Filomat 31:6 (2017), 1773–1779
DOI 10.2298/FIL1706773M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and Structure of the Common Fixed Points Based on TVS

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Abstract. In this paper, we investigate the common fixed point property for commutative nonexpansive mappings on τ-compact convex sets in normed and Banach spaces, where τ is a Hausdorff topological vector space topology that is weaker than the norm topology. As a consequence of our main results, we obtain that the set of common fixed points of any commutative family of nonexpansive self-mappings of a nonempty *clm*-compact (resp. weak* compact) convex subset C of $L_1(\mu)$ with a σ-finite μ (resp. the James space J_0) is a nonempty nonexpansive retract of C.

1. Introduction

Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. E is said to have the fixed point property with respect to τ (τ-fpp) if the following holds: For each nonempty, norm bounded, τ-compact, convex subset C of E, every nonexpansive mapping $T : C \to C$ (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) has a fixed point. We say that a nonempty closed and convex subset C of E has the fpp if every nonexpansive mapping $T : C \to C$ has a fixed point, and also C is said to have the τ-fpp, if each nonempty, norm bounded, τ-compact, convex subset of C has the fpp. It is not clear that a set having the fpp must be bounded. It is known that the norm boundedness assumption for τ-compact, convex sets is redundant (see, for example, [12]). An standard example of such a τ is where τ is the weak topology on E. Another example is where E is a dual Banach space and τ is the weak* topology. Yet another example is when E is $L_1(\mu)$ and τ is the topology *clm* of convergence locally in measure (see, e.g., [18]).

Determining conditions on a Banach space E so that it has the fixed point property has been of considerable interest for many years. Kirk [13] proved that a weakly compact convex subset of a Banach space with weak normal structure has the fpp. It is known that every compact convex subset of a Banach space has normal structure. Kirk's proof of his result also yields that a weak* compact convex subset of a Banach space with weak* normal structure has the fixed point property (see [19]). The condition above that C has normal structure can not be dropped. In fact, Alspach [2] showed that $L_1[0, 1]$ fails the weak fpp.

We say that E has the (common) τ-fpp for commutative semigroups if whenever $\mathcal{S} = \{T_s : s \in S\}$ is a commutative semigroup of nonexpansive self-mappings on a nonempty, τ-compact, convex subset of E, then the common fixed point set of $\mathcal{S}$, $Fix(\mathcal{S})$, is nonempty. Bruck [4] showed that a Banach space E having the weak fpp has the weak fpp for commutative semigroups. We refer to [23] for a simple proof

2010 *Mathematics Subject Classification*. Primary 47H20; Secondary 47H10
Keywords. Retraction, nonexpansive mapping, *clm*-topology, τ-fixed point property, τ-Opial condition
Received: 07 April 2015; Accepted: 03 March 2016
Communicated by Dragan S. Djordjević
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to a more general version of Bruck's result. For a dual Banach space E satisfying the weak* fpp, it is still unknown whether E has the weak* fpp for commutative semigroups. Very recently, Borzdynski and Wisnicki [3] proved that if $\mathcal{S}$ is a commuting family of weak* continuous nonexpansive mappings acting on a weak* compact convex subset C of the dual Banach space E, then the set of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of C. This partially answers a long-standing open problem posed by Lau in [15] (see also [17]). Examples of Banach spaces with the weak* fpp for commutative semigroups include ℓ_1, trace class operators on a Hilbert space, Hardy space H^1 and the Fourier algebra of a compact group (see [16, 19–22]).

In this paper, by using the retraction tool, we study the common fixed point property for commutative nonexpansive mappings on τ-compact convex sets in E, where τ is a Hausdorff topological vector space topology that is weaker than the norm topology. In Section 2, we shall prove the following: Let E be a Banach space, τ a Hausdorff topological vector space topology on E that is weaker than the norm topology and the norm of E is lsc with respect to τ, and C be a nonempty, τ-compact, separable, convex subset of E which has the τ-fpp. Then any commutative family of nonexpansive self-mappings of C has a common fixed point and the set of common fixed points is a nonexpansive retract of C. In Section 4, we obtain the same result by replacing the separability with the τ-Opial condition. As a consequence, we shall show that the set of common fixed points of any commutative family of nonexpansive self-mappings of a nonempty *clm*-compact (resp. weak* compact) convex subset C of $L_1(\mu)$ with a σ-finite μ (resp. the James space J_0) is a nonempty nonexpansive retract of C.

