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The Relation Between Nabla Fractional Differences and Nabla Integer Differences

Jia Baoguo[a], Lynn Erbe[b], Christopher Goodrich[c], Allan Peterson[b]

[a]*School of Mathematics and Computational Science, Sun Yat-Sen University Guangzhou, China, 510275*
[b]*Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, U.S.A.*
[c]*Department of Mathematics Creighton Preparatory School, Omaha, NE 68114, U.S.A.*

Abstract. In this paper we obtain two interrelated results. The first result is the following inequality:

Theorem. Assume that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies $\nabla_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, $\nu > 0, \nu \notin \mathbb{N}_1$, and choose $N \in \mathbb{N}_1$, such that $N - 1 < \nu < N$. Then for each $k \in \mathbb{N}_{a+N}$, we have

$$\nabla^{N-1} f(a+k) \geq -\sum_{i=0}^{N-2} H_{-\nu+i}(a+k, a+i)\nabla^i f(a+i+1) - \sum_{i=N}^{k-1} H_{-\nu+N-2}(a+k, a+i-1)\nabla^{N-1} f(a+i),$$

where

$$H_{-\nu+N-2}(a+k, a+i-1) = \frac{(k-i+1)^{\overline{-\nu+N-2}}}{\Gamma(-\nu+N-1)} < 0.$$

As an application of the above inequality we prove the following result:

Theorem. Assume that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies $\nabla_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, where $5 < \nu < 6$. Then $\nabla^5 f(t) \geq 0$, for $t \in \mathbb{N}_{a+6}$.

This demonstrates that, in some sense, the positivity of the ν-th order fractional difference has a strong connection to the positivity of an integer-order difference of the function f.

1. Introduction

Discrete fractional calculus has generated much interest in recent years. Some of the work has employed the forward or delta difference operator, and these studies have included a variety of areas such as initial and boundary value problems, operational properties of the fractional difference, and applications of the discrete fractional calculus. We refer the reader to [1, 3, 4, 14, 19], for example, and more recently to

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First author is supported by the National Natural Science Foundation of China (No.11271380)
Email addresses: mcsjbg@mail.sysu.edu.cn (Jia Baoguo), lerbe2@math.unl.edu (Lynn Erbe), cgood@prep.creighton.edu (Christopher Goodrich), apeterson1@math.unl.edu (Allan Peterson)

[2, 5, 6, 11–13, 15, 16, 18, 23]; see also the book by Kelley and Peterson [20], which provides a thorough overview of the integer-order difference calculus with delta difference.

On the other hand, many other works have developed the backward or nabla difference, and for this we refer the readers to [9, 11]. There has also been some work [4, 21] to develop relations between the forward and backward fractional operators, Δ^ν and ∇^ν, and, furthermore, fractional calculus on time scales [7] – see also the monographs by Bohner and Peterson [8, 9] for a survey of the calculus on time scales. This present work is motivated by C. S. Goodrich [17], who obtained convexity results for the delta fractional difference (see also for monotonicity results [10] and [21]).

We will be interested in functions defined on sets of the form

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\}.$$

Our results could also be stated for functions on the finite set

$$\mathbb{N}_a^b := \{a, a+1, a+2, \cdots, b\},$$

where $a, b \in \mathbb{R}$ and $b-a$ is a nonnegative integer. We leave it to the reader to consider our results for this case. For $f : \mathbb{N}_a \to \mathbb{R}$ we define the nabla (backwards difference) operator by

$$\nabla f(t) = f(t) - f(t-1) \quad t \in \mathbb{N}_{a+1}.$$

We will briefly compare our results for the nabla case to the delta case, where the delta (forward difference) operator is defined by

$$\Delta f(t) = f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

With this context in mind, in this paper we obtain a couple of interrelated results. The first of these results is the following inequality:

Theorem 1.1. *Assume that $f : \mathbb{N}_{a+1} \to \mathbb{R}$ satisfies $\nabla_{a+1}^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, $\nu > 0, \nu \notin \mathbb{N}_1$, and choose $N \in \mathbb{N}_1$, such that $N-1 < \nu < N$. Then for each $k \in \mathbb{N}_N$ we have*

$$\nabla^{N-1} f(a+k) \geq -\sum_{i=0}^{N-2} H_{-\nu+i}(a+k, a+i)\nabla^i f(a+i+1) - \sum_{i=N}^{k-1} H_{-\nu+N-2}(a+k, a+i-1)\nabla^{N-1} f(a+i), \tag{1}$$

where, for $N \leq i \leq k-1$,

$$H_{-\nu+N-2}(a+k, a+i-1) = \frac{(k-i+1)^{\overline{-\nu+N-2}}}{\Gamma(-\nu+N-1)} < 0.$$

As an application, we prove the following theorem.

Theorem 1.2. *Assume that $f : \mathbb{N}_{a+1} \to \mathbb{R}$ satisfies $\nabla_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, where $5 < \nu < 6$. Then $\nabla^5 f(t) \geq 0$, for $t \in \mathbb{N}_{a+6}$.*

As a consequence of Theorem 1.2, we demonstrate an interesting corollary, which essentially states that the positivity of the ν-th order fractional nabla difference, for $5 < \nu < 6$, implies certain positivity results about the integer-order nabla differences $\nabla^i f(t)$, for $i \in \mathbb{N}_0^4$ – see Corollary 3.9 for the details.

Finally, in Section 4, we provide analogous results in the cases where either $3 < \nu < 4$ or $4 < \nu < 5$. Since the proofs of the corresponding results are similar to the case in which $5 < \nu < 6$, we omit the proofs. In any case, the statement of these results, as provided in Section 4, are as follows.

Theorem 1.3. *Assume that $f : \mathbb{N}_{a+1} \to \mathbb{R}$ satisfies $\nabla_a^\nu f(t) \geq 0$, for $t \in \mathbb{N}_{a+1}$, where $4 < \nu < 5$. Then $\nabla^4 f(t) \geq 0$ for $t \in \mathbb{N}_{a+5}$.*

Theorem 1.4. *Assume that* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$, *where* $3 < \nu < 4$. *Then* $\nabla^3 f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+4}$.

