_n Ax_n = \lim_n Sx_n = 0$$

$$\lim_n By_n = \lim_n Ty_n = 0,$$

then (A,B) is strongly tangential w.r.t (S,T).
For the inequality (2) in Theorem 2.4 with $\psi(t) = t$ and $\varphi(t) = \frac{1}{3}t$, we have the following cases:

1. *For $(x,y) \in \{(0,0),(0,2)\}$, we have $\sigma(S(0),T(0)) = \sigma(S(0),T(2)) = 0$, so obviously that (2) is satisfied.*
2. *For $(x,y) \in \{(0,1),(1,0),(2,2),(2,0),(1,2)\}$, we have:*

$$\sigma(Sx, Ty) = 1 \leq \frac{4}{3} = \frac{2}{3}\sigma(Ax, By).$$

3. *For $(x,y) \in \{(2,1),(1,1)\}$, we have*

$$\sigma(S(2), T(1)) = \sigma(S(1), T(1)) = 1 \leq 2 = \frac{2}{3}\sigma(A1, By).$$

Consequently all hypotheses of Theorem 2.4 are satisfied and 0 is the unique common fixed point .

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