2. The τ-fpp for Commutative Mappings on Separable Subsets

Recall some general concepts and definitions. Let E be a normed space and C be a nonempty subset of E. A mapping T on C is said to be a retraction if $T^2 = T$. A subset F of C is called a nonexpansive retract of C if either $F = \emptyset$ or there exists a retraction of C onto F which is a nonexpansive mapping. Nonexpansive retract plays an important role in the study of the structure of fixed point sets of nonexpansive mappings. We refer the reader to [4–7] for more information concerning nonexpansive retracts.

Let E be a normed space, and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. The purpose of this section is to study the τ-fpp (which implies the weak* fpp) for commutative semigroups of nonexpansive mappings. In fact, we give some partial answers to the following question:
If a dual Banach space E has the τ-fpp, does E have the τ-fpp for commuting semigroups?

The following theorem is essential to get the main results.

Theorem 2.1. *Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ. Let C be a nonempty, τ-compact, convex subset of E with the τ-fpp. If $\{T_1, \ldots, T_n\}$ is a commutative family of nonexpansive mappings on C, then $\cap_{i=1}^{n} Fix(T_i)$ is a nonempty nonexpansive retract of C.*

Proof. First, we prove that for each nonexpansive mapping $T : C \to C$, $Fix(T)$ is a nonempty nonexpansive retract of C. To this purpose, consider C^C with the product topology induced by the topology τ on C. Then by Tychonoff's theorem C^C is compact. Now, consider a nonexpansive mapping $T : C \to C$ and define

$$\mathfrak{R} := \{S \in C^C : S \text{ is nonexpansive}, Fix(T) \subset Fix(S)\}.$$

We show that $\mathfrak{R}$ is closed in C^C. Suppose that $\{U_\lambda : \lambda \in \Lambda\}$ is a net in $\mathfrak{R}$ which converges to U in C^C. Then for $z \in Fix(T)$, $U_\lambda(z) = z$ so $U(z) = \tau - \lim_\lambda U_\lambda(z) = z$. By the lower semi-continuity of the norm with respect to τ, for any x, y in C, $\|Ux - Uy\| \leq \liminf_\lambda \|U_\lambda x - U_\lambda(y)\| \leq \|x - y\|$. So we have shown that $U \in \mathfrak{R}$, hence that $\mathfrak{R}$ is closed in C^C. Since C^C is compact, therefore $\mathfrak{R}$ is compact (the topology on $\mathfrak{R}$ is that of τ-pointwise convergence). Define a preorder $\preceq$ in $\mathfrak{R}$ by $S \preceq U$ if $\|Sx - Sy\| \leq \|Ux - Uy\|$ for all $x, y \in C$ and using the Bruck's method [7] we obtain a minimal element $R \in \mathfrak{R}$. Indeed, by considering Zorn's lemma it suffices to show every linearly ordered subset of $\mathfrak{R}$ has a lower bound in $\mathfrak{R}$. If $\{U_\lambda\}$ is a linearly ordered subset of $\mathfrak{R}$ by $\preceq$, the family of sets $\{S \in \mathfrak{R} : S \preceq U_\lambda\}$ is linearly ordered by inclusion. The proof that $\mathfrak{R}$ is closed in C^C can be repeated to show that these sets are closed in $\mathfrak{R}$, and hence compact. So there exists $U \in \bigcap_\lambda \{S \in \mathfrak{R} : S \preceq U_\lambda\}$ with $U \preceq U_\lambda$ for each λ. Now, we have shown the existence of a minimal element $P \in \mathfrak{R}$ in the following sense:

$$\text{if } S \in \mathfrak{R} \text{ and } \|S(x) - S(y)\| \leq \|P(x) - P(y)\|, \forall x, y \in C,$$

$$\text{then } \|S(x) - S(y)\| = \|P(x) - P(y)\|. \quad (*)$$

We shall prove that $P(x) \in Fix(T)$ for all $x \in C$. For a given $x \in C$, consider the set $K = \{S(P(x)) : S \in \mathfrak{R}\}$. Then K is a nonempty τ-compact convex subset of C, because $\mathfrak{R}$ is convex and compact. On the other hand, $TS \in \mathfrak{R}$, $\forall S \in \mathfrak{R}$. Therefore, we have $T(K) \subseteq K$, and then, by the τ-fpp, there exists $h \in \mathfrak{R}$ with $h(P(x)) \in Fix(T)$. Let $y = h(P(x))$. Then $P(y) = h(y) = y$, and by using the minimality of P, we have $\|P(x) - y\| = \|P(x) - P(y)\| = \|h(P(x)) - h(P(y))\| = \|h(P(x)) - y\| = 0$. So $P(x) = y \in Fix(T)$. Since this is so for each $x \in C$ and P belongs to $\mathfrak{R}$, it follows that $P^2 = P$. So, we have shown that $Fix(T)$ is a nonexpansive retract of C. Now, let $\{T_1, \ldots, T_n\}$ be a commuting family of nonexpansive mappings on C. We prove that $\cap_{i=1}^n Fix(T_i)$ is a nonempty nonexpansive retract of C. The proof is by induction n. If $n = 1$, then $Fix(T_1)$ is a nonempty nonexpansive retract of C by the above discussion. Now suppose that $\cap_{j=1}^n Fix(T_j)$ is a nonempty nonexpansive retract of C and $R : C \to \cap_{j=1}^n Fix(T_j)$ a nonexpansive retraction. Then, it is easy to check that $Fix(T_{n+1}R) = \cap_{j=1}^{n+1} Fix(T_j)$ (see, e.g., [4] for the details). Another application of the first part of the proof implies that $Fix(T_{n+1}R) = \cap_{j=1}^{n+1} Fix(T_j)$ is a nonempty nonexpansive retract of C, which completes the induction. □

We will also need the following lemma, due to Bruck [4], as an intermediary step.

Lemma 2.2. *If C is a bounded closed convex subset of a Banach space E and $\{F_n\}$ is a descending sequence of nonempty nonexpansive retracts of C, then $\cap_{n=1}^\infty F_n$ is the fixed point set of some nonexpansive $r : C \to C$.*

Theorem 2.3. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose the norm of E is lsc with respect to τ, and C be a nonempty, τ-compact, separable, convex subset of E which has the τ-fpp. Then any commutative family of nonexpansive self-mappings of C has a common fixed point and the set of common fixed points is a nonexpansive retract of C.*

Proof. Let $\mathcal{S} = \{T_i\}_{i \in I}$ be a commutative family of nonexpansive mappings on C, and let $\mathcal{F}$ be the family of the finite intersections of fixed point sets of mappings in the commutative family $\mathcal{S}$. We have shown, in Theorem 2.1, that $\mathcal{F}$ is a family of nonempty nonexpansive retracts of C, and $\mathcal{F}$ is obviously directed by $\supset$. Now, since C is separable, there is a countable subfamily $\mathcal{F}'$ of $\mathcal{F}$ such that

$$Fix(\mathcal{S}) = \bigcap \{F : F \in \mathcal{F}\} = \bigcap \{F : F \in \mathcal{F}'\}.$$

Using the fact that $\mathcal{F}$ is directed by $\supset$ we can therefore find a descending sequence $\{F_n\}$ in $\mathcal{F}$ with $Fix(\mathcal{S}) = \bigcap_n F_n$. But, by Lemma 2.2, $Fix(\mathcal{S}) = \bigcap_n F_n = Fix(r)$, for some nonexpansive $r : C \to C$. Since C has the τ-fpp, Theorem 2.1 implies $Fix(\mathcal{S})$ is a nonempty nonexpansive retract of C. This completes the proof. □

