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A Class of Drazin Inverses in Rings

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Abstract. A strongly direct sum decomposition given by a strongly nil-clean endomorphism motivates us to introduce a class of new Drazin inverses, which is called strong Drazin inverse. In this paper, some basic properties of the strong Drazin inverse are obtained. We show the Cline's Formula and Jacobson's Lemma for the strong Drazin inverse. Further, some additive results for the strong Drazin inverse are presented under additional conditions that $ab = 0$ or $ab = ba$.

1. Introduction

Throughout this paper, R denotes an associative ring with identity. $\mathrm{N}(R)$ will denote the set of all nilpotent elements in R. For any element $a \in R$, we define the commutant and double commutant of a by $\mathrm{comm}(a) = \{x \in R : ax = xa\}$, $\mathrm{comm}^2(a) = \{x \in R : xy = yx$ for any $y \in \mathrm{comm}(a)\}$, respectively. Other standard notations in ring theory follow those in [18].

An element a in a ring R is called strongly π-regular if $a^n \in Ra^{n+1} \cap a^{n+1}R$ for some $n \geq 1$. The strongly π-regularity plays an important role in ring theory and C^*-algebras (see [1, 4, 12, 15]). Azumaya [3] proved that $a \in R$ is strongly π-regular if and only if there exists $x \in R$ such that $ax = xa$ and $a^n = a^{n+1}x$ for some $n \geq 1$. Further, we may choose x here to satisfy $x^2a = x$. As it turned out, the notion of a strongly π-regular element is identical to the notion of a Drazin invertible element, which was first introduced by M. P. Drazin [11] in 1958. The theory of Drazin inverses is very useful in linear algebra, functional analysis, matrix computations, differential equations and in various applications of matrices (see [6, 20, 25, 27, 28]).

Given a right R-module M_R, and let $\varphi \in \mathrm{End}(M_R)$. Then φ is Drazin invertible if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $\varphi|_A \in \mathrm{End}(A_R)$ is an isomorphism and $\varphi|_B \in \mathrm{End}(B_R)$ is nilpotent [21]. Moreover, Disel [10] introduced the notion of a strongly nil-clean element and gave a characterization of the strongly nil-clean endomorphism as follows: $\varphi \in \mathrm{End}(M_R)$ is strongly nil-clean if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $(1 - \varphi)|_A \in \mathrm{End}(A_R)$ and $\varphi|_B \in \mathrm{End}(B_R)$ are both nilpotent. By straightforward observing of the direct sum decompositions, the strong nil-cleanness is Drazin invertible. This motivates us to introduce a class of new Drazin inverses corresponding to the strong nil-cleanness. An element a in a ring R is said to be strongly Drazin invertible (s-Drazin invertible, for short) if there exists

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an element $x \in R$ such that $x^2a = x,\ ax = xa$ and $a - ax$ is nilpotent. Here, x is unique (denoted by a'), and is called a strong Drazin inverse for $a \in R$.

In Section 3, we show Cline's Formula and Jacobson's Lemma for the s-Drazin inverse. Further, some additive results for the strong Drazin inverse are presented under additional conditions in Section 4. For example, we prove that if $a,\ b$ are s-Drazin invertible and $ab = 0$, then $a + b$ is s-Drazin invertible and the s-Drazin inverse can be expressed by a, b, a' and b'. Moreover, let $a,\ b \in R$ be s-Drazin invertible and $ab = ba$. Then $a + b$ is Drazin invertible if and only if $1 + a'b$ is Drazin invertible.

2. Strong Drazin Inverses in Rings

In the ring-theoretic setting, the notion of a Drazin invertible element is identical to the notion of a strongly π-regular element by Azumaya's Lemma [3]. The following important result about strongly π-regular endomorphisms is proved in [2, 21].