All in all, these results demonstrate that, in some sense, the positivity of the ν-th order fractional difference has a strong connection to the positivity of an integer order difference of the function f.

Recently, for the delta fractional difference Baoguo, Erbe, and Peterson [22] obtained the following result.

Theorem 1.5. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$, *that* $\Delta_a^\nu f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+1}$, *where* $5 < \nu < 6$, $(-1)^{6-i}\Delta^i f(a) \geq 0$, *for* $i = 0, 1, \cdots, 4$ *and* $\Delta^5 f(a) \geq 0$. *Then* $\Delta^5 f(t) \geq 0$, *for* $t \in \mathbb{N}_a$.

Comparing Theorem 1.5 with Theorems 3.8, 4.3, and 4.4, we see that there are substantial differences between the nabla fractional difference and the corresponding delta fractional difference. In particular, as the statement of Theorem 1.5 above indicates, in the delta fractional difference case one must assume additional, auxiliary hypotheses in order to ensure that the desired conclusion, i.e. $\Delta^5 f(t) \geq 0$, holds. Moreover, these auxiliary hypotheses are not so natural since one must, in particular, assume that f satisfies $\Delta^5 f(a) \geq 0$ as well as the collection of the conditions $(-1)^{6-i}\Delta^i f(a) \geq 0$, for $i \in \mathbb{N}_0^4$.

Finally, it should be pointed out that the computational methods in Section 3 are also valid for $N > 6$. However, for the sake of brevity, we focus in this article on the case where $N \leq 6$.

2. A Basic Inequality for the Nabla Difference Sum

Let the map $x \mapsto \Gamma(x)$ denote the gamma function. We then define the rising factorial function (see [18, equation (3.2)]) by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)},$$

for those values of t and r such that the right hand sides of these equations are well defined. We also use the standard extensions of their domains to define these functions to be zero when the numerator is well defined, but the denominator is not defined. The nabla fractional Taylor monomial of degree ν based at a (see [18, Definition 3.56]) is defined by

$$H_\nu(t,a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

The following lemma is from [18, Theorem 3.62]. We will frequently use this result in the sequel.

Lemma 2.1. *Assume that* $f : \mathbb{N}_{a+1} \to \mathbb{R}, \nu > 0, \nu \notin \mathbb{N}_1$, *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu < N$. *Then*

$$\nabla_a^\nu f(t) = \int_a^t H_{-\nu-1}(t, \rho(\tau)) f(\tau) \nabla\tau := \sum_{\tau=a+1}^{t} H_{-\nu-1}(t, \rho(\tau)) f(\tau) \nabla\tau,$$

for $t \in \mathbb{N}_{a+1}$.

We now prove the following important inequality.

Theorem 2.2. *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *where* $\nu > 0$ *and* $\nu \notin \mathbb{N}_1$. *Choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu < N$. *Then for* $t = k \in \mathbb{N}_N$

$$\nabla^{N-1} f(a+k) \geq -\sum_{i=0}^{N-2} H_{-\nu+i}(a+t, a+i)\nabla^i f(a+i+1) - \sum_{i=N}^{k-1} H_{-\nu+N-2}(a+k, a+i-1)\nabla^{N-1} f(a+i), \tag{2}$$

where

$$H_{-\nu+N-2}(a+k, a+i-1) = \frac{(k-i+1)^{\overline{-\nu+N-2}}}{\Gamma(-\nu+N-1)} < 0. \tag{3}$$

Proof. For simplicity, we only prove this theorem for the case $a = 0$. So, using Lemma 2.1, the power rule (see [18, Theorem 3.57])

$$\nabla_\tau H_{-\nu}(t,\tau) = -H_{-\nu-1}(t,\rho(\tau)),$$

and integration by parts, we have

$$\begin{aligned}
0 &\le \int_0^t H_{-\nu-1}(t,\rho(\tau))f(\tau)\nabla\tau \\
&= H_{-\nu-1}(t,0)f(1) + \int_1^t H_{-\nu-1}(t,\rho(\tau))f(\tau)\nabla\tau \\
&= H_{-\nu-1}(t,0)f(1) - \left[H_{-\nu}(t,\tau)f(\tau)\right]_{\tau=1}^t + \int_1^t H_{-\nu}(t,\rho(\tau))\nabla f(\tau)\nabla\tau \\
&= \Big[H_{-\nu-1}(t,0) + H_{-\nu}(t,1)\Big]f(1) + \int_1^t H_{-\nu}(t,\rho(\tau))\nabla f(\tau)\nabla\tau \\
&= H_{-\nu}(t,0)f(1) + \int_1^t H_{-\nu}(t,\rho(\tau))\nabla f(\tau)\nabla\tau,
\end{aligned}$$

where we used the formula

$$H_{-\nu-1}(t,0) + H_{-\nu}(t,1) = H_{-\nu}(t,0);$$

this formula is a special case of a formula in [18, Theorem 3.96], but it is easy to prove directly. Hence, another application of integration by parts gives the following estimate

$$\begin{aligned}
0 &\le H_{-\nu}(t,0)f(1) - \left[H_{-\nu+1}(t,\tau)\nabla f(\tau)\right]_{\tau=1}^t + \int_1^t H_{-\nu+1}(t,\rho(\tau))\nabla^2 f(\tau)\nabla\tau \\
&= H_{-\nu}(t,0)f(1) + H_{-\nu+1}(t,1)\nabla f(1) + \int_1^t H_{-\nu+1}(t,\rho(\tau))\nabla^2 f(\tau)\nabla\tau \\
&= H_{-\nu}(t,0)f(1) + H_{-\nu+1}(t,1)\nabla f(1) + \int_1^2 H_{-\nu+1}(t,\rho(\tau))\nabla^2 f(\tau)\nabla\tau + \int_2^t H_{-\nu+1}(t,\rho(\tau))\nabla^2 f(\tau)\nabla\tau \\
&= H_{-\nu}(t,0)f(1) + H_{-\nu+1}(t,1)\nabla f(1) + H_{-\nu+1}(t,1)\nabla^2 f(2) + \int_2^t H_{-\nu+1}(t,\rho(\tau))\nabla^2 f(\tau)\nabla\tau \\
&= H_{-\nu}(t,0)f(1) + H_{-\nu+1}(t,1)\nabla f(2) + \int_2^t H_{-\nu+1}(t,\rho(\tau))\nabla^2 f(\tau)\nabla\tau,
\end{aligned}$$

where we use both that $H_{-\nu}(t,t) = 0$ and that $H_{-\nu+1}(t,t) = 0$.