An standard example of such a pair (E, τ) is where E is a dual Banach space and τ is the weak* topology. Thus, Theorem 2.3 yields the following result:

Theorem 2.4. *Let E be a dual Banach space and suppose C is a nonempty, weak* compact, separable, convex subset of E which has the weak*-fpp. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

Example 2.5. *The James space J_0, ℓ_1 and L_1 over a separable measure space are separable [1]. Moreover, it is well known that in ℓ_1, and in the James space J_0, a nonexpansive self-mapping of a weak* compact convex subset has a fixed point [11]. Thus, by Theorem 2.4, we deduce that for any nonempty, weak* compact, convex subset C of ℓ_1, or the James space J_0, the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

For commutative sequences of nonexpansive self-mappings it is possible to say even more (cf. [24]).

Theorem 2.6. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose the norm of E is lsc with respect to τ, and C be a nonempty, τ-compact, convex subset of E which has the τ-fpp. Then the set of common fixed points of any commutative sequence of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

Proof. Let $\{T_n\}$ be a commutative sequence of nonexpansive mappings on C. Then, by Theorem 2.1, for each natural number n, $F_n = \cap_{j=1}^{n} Fix(T_j)$ is a nonempty nonexpansive retract of C. Thus, applying Lemma 2.2, we deduce that $\cap_{n=1}^{\infty} F_n = \cap_{n=1}^{\infty} Fix(T_n)$ is the fixed point set of some nonexpansive mapping $r : C \to C$. Therefore, by Theorem 2.1, $Fix(r) = \cap_{n=1}^{\infty} Fix(T_n)$ is a nonempty nonexpansive retract of C. □

3. Fixed Point Property with Respect to *clm*-Topology in $L_1(\mu)$

In this section, we will use the topology of convergence in measure which we now recall for the convenience of the reader. Let (Ω, Σ, μ) be a positive σ-finite measure space and $L_0(\mu)$ be the set of all scalar-valued Σ-measurable functions on Ω. The topological vector space topology *clm*, of convergence locally in measure on $L_0(\mu)$, is generated by the following translation-invariant metric: Let $(A_n)_{n=1}^{\infty}$ be a $\widetilde{\Sigma}$-partition of Ω, where $\widetilde{\Sigma} := \{A \in \Sigma : \ \mu(A) \in (0, \infty)\}$. Define d_0 by

$$d_0(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(A_n)} \int_{A_n} \frac{|f - g|}{1 + |f - g|} d\mu, \text{ for all } f, g \in L_0(\mu).$$

If $\mu(\Omega) < \infty$, then the simpler metric

$$d_0(f, g) = \int_{\Omega} \frac{|f - g|}{1 + |f - g|} d\mu, \text{ for all } f, g \in L_0(\mu),$$

generates the *clm* topology. In this case, we simply refer to *clm* as the topology of convergence in measure, denoted by *cm*. $L_0(\mu)$ is complete with respect to the above metric. For sequences, the *clm*-topology reduces, in a sense, to almost everywhere convergence. Indeed, any sequence in $L_0(\mu)$ that converges almost everywhere to $f \in L_0(\mu)$ must converge to f locally in measure. On the other hand, every *clm*-convergent sequence of scalar-valued measurable functions has a subsequence that converges almost everywhere to the same limit function. Note that when we discuss $L_1(\mu)$, *clm* or *cm* will denote the topologies introduced above, restricted to $L_1(\mu)$. Further, the $L_1(\mu)$-norm is *clm*-lower semicontinuous. This follows from Fatou's lemma and the fact that *clm* is a metric topology. Thus, an example of the pair (E, τ) in Section 2 is where E is $L_1(\mu)$ and τ is the topology *clm* of convergence locally in measure.