Lemma 2.1. *Let R be a ring, and let M_R be a right R-module. Then the following are equivalent:*
(1) $\varphi \in \mathrm{End}(M_R)$ is Drazin invertible.
(2) $\varphi \in \mathrm{End}(M_R)$is strongly π-regular.
(3) $M = \mathrm{Im}(\varphi^n) \oplus \mathrm{Ker}(\varphi^n)$ for some $n \geq 1$.
(4) there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $\varphi|_A \in \mathrm{End}(A_R)$ is an isomorphism and $\varphi|_B \in \mathrm{End}(B_R)$ is nilpotent.

We express Lemma 2.1(4) with the following diagram:

$$\begin{tikzcd} M = & A \arrow[d, "\varphi|_A"', "\cong"] & \bigoplus & B \arrow[d, "\varphi|_B"', "nilpotent"] \\ M = & A & \bigoplus & B \end{tikzcd}$$

An element a in a ring R is said to be **strongly nil-clean** if there exits an idempotent $p \in \mathrm{comm}(a)$ such that $a - p$ is nilpotent. Replacing the automorphism $\varphi|_A$ as above by a nilpotent endomorphism $(1 - \varphi)|_A$, A.J. Diesl [10] gave a characterization of the strongly nil-clean endomorphism.

Lemma 2.2. *Let R be a ring, and let M_R be a right R-module. Then the following are equivalent:*
(1) $\varphi \in \mathrm{End}(M_R)$ is strongly nil-clean.
(2) there exists $\chi \in \mathrm{End}(M_R)$ such that $\chi^2\varphi = \chi,\ \varphi\chi = \chi\varphi$ and $\varphi - \varphi\chi$ is nilpotent.
(3) $M = \mathrm{Ker}((1 - \varphi)^n) \oplus \mathrm{Ker}(\varphi^n)$ for some $n \geq 1$.
(4) there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $(1 - \varphi)|_A \in \mathrm{End}(A_R)$ and $\varphi|_B \in \mathrm{End}(B_R)$ are both nilpotent.
In this case, φ is also Drazin invertible in $\mathrm{End}(M_R)$.

Before giving the proof of Lemma 2.2, we first include a diagram corresponding to Lemma 2.2(4) which may be compared with the diagram following Lemma 2.1.

$$\begin{tikzcd} M = & A \arrow[d, "(1-\varphi)|_A"', "nilpotent"] & \bigoplus & B \arrow[d, "\varphi|_B"', "nilpotent"] \\ M = & A & \bigoplus & B \end{tikzcd}$$

Proof of Lemma 2.2. (1) $\Leftrightarrow$ (4) is due to A.J. Diesl (see [10]).

(1) $\Rightarrow$ (2). Suppose that φ is strongly nil-clean. Then there exists $\rho^2 = \rho \in \mathrm{End}(M_R)$ such that $\varphi\rho = \rho\varphi$ and $\varphi - \rho$ is nilpotent. Set

$$\chi = (\varphi + 1 - \rho)^{-1}\rho.$$

The hypothesis $\varphi\rho = \rho\varphi$ implies that φ, ρ and $(\varphi + 1 - \rho)^{-1}$ commute with each other. Then, in terms of $\rho^2 = \rho$,

$$\varphi\chi = \chi\varphi = (\varphi + 1 - \rho)^{-1}\rho(\varphi + 1 - \rho) = \rho.$$

Hence, we have $\chi^2\varphi = \chi(\varphi\chi) = \chi\rho = \chi$ and $\varphi - \varphi\chi = \varphi - \rho$ is nilpotent.