Continuing in this manner we get by mathematical induction for $i \in \mathbb{N}_0$ that

$$\begin{aligned}
0 &\le \int_0^t H_{-\nu-1}(t,\rho(\tau))f(\tau)\nabla\tau \\
&= H_{-\nu}(t,0)f(1) + H_{-\nu+1}(t,1)\nabla f(2) + \cdots + H_{-\nu+i}(t,i)\nabla^i f(i+1) \qquad (4)\\
&\quad + \int_{i+1}^t H_{-\nu+i}(t,\rho(\tau))\nabla^{i+1} f(\tau)\nabla\tau,
\end{aligned}$$

for $t \in \mathbb{N}_{i+1}$. Taking $i = N-2$ in (4), we get that for $t \in \mathbb{N}_N$

$$\int_0^t H_{-\nu-1}(t,\rho(\tau))f(\tau)\nabla\tau = \sum_{j=0}^{N-2} H_{-\nu+j}(t,j)\nabla^j f(j+1) + \int_{N-1}^t H_{-\nu+N-2}(t,\rho(\tau))\nabla^{N-1} f(\tau)\nabla\tau. \qquad (5)$$

Then putting $t = k$ in (5), we have

$$\begin{aligned} 0 &\leq \int_0^t H_{-\nu-1}(t,\rho(\tau))f(\tau)\nabla\tau \\ &= \sum_{i=0}^{N-2} H_{-\nu+i}(k,i)\nabla^i f(i+1) + \sum_{i=N}^{k-1} H_{-\nu+N-2}(k,i-1)\nabla^{N-1}f(i) + \nabla^{N-1}f(k), \end{aligned}$$

for $k \in \mathbb{N}_N$, where in the last step we used $H_{-\nu+N-2}(k,k-1) = 1$. Solving this last inequality for $\nabla^{N-1}f(k)$ we obtain the desired inequality (1) for $a = 0$.

Finally, we show that (3) holds for $a = 0$. This follows by noting that for each $N \leq i \leq k-1$ we have

$$\begin{aligned} H_{-\nu+N-2}(k,i-1) &= \frac{(k-i+1)^{\overline{-\nu+N-2}}}{\Gamma(-\nu+N-1)} \\ &= \frac{\Gamma(k-i-\nu+N-1)}{\Gamma(k-i+1)\Gamma(-\nu+N-1)} \\ &= \frac{1}{(k-i)!}\prod_{j=-1}^{k-i-2}(-\nu+N+j) \\ &< 0, \end{aligned}$$

since $N-1 < \nu < N$, and this completes the proof. □

3. Preliminary Lemmas and Main Result

In this section we state and prove the main result of our paper. Prior to this, however, we need to establish some preliminary lemmas. To begin, we notice that the following lemma may be established.

Lemma 3.1. *Assume that $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nabla_a^\nu f(t) \geq 0$, for $t \geq \mathbb{N}_{a+1}$. Then the following inequalities hold:*

$$f(a+1) \geq 0, \qquad H_{-\nu}(a+2,a)f(a+1) + \nabla f(a+2) \geq 0,$$

$$\begin{gathered} H_{-\nu}(a+3,a)f(a+1) + H_{-\nu+1}(a+3,a+1)\nabla f(a+2) + \nabla^2 f(a+3) \geq 0, \\ \sum_{i=0}^{2} H_{-\nu+i}(a+4,a+i)\nabla^i f(a+i+1) + \nabla^3 f(a+4) \geq 0, \\ \sum_{i=0}^{3} H_{-\nu+i}(a+5,a+i)\nabla^i f(a+i+1) + \nabla^4 f(a+5) \geq 0, \\ \sum_{i=0}^{4} H_{-\nu+i}(a+6,a+i)\nabla^i f(a+i+1) + \nabla^5 f(a+6) \geq 0. \end{gathered}$$

Proof. We just prove this result for the case $a = 0$. Using (4) and letting both $i \in \mathbb{N}_0^5$ and $t = i+1$, the result follows by noticing that $H_{-\nu+i}(i+1,i) = 1$. □

We next establish another lemma. This particular result is a key observation in establishing the main result, Theorem 3.8, of this section. It is also of some independent interest.

Lemma 3.2. *Assume $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $5 < \nu < 6$, and $\nabla_a^\nu f(t) \geq 0$, for $t \in \mathbb{N}_{a+1}$. Then each of the following inequalities holds:*

1. $\nabla^5 f(a+6) \geq -H_{-\nu}(a+6,a)f(a+1) \geq 0$;
2. $\nabla^4 f(a+5) \geq \quad H_{-\nu}(a+5,a)f(a+1) \geq 0$;
3. $\nabla^3 f(a+4) \geq -H_{-\nu}(a+4,a)f(a+1) \geq 0$;
4. $\nabla^2 f(a+3) \geq \quad H_{-\nu}(a+3,a)f(a+1) \geq 0$;
5. $\nabla f(a+2) \geq -H_{-\nu}(a+2,a)f(a+1) \geq 0$;
6. $f(a+1) \geq 0$.