We remark that one can show that in every $L_1(\mu)$, μ σ-finite, *clm*-compact sets must be norm separable. Besides, Lami Dozo and Turpin [14] showed that $L_1(\mu)$ has the fpp with respect to the topology *clm* of convergence locally in measure. Combining the above discussion with Theorem 2.3 yields the following result:

Theorem 3.1. *Let C be a nonempty, clm-compact, convex subset of $L_1(\mu)$, where μ is σ-finite. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

In the particular case that μ is finite, Theorem 3.1 is a consequence of a result in [22] (see also [21]).

4. The τ-fpp for Commutative Mappings Under the τ-Opial Condition

In this section, we study the common fpp for commutative semigroups by considering conditions under which the fixed point sets of nonexpansive mappings are τ-closed. For our purposes it will be convenient to prove the following result:

Theorem 4.1. *Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ. Let C be a nonempty, τ-compact, convex subset of E with the τ-fpp. Then any commutative family of nonexpansive self-mappings of C such that their fixed point sets are τ-closed has a common fixed point and the set of common fixed points is a nonexpansive retract of C.*

Proof. Let $\mathcal{S} = \{T_i\}_{i\in I}$ be a commutative family of nonexpansive mappings on C, and let $\mathcal{F}$ be the family of the finite intersections of fixed point sets of mappings in the commutative family $\mathcal{S}$. Since, by assumption, the fixed point sets are τ-closed, Theorem 2.1 implies that $\mathcal{F}$ is a family of nonempty τ-compact subsets of C that is directed by $\supset$. Hence,

$$Fix(\mathcal{S}) = \bigcap\{F : F \in \mathcal{F}\} \neq \emptyset.$$

Now, defining

$$\mathfrak{R} := \{T \in C^C : T \text{ is nonexpansive}, Fix(\mathcal{S}) \subset Fix(T)\},$$

and using an argument quite similar to the one used in the proof of Theorem 2.1, it is easy to show that $\mathfrak{R}$ contains a nonexpansive retraction P from C onto $Fix(\mathcal{S})$. □

Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. We say that a nonempty set $C \subset E$ satisfies the τ-Opial condition if whenever the bounded sequence $\{x_n\}$ of elements of C converges to $x \in C$, with respect to τ, we have

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|,$$

for $y \in C \setminus \{x\}$. We say that C satisfies the τ-Opial condition for nets, if whenever the bounded net $\{x_\alpha\}_{\alpha\in I}$ of elements of C converges to $x \in C$, with respect to τ, we have

$$\liminf_{\alpha\in I} \|x_\alpha - x\| < \liminf_{\alpha\in I} \|x_\alpha - y\|,$$

for $y \in C \setminus \{x\}$.

Spaces with the weak Opial condition have weak normal structure [10] and hence the weak fpp. Similarly dual spaces with the weak* Opial condition can contain no nontrivial separable weak* sequentially compact convex diameteral sets; in particular, separable duals with the weak* Opial condition have weak* normal structure [18], and hence the weak* fpp. Using the following argument, it is possible to say more:

Suppose T is a nonexpansive self mapping of a nonempty bounded closed convex subset C of a Banach space E. It is well known that T admits an approximate fixed point sequence; that is, a sequence (x_n) in C with $\|Tx_n - x_n\| \to 0$. Now suppose C is, in addition, τ-compact, where τ is a Hausdorff topological vector space topology on E that is weaker than the norm topology. If the norm of E satisfies τ-Opial condition for nets (for sequences), then the fixed point set of T is nonempty and τ-compact (provided C is metrizable with respect to τ). In fact, if (x_i) is net (sequence) in C such that converges to x, with respect to τ, and $\|Tx_i - x_i\| \to 0$, then

$$\limsup_i \|Tx - x_i\| = \liminf_i \|Tx - Tx_i\| \leq \liminf_i \|x - x_i\|,$$

contradicting the τ-Opial condition unless $Tx = x$. Moreover, if $\{z_i\}$ is a net (sequence) in $Fix(T)$ converging in τ to some $z \in C$, then

$$\liminf_i \|z_i - Tz\| = \liminf_i \|Tz_i - Tz\| \leq \liminf_i \|z_i - z\|,$$

contradicting the τ-Opial condition for nets (for sequences) unless $Tz = z$. This implies that if either the norm of E satisfies τ-Opial condition for nets, or the norm of E satisfies τ-Opial condition for sequences and C is metrizable with respect to τ, then $Fix(T)$ is nonempty and τ-closed. Thus, Banach spaces with the τ-Opial condition for nets (for sequences) have the τ-fpp (provided τ is metrizable), and in this case the fixed point sets of nonexpansive mappings are τ-compact.