(2) ⇒ (3). Suppose that $(\varphi - \varphi\chi)^n = 0$ for some $n \geq 1$, and let

$$(1 - \varphi)^n = 1 - a_1\varphi - \cdots - a_n\varphi^n,$$

where $a_i \in R$ for $i = 1, 2, \cdots, n$. Let $m \in \mathrm{Ker}((1 - \varphi)^n) \cap \mathrm{Ker}(\varphi^n)$. Then $(1 - \varphi)^n(m) = 0 = \varphi^n(m)$. This implies that

$$m = a_1\varphi(m) + \cdots + a_{n-1}\varphi^{n-1}(m).$$

Again by $\varphi^n(m) = 0$, we have $\varphi^{n-1}(m) = a_1\varphi^n(m) + \cdots + a_{n-1}\varphi^{2n-2}(m) = 0$. Similarly, we can get $\varphi^i(m) = 0$, $(i = n - 2, \cdots, 2, 1)$, and so $m = 0$.

Next, we will show that, for any $m \in M$, $m = \varphi\chi(m) + (1 - \varphi\chi)(m)$ where $\varphi\chi(m) \in \mathrm{Ker}((1 - \varphi)^n)$ and $(1 - \varphi\chi)(m) \in \mathrm{Ker}(\varphi^n)$. Note that $\varphi\chi = \chi\varphi$, $(\varphi - \varphi\chi)^n = 0$ and $\varphi\chi$ is an idempotent endomorphism. Then

$$(1 - \varphi)^n\varphi\chi(m) = (\varphi\chi - \varphi^2\chi)^n(m) = (\varphi^2\chi^2 - \varphi^2\chi)^n(m) = (\varphi\chi - \varphi)^n\varphi\chi(m) = 0,$$

and

$$\varphi^n(1 - \varphi\chi)(m) = [(\varphi - \varphi\chi)(1 - \varphi\chi)]^n(m) = (\varphi - \varphi\chi)^n(1 - \varphi\chi)(m) = 0,$$

as required.

(3) ⇒ (1) is completed easily by taking $A = \mathrm{Ker}((1 - \varphi)^n)$ and $B = \mathrm{Ker}(\varphi^n)$.

Finally, note that $(1 - \varphi)|_A$ being nilpotent implies $\varphi|_A$ to be isomorphic on A. Comparing the diagrams corresponding Lemma 2.1(4) and Lemma 2.2(4), one can give a straightforward proof for the remaining part. □

Definition 2.3. *An element a in a ring R is said to be* **strongly Drazin invertible** *(s-Drazin invertible, for short) if there exists an element $x \in R$ such that*

$$x^2a = x, \ ax = xa \text{ and } a - ax \text{ is nilpotent}.$$

In this case, x is called a **strong Drazin inverse** *of a. The set of all strongly Drazin invertible elements in R will be denoted by R^{sD}.*

Next, we will prove the s-Drazin inverse to be unique. The initial idea of the following result was given in [11] by M. P. Drazin.

Theorem 2.4. *Let R be a ring and $a \in R$. Then a has at most one s-Drazin inverse in R, and, if the s-Drazin inverse of a exists, it commutes with every element of R which commutes with a.*

Proof. Let x be the s-Drazin inverses of a. Note that $ax = xa$ and $x^2a = x$. Then we have $(ax)^2 = ax$. Since $a - ax$ is nilpotent, it follows that $a(1 - ax) = (a - ax)(1 - ax)$ is also nilpotent. So, x is the Drazin inverse of a. Since the Drazin inverse of a is unique and double commutes with a, we conclude that the s-Drazin inverse x of a is unique and double commutes with a. □

Given any s-Drazin invertible element a of a ring R. We will denote the s-Drazin inverse of a by $a' \in R$. The least positive integer n for $(a - aa')^n = 0$ is called the **strong Drazin index** of a, denoted by $\mathrm{ind}(a)$.

Example 2.5. *(1) All nilpotent elements and idempotent elements of a ring R are s-Drazin invertible. In fact, let $e^2 = e \in R$ and let $a^n = 0$ for $n \geq 1$. It is verified directly that $e' = e$ with $\mathrm{ind}(e) = 1$ and $a' = 0$ with $\mathrm{ind}(a) = n$. However, for an integral domain ring R, we can get $R^{sD} = \{0, 1\}$. In particular, 2 is invertible but not s-Drazin invertible in complex number field $\mathbb{C}$.*

(2) Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Then every element of the 2×2 upper triangular matrix ring $\mathrm{T}_2(\mathbb{Z}_2)$ is s-Drazin invertible. We will show a general result about the s-Drazin invertiblity of the supper triangular matrix in the Section 4 (see Lemma 4.1).