Proof. For simplicity we just prove this theorem for the case $a = 0$. In this proof we will use the following notation. If

$$\mathbf{Y} := (y_1, y_2, y_3, y_4, y_5, y_6)^T,$$

then we write $\mathbf{Y} \geq \mathbf{0}$, where $\mathbf{0}$ is the 6×1 column vector $\mathbf{0} := (0,0,0,0,0,0)^T$, if and only if $y_i \geq 0$ for $1 \leq i \leq 6$. If we let

$$\mathbf{X} := (\nabla^5 f(6), \nabla^4 f(5), \nabla^3 f(4), \nabla^2 f(3), \nabla f(2), f(1))^T,$$

then by Lemma 3.1, we have that

$$\mathbf{A}_1\mathbf{X} \geq \mathbf{0},$$

where $\mathbf{A}_1 \in \mathbb{R}^{6\times 6}$ is the matrix defined by

$$\mathbf{A}_1 := \begin{pmatrix} 1 & H_{-\nu+4}(6,4) & H_{-\nu+3}(6,3) & H_{-\nu+2}(6,2) & H_{-\nu+1}(6,1) & H_{-\nu}(6,0) \\ 0 & 1 & H_{-\nu+3}(5,3) & H_{-\nu+2}(5,2) & H_{-\nu+1}(5,1) & H_{-\nu}(5,0) \\ 0 & 0 & 1 & H_{-\nu+2}(4,2) & H_{-\nu+1}(4,1) & H_{-\nu}(4,0) \\ 0 & 0 & 0 & 1 & H_{-\nu+1}(3,1) & H_{-\nu}(3,0) \\ 0 & 0 & 0 & 0 & 1 & H_{-\nu}(2,0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Our goal is to use elementary row operations to obtain from $\mathbf{A}_1\mathbf{X} \geq \mathbf{0}$ a suitable matrix-vector inequality of the form $\mathbf{B}\mathbf{X} \geq \mathbf{0}$, where $\mathbf{B}$ is chosen in such a way that the desired conclusion immediately follows.

To facilitate the computations that follow, given a matrix $\mathbf{C}_i \in \mathbb{R}^{6\times 6}$, for use in the sequel we shall put

$$\mathbf{C}_{i,j} := \mathbf{e}_j\mathbf{C}_i,$$

where $\mathbf{e}_j \in \mathbb{R}^6$ is the j-th basis vector (interpreted as a row matrix) in the standard, ordered basis for $\mathbb{R}^6$; that is to say, the matrix $\mathbf{C}_{i,j}$ is the j-th row vector of matrix $\mathbf{C}_i$. Finally, we recall that the matrix $\mathbf{e}_j^T\mathbf{e}_i$, for $1 \leq i, j \leq 6$, has the property that for any matrix $\mathbf{A} \in \mathbb{R}^{6\times 6}$ the matrix

$$\mathbf{e}_j^T\mathbf{e}_i\mathbf{A} \in \mathbb{R}^{6\times 6}$$

is a matrix, whose j-th row is the i-th row vector of $\mathbf{A}$ and each of whose other rows is the zero vector in $\mathbb{R}^6$.

So, with these observations in mind, we begin by noticing that

$$-H_{-\nu+4}(6,4) = \nu - 5 > 0.$$

Thus, if we then put

$$\mathbf{A}_2 := \underbrace{\mathbf{e}_1^T\left(-H_{-\nu+4}(6,4)\underbrace{\mathbf{A}_{1,2}}_{=\mathbf{e}_2\mathbf{A}_1}\right)}_{\in\mathbb{R}^{6\times 6}} + \mathbf{A}_1,$$

we obtain that

$$\mathbf{A}_2\mathbf{X} \geq \mathbf{A}_1\mathbf{X} \geq \mathbf{0},$$

where

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & \frac{-\Gamma(-\nu+6)}{2\Gamma(-\nu+4)} & \frac{-\Gamma(-\nu+6)}{3\Gamma(-\nu+3)} & \frac{-\Gamma(-\nu+6)}{2!\cdot 4\Gamma(-\nu+2)} & \frac{-\Gamma(-\nu+6)}{3!\cdot 5\Gamma(-\nu+1)} \\ 0 & 1 & H_{-\nu+3}(5,3) & H_{-\nu+2}(5,2) & H_{-\nu+1}(5,1) & H_{-\nu}(5,0) \\ 0 & 0 & 1 & H_{-\nu+2}(4,2) & H_{-\nu+1}(4,1) & H_{-\nu}(4,0) \\ 0 & 0 & 0 & 1 & H_{-\nu+1}(3,1) & H_{-\nu}(3,0) \\ 0 & 0 & 0 & 0 & 1 & H_{-\nu}(2,0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next notice that $-H_{-\nu+3}(5,3) = \nu - 4 > 0$ and that $\frac{\Gamma(-\nu+6)}{2\Gamma(-\nu+4)} > 0$. Using these observations, we next define the matrix $\mathbf{A}_3$ by

$$\mathbf{A}_3 := \mathbf{e}_2^T\left(-H_{-\nu+3}(5,3)\mathbf{A}_{2,3}\right) + \mathbf{e}_1^T\left(\frac{\Gamma(-\nu+6)}{2\Gamma(-\nu+4)}\mathbf{A}_{2,3}\right) + \mathbf{A}_2.$$

We thus obtain the vector inequality

$$\mathbf{A}_3\mathbf{X} \geq \mathbf{A}_2\mathbf{X} \geq \mathbf{0},$$

where

$$\mathbf{A}_3 = \begin{pmatrix} 1 & 0 & 0 & \frac{\Gamma(-\nu+6)}{6\Gamma(-\nu+3)} & \frac{\Gamma(-\nu+6)}{8\Gamma(-\nu+2)} & \frac{\Gamma(-\nu+6)}{20\Gamma(-\nu+1)} \\ 0 & 1 & 0 & \frac{-\Gamma(-\nu+5)}{2\Gamma(-\nu+3)} & \frac{-\Gamma(-\nu+5)}{1!\cdot 3\Gamma(-\nu+2)} & \frac{-\Gamma(-\nu+5)}{2!\cdot 4\Gamma(-\nu+1)} \\ 0 & 0 & 1 & H_{-\nu+2}(4,2) & H_{-\nu+1}(4,1) & H_{-\nu}(4,0) \\ 0 & 0 & 0 & 1 & H_{-\nu+1}(3,1) & H_{-\nu}(3,0) \\ 0 & 0 & 0 & 0 & 1 & H_{-\nu}(2,0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now notice that $-H_{-\nu+2}(4,2) = \nu - 3 > 0$. Moreover, it may be easily shown that each of the following inequalities holds as well.