Combining the above facts with Theorem 4.1, we obtain the following results:

Theorem 4.2. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ and satisfies τ-Opial condition. Let C be a nonempty, τ-compact, convex subset of E and let C be metrizable with respect to τ. Then any commuting family of nonexpansive self-mappings of C has a common fixed point and the set of common fixed points is a nonexpansive retract of C.*

Let (Ω, Σ, μ) be a finite measure space. It follows from Proposition 5.2 in [21] (see also Lemma 2.6 in [22]) that $L_1(\mu)$ satisfies *cm*-Opial condition. Further, *cm*-topology is metrizable and the norm of $L_1(\mu)$ is lsc with respect to the topology of convergence in measure. Thus, $L_1(\mu)$ is an example that satisfies the assumptions of Theorem 4.2.

Theorem 4.3. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ and satisfies τ-Opial condition for nets. Let C be a nonempty, τ-compact, convex subset of E. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

Corollary 4.4. *Let C be a nonempty, weak* compact and convex subset of a dual Banach space E. Suppose that C satisfies weak*-Opial condition for nets. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

Let E be a Banach space and let Γ be a nonempty subspace of its dual E^*. If

$$\sup\{x^*(x) : \ x^* \in \Gamma, \ \|x^*\| = 1\} = \|x\|,$$

for each $x \in E$, then we say that Γ is a norming set for E. It is obvious that a norming set generates a Hausdorff linear topology $\sigma(E, \Gamma)$ which is weaker than the weak topology $\sigma(E, E^*)$. It is worth noting here that $n(E) \subseteq E^{**}$ is a norming set for E^*, where n is a natural embedding of E into E^{**}, and hence, for $\Gamma = n(E)$, $\sigma(E, \Gamma)$ is the weak* topology on E^*. Throughout, Γ denotes a norming set for E. It is easy to observe that the norm of E is lower semicontinuous with respect to the $\sigma(E, \Gamma)$-topology [8]. It is shown in [8] that if E is a Banach space, Γ is a norming set for E and C is a nonempty, bounded and Γ-sequentially compact subset of E, then in C the Γ-Opial condition for nets is equivalent to the Γ-Opial condition. Thus, we obtain the following:

Corollary 4.5. *Let E be a Banach space, Γ be a norming set for E, C be a nonempty, Γ-compact and Γ-sequentially compact convex subset of E. Suppose that C satisfies Γ-Opial condition. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

Corollary 4.6. *Let E be a dual Banach space with a separable predual space, C be a nonempty, weak* compact and convex subset of E. Suppose that C satisfies weak*-Opial condition. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C.*

Finally, it is worth mentioning that some of the well-known classical dual Banach spaces satisfy weak*-Opial condition.

Example 4.7. *The following dual Banach spaces satisfy the weak*-Opial condition for nets:*
(i) ℓ_1;
(ii) the James space J_0;
(iii) $B(G)$, the Fourier-Stieltjes algebra of a compact group G;
see [1, 9, 16, 19], for details. Hence, by Corollary 4.4 (or 4.6), the common fixed point set of any commutative family of nonexpansive self-mappings of a nonempty weak compact convex subset C in the above spaces is a nonempty nonexpansive retract of C.*

Problem: Can Theorems 4.2 and 4.3 be extended to left reversible or amenable semigroups?

Acknowledgments

The authors would like to thank Professor Anthony To-Ming Lau, for his helpful suggestions and interest. The authors are also very grateful to the anonymous referee.

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