$$\frac{\Gamma(-\nu+5)}{2\Gamma(-\nu+3)} > 0, \quad -\frac{\Gamma(-\nu+6)}{6\Gamma(-\nu+3)} > 0.$$

So, we may then construct the matrix $\mathbf{A}_4$ by setting

$$\begin{aligned}\mathbf{A}_4 &:= \mathbf{e}_3^T\left(-H_{-\nu+2}(4,2)\mathbf{A}_{3,4}\right) + \mathbf{e}_2^T\left(\frac{\Gamma(-\nu+5)}{2\Gamma(-\nu+3)}\mathbf{A}_{3,4}\right) \\ &\quad + \mathbf{e}_1^T\left(-\frac{\Gamma(-\nu+6)}{6\Gamma(-\nu+3)}\mathbf{A}_{3,4}\right) + \mathbf{A}_3,\end{aligned}$$

which thus yields the inequality

$$\mathbf{A}_4\mathbf{X} \geq \mathbf{A}_3\mathbf{X} \geq \mathbf{0},$$

where

$$\mathbf{A}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{-\Gamma(-\nu+6)}{24\Gamma(-\nu+2)} & \frac{-\Gamma(-\nu+6)}{30\Gamma(-\nu+1)} \\ 0 & 1 & 0 & 0 & \frac{\Gamma(-\nu+5)}{6\Gamma(-\nu+2)} & \frac{\Gamma(-\nu+5)}{8\Gamma(-\nu+1)} \\ 0 & 0 & 1 & 0 & \frac{-\Gamma(-\nu+4)}{2\Gamma(-\nu+2)} & \frac{-\Gamma(-\nu+4)}{3\Gamma(-\nu+1)} \\ 0 & 0 & 0 & 1 & H_{-\nu+1}(3,1) & H_{-\nu}(3,0) \\ 0 & 0 & 0 & 0 & 1 & H_{-\nu}(2,0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

At last, we observe both that $-H_{-\nu+2}(3,1) = \nu - 2 > 0$ and that each of the following inequalities holds:

$$\frac{\Gamma(-\nu+4)}{2\Gamma(-\nu+2)} > 0 \quad -\frac{\Gamma(-\nu+5)}{6\Gamma(-\nu+2)} > 0 \quad \frac{\Gamma(-\nu+6)}{24\Gamma(-\nu+2)} > 0.$$

Consequently, we may put

$$\begin{aligned}\mathbf{A}_5 &:= \mathbf{e}_4^T\left(-H_{-\nu+1}(3,1)\mathbf{A}_{4,5}\right) + \mathbf{e}_3^T\left(\frac{\Gamma(-\nu+4)}{2\Gamma(-\nu+2)}\mathbf{A}_{4,5}\right)\\ &\quad + \mathbf{e}_2^T\left(-\frac{\Gamma(-\nu+5)}{6\Gamma(-\nu+2)}\mathbf{A}_{4,5}\right) + \mathbf{e}_1^T\left(\frac{\Gamma(-\nu+6)}{24\Gamma(-\nu+2)}\mathbf{A}_{4,5}\right) + \mathbf{A}_4.\end{aligned}$$

And this at last yields the inequality

$$\mathbf{A}_5\mathbf{X} \geq \mathbf{A}_4\mathbf{X} \geq \mathbf{0},$$

where it can be checked that

$$\mathbf{A}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{\Gamma(-\nu+6)}{120\Gamma(-\nu+1)} \\ 0 & 1 & 0 & 0 & 0 & \frac{-\Gamma(-\nu+5)}{24\Gamma(-\nu+1)} \\ 0 & 0 & 1 & 0 & 0 & \frac{\Gamma(-\nu+4)}{6\Gamma(-\nu+1)} \\ 0 & 0 & 0 & 1 & 0 & \frac{-\Gamma(-\nu+3)}{2\Gamma(-\nu+1)} \\ 0 & 0 & 0 & 0 & 1 & H_{-\nu}(2,0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that after some routine simplification we obtain that

$$\mathbf{A}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & H_{-\nu}(6,0) \\ 0 & 1 & 0 & 0 & 0 & -H_{-\nu}(5,0) \\ 0 & 0 & 1 & 0 & 0 & H_{-\nu}(4,0) \\ 0 & 0 & 0 & 1 & 0 & -H_{-\nu}(3,0) \\ 0 & 0 & 0 & 0 & 1 & H_{-\nu}(2,0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, the inequality $\mathbf{A}_5\mathbf{X} \geq \mathbf{0}$ implies immediately that each of the following inequalities holds.

$$\begin{aligned}\nabla^5 f(6) &\geq -H_{-\nu}(6,0)f(1) \geq 0\\ \nabla^4 f(5) &\geq H_{-\nu}(5,0)f(1) \geq 0\\ \nabla^3 f(4) &\geq -H_{-\nu}(4,0)f(1) \geq 0\\ \nabla^2 f(3) &\geq H_{-\nu}(3,0)f(1) \geq 0\\ \nabla f(2) &\geq -H_{-\nu}(2,0)f(1) \geq 0\\ f(1) &\geq 0\end{aligned}$$

This completes the proof. □

We next state and prove four different technical lemmas, each of which will be used in the sequel – particularly, in the proof of Lemma 3.7.

Lemma 3.3. *Assume $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$, and $5 < \nu < 6$. Then for $t \in \mathbb{N}_{a+6}$*

$$-\left[H_{-\nu+3}(t,a+3) - H_{-\nu+4}(t,a+4)H_{-\nu+3}(5,3)\right] = \frac{\Gamma(-\nu+t)}{(t-6)!(t-4)\Gamma(-\nu+4)} > 0. \tag{6}$$

Proof. We omit the straight forward proof of this lemma. □

Lemma 3.4. *Assume* $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$, *and* $5 < \nu < 6$. *Then for* $t \in \mathbb{N}_{a+6}$

$$\begin{aligned}
&-\left[H_{-\nu+2}(t, a+2) + \frac{\Gamma(-\nu+t)H_{-\nu+2}(4,2)}{(t-6)!(t-4)\Gamma(-\nu+4)} - H_{-\nu+4}(t, a+4)H_{-\nu+2}(5,2)\right] \\
&\qquad = -\frac{\Gamma(-\nu+t)}{2(t-6)!(t-3)\Gamma(-\nu+3)} > 0.
\end{aligned} \tag{7}$$

Proof. For simplicity we only consider the case $a = 0$. We merely observe that

$$\begin{aligned}
&-\left[H_{-\nu+2}(t, 2) + \frac{\Gamma(-\nu+t)H_{-\nu+2}(4,2)}{(t-6)!(t-4)\Gamma(-\nu+4)} - H_{-\nu+4}(t, 4)H_{-\nu+2}(5,2)\right] \\
&= \frac{\Gamma(-\nu+t)}{(t-6)!\Gamma(-\nu+3)}\left[\frac{-1}{(t-3)(t-4)(t-5)} - \frac{1}{t-4} + \frac{1}{2(t-5)}\right] \\
&= -\frac{\Gamma(-\nu+t)}{2(t-6)!(t-3)\Gamma(-\nu+3)} > 0.
\end{aligned}$$

And this completes the proof. □

Lemma 3.5. *Assume* $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$, *and* $5 < \nu < 6$. *Then for* $t \in \mathbb{N}_{a+6}$

$$\begin{aligned}
&-H_{-\nu+1}(t, a+1) - \frac{\Gamma(-\nu+t)H_{-\nu+1}(4,1)}{(t-6)!(t-4)\Gamma(-\nu+4)} \\
&+ H_{-\nu+4}(t, a+4)H_{-\nu+1}(5,1) + \frac{\Gamma(-\nu+t)H_{-\nu+1}(3,1)}{2(t-6)!(t-3)\Gamma(-\nu+3)} \\
&> 0.
\end{aligned} \tag{8}$$

Proof. For simplicity we only consider the case $a = 0$. We observe that

$$\begin{aligned}
&-H_{-\nu+1}(t, 1) - \frac{\Gamma(-\nu+t)H_{-\nu+1}(4,1)}{(t-6)!(t-4)\Gamma(-\nu+4)} + H_{-\nu+4}(t, 4)H_{-\nu+1}(5,1) + \frac{\Gamma(-\nu+t)H_{-\nu+1}(3,1)}{2(t-6)!(t-3)\Gamma(-\nu+3)} \\
&= \frac{\Gamma(-\nu+t)}{\Gamma(-\nu+2)(t-6)!} \cdot \frac{t^3 - 12t^2 + 47t - 60}{6(t-2)(t-3)(t-4)(t-5)} \\
&= \frac{\Gamma(-\nu+t)}{6\Gamma(-\nu+2)(t-6)!(t-2)} > 0,
\end{aligned}$$

which establishes the claim. □

Lemma 3.6. *Assume* $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$, *and* $5 < \nu < 6$. *Then for* $t \in \mathbb{N}_{a+6}$

$$\begin{aligned}
&-\frac{\Gamma(-\nu+t)H_{-\nu}(2,0)}{6\Gamma(-\nu+2)(t-6)!(t-2)} - H_{-\nu}(t, a) - \frac{\Gamma(-\nu+t)H_{-\nu}(4,0)}{(t-6)!(t-4)\Gamma(-\nu+4)} \\
&+ H_{-\nu+4}(t, a+4)H_{-\nu}(5,0) + \frac{\Gamma(-\nu+t)H_{-\nu}(3,0)}{2(t-6)!(t-3)\Gamma(-\nu+3)} \\
&= -\frac{\Gamma(-\nu+t)}{6\Gamma(-\nu+1)((t-6)!(t-1))}.
\end{aligned} \tag{9}$$

Proof. We observe that

$$\begin{aligned}
&-\frac{\Gamma(-\nu+t)H_{-\nu}(2,0)}{6\Gamma(-\nu+2)(t-6)!(t-2)} - H_{-\nu}(t,0) - \frac{\Gamma(-\nu+t)H_{-\nu}(4,0)}{(t-6)!(t-4)\Gamma(-\nu+4)}\\
&+ H_{-\nu+4}(t,4)H_{-\nu}(5,0) + \frac{\Gamma(-\nu+t)H_{-\nu}(3,0)}{2(t-6)!(t-3)\Gamma(-\nu+3)}\\
&= -\frac{\Gamma(-\nu+t)}{\Gamma(-\nu+1)((t-6)!)} \cdot \frac{t^4-14t^3+71t^2-154t+120}{6(t-1)(t-2)(t-3)(t-4)(t-5)}\\
&= -\frac{\Gamma(-\nu+t)}{\Gamma(-\nu+1)((t-6)!)} \cdot \frac{(t-2)(t-3)(t-4)(t-5)}{6(t-1)(t-2)(t-3)(t-4)(t-5)}\\
&= -\frac{\Gamma(-\nu+t)}{6\Gamma(-\nu+1)((t-6)!(t-1))},
\end{aligned}$$

which completes the proof. □

With Lemmas 3.3–3.6 in hand, we now prove one final preliminary lemma.

Lemma 3.7. *Assume $f:\mathbb{N}_{a+1}\to\mathbb{R}$, $5<\nu<6$, and $\nabla_a^\nu f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$. Then for $t\in\mathbb{N}_{a+6}$*

$$-\sum_{i=0}^{4} H_{-\nu+i}(t,a+i)\nabla^i f(a+i+1) \geq -\frac{\Gamma(-\nu+t)f(a+1)}{6\Gamma(-\nu+1)(t-6)!(t-1)} \geq 0. \tag{10}$$

Proof. We will just prove this result for $a=0$. From Lemma 3.1, we have both that

$$\nabla^4 f(5) \geq -\sum_{i=0}^{3} H_{-\nu+i}(5,i)\nabla^i f(i+1) \tag{11}$$

and that

$$\nabla^3 f(4) \geq -\sum_{i=0}^{2} H_{-\nu+i}(4,i)\nabla^i f(i+1). \tag{12}$$

So using both (11) and $-H_{-\nu+4}(t,4)>0$, we have

$$\begin{aligned}
&-\sum_{i=0}^{4} H_{-\nu+i}(t,i)\nabla^i f(i+1) = -\sum_{i=0}^{2} H_{-\nu+i}(t,i)\nabla^i f(i+1) - [H_{-\nu+3}(t,3)\nabla^3 f(4) + H_{-\nu+4}(t,4)\nabla^4 f(5)]\\
&\overset{(11)}{\geq} -\sum_{i=0}^{2} H_{-\nu+i}(t,i)\nabla^i f(i+1) - [H_{-\nu+3}(t,3) + H_{-\nu+4}(t,4)H_{-\nu+3}(5,3)]\nabla^3 f(4)\\
&\quad + H_{-\nu+4}(t,4)\sum_{i=0}^{2} H_{-\nu+i}(5,i)\nabla^i f(i+1)\\
&\overset{(6),(12)}{\geq} -\sum_{i=0}^{2} H_{-\nu+i}(t,i)\nabla^i f(i+1) - \frac{\Gamma(-\nu+t)}{(t-6)!(t-4)\Gamma(-\nu+4)}\sum_{i=0}^{2} H_{-\nu+i}(4,i)\nabla^i f(i+1)\\
&\quad + H_{-\nu+4}(t,4)\sum_{i=0}^{2} H_{-\nu+i}(5,i)\nabla^i f(i+1)\\
&= -\sum_{i=0}^{1} H_{-\nu+i}(t,i)\nabla^i f(i+1) - \frac{\Gamma(-\nu+t)}{(t-6)!(t-4)\Gamma(-\nu+4)}\sum_{i=0}^{1} H_{-\nu+i}(4,i)\nabla^i f(i+1)\\
&+ H_{-\nu+4}(t,4)\sum_{i=0}^{1} H_{-\nu+i}(5,i)\nabla^i f(i+1) - \left[H_{-\nu+2}(t,2) + \frac{\Gamma(-\nu+t)H_{-\nu+2}(4,2)}{(t-6)!(t-4)\Gamma(-\nu+4)} - H_{-\nu+4}(t,4)H_{-\nu+i}(5,2)\right]\nabla^2 f(3).
\end{aligned}$$

Furthermore, using (7) it follows that

$$\begin{aligned}
&-\sum_{i=0}^{4} H_{-\nu+i}(t,i)\nabla^i f(i+1)\\
&\geq -\sum_{i=0}^{1} H_{-\nu+i}(t,i)\nabla^i f(i+1) - \frac{\Gamma(-\nu+t)}{(t-6)!(t-4)\Gamma(-\nu+4)}\sum_{i=0}^{1} H_{-\nu+i}(4,i)\nabla^i f(i+1)\\
&\quad + H_{-\nu+4}(t,4)\sum_{i=0}^{1} H_{-\nu+i}(5,i)\nabla^i f(i+1) - \frac{\Gamma(-\nu+t)}{2(t-6)!(t-3)\Gamma(-\nu+3)}\nabla^2 f(3).
\end{aligned}$$

Next recalling that $\nabla^2 f(3) \geq -[H_{-\nu+1}(3,1)\nabla f(2) + H_{-\nu}(3,0)f(1)]$, we estimate

$$\begin{aligned}
&-\sum_{i=0}^{4} H_{-\nu+i}(t,i)\nabla^i f(i+1)\\
&\geq -\sum_{i=0}^{1} H_{-\nu+i}(t,i)\nabla^i f(i+1) - \frac{\Gamma(-\nu+t)}{(t-6)!(t-4)\Gamma(-\nu+4)}\sum_{i=0}^{1} H_{-\nu+i}(4,i)\nabla^i f(i+1)\\
&\quad + H_{-\nu+4}(t,4)\sum_{i=0}^{1} H_{-\nu+i}(5,i)\nabla^i f(i+1) + \frac{\Gamma(-\nu+t)}{2(t-6)!(t-3)\Gamma(-\nu+3)}[H_{-\nu+1}(3,1)\nabla f(2) + H_{-\nu}(3,0)f(1)]\\
&= \left[-H_{-\nu+1}(t,1) - \frac{\Gamma(-\nu+t)H_{-\nu+1}(4,1)}{(t-6)!(t-4)\Gamma(-\nu+4)} + H_{-\nu+4}(t,4)H_{-\nu+1}(5,1) + \frac{\Gamma(-\nu+t)H_{-\nu+1}(3,1)}{2(t-6)!(t-3)\Gamma(-\nu+3)}\right]\nabla f(2)\\
&\quad + \left[-H_{-\nu}(t,0) - \frac{\Gamma(-\nu+t)H_{-\nu}(4,0)}{(t-6)!(t-4)\Gamma(-\nu+4)} + H_{-\nu+4}(t,4)H_{-\nu}(5,0) + \frac{\Gamma(-\nu+t)H_{-\nu}(3,0)}{2(t-6)!(t-3)\Gamma(-\nu+3)}\right]f(1).
\end{aligned}$$

Now, inequality (8) implies the estimate

$$\begin{aligned}
&-\sum_{i=0}^{4} H_{-\nu+i}(t,i)\nabla^i f(i+1)\\
&\geq \frac{\Gamma(-\nu+t)}{6\Gamma(-\nu+2)(t-6)!(t-2)}\nabla f(2) + \left[-H_{-\nu}(t,0) - \frac{\Gamma(-\nu+t)H_{-\nu}(4,0)}{(t-6)!(t-4)\Gamma(-\nu+4)}\right.\\
&\quad \left. + H_{-\nu+4}(t,4)H_{-\nu}(5,0) + \frac{\Gamma(-\nu+t)H_{-\nu}(3,0)}{2(t-6)!(t-3)\Gamma(-\nu+3)}\right]f(1).
\end{aligned}$$

Finally, since $\nabla f(2) \geq -H_{-\nu}(2,0)f(1)$, this together with (9) yields the inequality

$$\begin{aligned}
&-\sum_{i=0}^{4} H_{-\nu+i}(t,i)\nabla^i f(i+1)\\
&\geq -\frac{\Gamma(-\nu+t)H_{-\nu}(2,0)}{6\Gamma(-\nu+2)(t-6)!(t-2)}f(1) + \left[-H_{-\nu}(t,0) - \frac{\Gamma(-\nu+t)H_{-\nu}(4,0)}{(t-6)!(t-4)\Gamma(-\nu+4)}\right.\\
&\quad \left. + H_{-\nu+4}(t,4)H_{-\nu}(5,0) + \frac{\Gamma(-\nu+t)H_{-\nu}(3,0)}{2(t-6)!(t-3)\Gamma(-\nu+3)}\right]f(1)\\
&= -\frac{\Gamma(-\nu+t)}{6\Gamma(-\nu+1)((t-6)!(t-1))}f(1)\\
&\geq 0.
\end{aligned}$$

This completes the proof. □

We now prove the main result of this paper.

Theorem 3.8. *Assume that* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *and* $5 < \nu < 6$. *Then* $\nabla^5 f(t) \geq 0$, *for* $t \in \mathbb{N}_{a+6}$.

Proof. For simplicity, we let $a = 0$. We prove that $\nabla^5 f(i) \geq 0$, for $i \in \mathbb{N}_6$ by the principle of strong induction. From Lemma 3.2, this holds for $i = 6$. Suppose that $\nabla^5 f(i) \geq 0$, for $i = 6, 7, \cdots, k-1$. From Lemma 3.3 and Theorem 2.2, we have $\nabla^5 f(k) \geq 0$. Hence, the proof is complete. □

From Lemma 3.2 and Theorem 3.8, we can get the following interesting corollary. Essentially, this corollary states that if $\nabla_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, and $5 < \nu < 6$, then this information actually implies the nonnegativity of i-th order nabla difference of f, for each $i \in \mathbb{N}_0^4$. Obviously, this does not occur in the integer-order setting. That is to say, just because, say, $\nabla^6 f(t) \geq 0$, this hardly implies that $\nabla^i f(t) \geq 0$, for each $i \in \mathbb{N}_0^5$; in fact, this sort of conclusion need not even eventually hold. That this somewhat unexpected result holds in the fractional-order setting is a direct consequence of the nonlocal structure of the fractional nabla difference.

Corollary 3.9. *Assume that* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *and* $5 < \nu < 6$. *Then* $\nabla^i f(t) \geq 0$, *for* $t \in \mathbb{N}_{a+i+1}$, $0 \leq i \leq 4$.

Proof. For simplicity, we again let $a = 0$. From Lemma 3.2 we have that $\nabla^4 f(5) \geq 0$. From Theorem 3.8, we have that $\nabla^4 f(t)$ is increasing, for $t \in \mathbb{N}_5$. So, it follows that

$$\nabla^4 f(t) \geq \nabla^4 f(5) \geq 0,$$

for $t \in \mathbb{N}_5$. Similarly, we have that

$$\nabla^i f(t) \geq \nabla^i f(i+1) \geq 0, \text{ for } t \in \mathbb{N}_{i+1},$$

for each $i \in \mathbb{N}_0^3$. This completes the proof. □

4. Additional Results in Case Either $3 < \nu < 4$ or $4 < \nu < 5$

In this section, we briefly show how the techniques of Sections 2 and 3 can generate additional results, similar to Theorem 3.8, in case either $3 < \nu < 4$ or $4 < \nu < 5$. Since the proofs of each of the results in this section is similar to those of Sections 2 and 3, we omit the proofs here for the sake of brevity.

Lemma 4.1. *Assume* $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$, *and* $4 < \nu < 5$. *Then we have that*

$$\nabla^4 f(a+5) \geq H_{-\nu}(a+5, a) f(a+1) \geq 0,$$

and

$$-\sum_{i=0}^{3} H_{-\nu+i}(t, a+i) \nabla^i f(a+i+1) \geq \frac{\Gamma(-\nu+t)}{6(t-5)!(t-1)\Gamma(-\nu+1)} f(a+1) \geq 0. \tag{13}$$

Lemma 4.2. *Assume* $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nabla_a^\nu f(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$, *and* $3 < \nu < 4$. *Then for* $t \in \mathbb{N}_{a+4}$, *we have both that*

$$\nabla^3 f(a+4) \geq -H_{-\nu}(a+4, a) f(a+1) \geq 0$$

and that

$$-\sum_{i=0}^{2} H_{-\nu+i}(t, a+i) \nabla^i f(a+i+1) \geq -\frac{\Gamma(-\nu+t)}{2(t-4)!(t-1)\Gamma(-\nu+1)} f(a+1) \geq 0.$$

Using Theorem 2.2 together with Lemmas 4.1–4.2, we can get the following theorems, whose proofs we omit.

Theorem 4.3. *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *where* $4 < \nu < 5$. *Then* $\nabla^4 f(t) \geq 0$, *for* $t \in \mathbb{N}_{a+5}$.

Theorem 4.4. *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *where* $3 < \nu < 4$. *Then* $\nabla^3 f(t) \geq 0$, *for* $t \in \mathbb{N}_{a+4}$.

Remark 4.5. *We note finally that similar techniques apply for the case* $N \geq 7$, *but we leave this to the interested reader.*

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