(3) Let $\mathrm{M}_n(\mathbb{C})$ be a $n \times n$ matrix ring over the complex number field $\mathbb{C}$. Note that $\mathrm{M}_n(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^n)$. Then, by Lemma 2.2, $A \in \mathrm{M}_n(\mathbb{C})$ is s-Drazin invertible if and only if there exists a direct sum decomposition $\mathbb{C}^n = V_1 \oplus V_2$ such that V_1 and V_2 are A-invariant and such that $A|_{V_1}$ and $(I - A)|_{V_2}$ are both nilpotent if and only if eigenvalues of A are only 0 or 1 if and only if A is similar to the Jordan matrix $J = diag(J_1, J_2, \cdots, J_s)$, where $J_1, \cdots, J_t$ and $I_{t+1} - J_{t+1}, \cdots, I_s - J_s$ are all nilpotent Jordan block for $1 \le t \le s$. In this case, there exists an invertible matrix P such that the s-Drazin inverse $A' = P^{-1} diag(O, \cdots, O, J_{t+1}^{-1}, \cdots, J_s^{-1})P$. More examples of s-Drazin invertible matrices over some special rings can be found in [7].

A generalized Drazin inverse was defined and investigated by Koliha [16] in complex Banach algebra. Koliha and Patrício [17] studied further quasipolar elements and generalized Drazin inverses of rings. We show the relations of various type of inverses by the following diagram. As we see in the Example 2.5(1), invertible elements or group invertible elements may not be s-Drazin invertible in general.

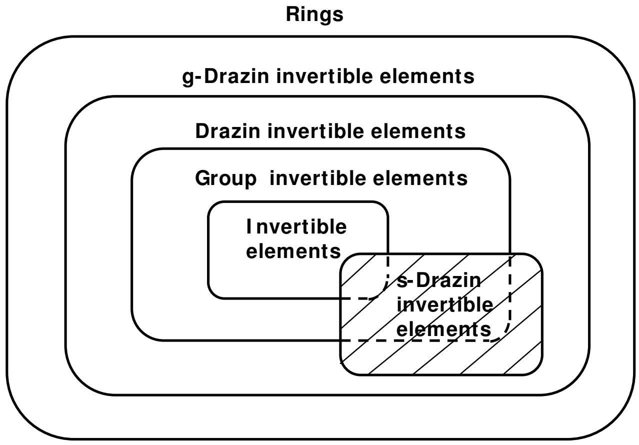

Fig. 1. Various types of inverses

Proposition 2.6. *Let R be a ring and let $a \in R$. If a is s-Drazin invertible, then a^n is s-Drazin invertible for $n \ge 2$. In this case, $(a^n)' = (a')^n$ and $\mathrm{ind}(a^n) \le \mathrm{ind}(a)$.*

Proof. Suppose that a is s-Drazin invertible. Note that

$$a^n - a^n(a')^n = (a - aa')[a^{n-1} + \cdots + (aa')^{n-1}].$$

Then we can check directly that $(a')^n$ is a s-Drazin inverse of a^n and $\mathrm{ind}(a^n) \le \mathrm{ind}(a)$. □

Proposition 2.7. *Let $a \in R$ be s-Drazin invertible. Then the s-Drazin inverse a' is also s-Drazin invertible. In this case, $(a')' = a^2a'$ and $\mathrm{ind}(a') \le \mathrm{ind}(a)$.*

Proof. Let a' be a s-Drazin inverse of a. Set $x = a^2a'$. It follows that $a'x = xa'$ from $aa' = a'a$. Note that $a - aa'$ is nilpotent and $(a')^2a = a'$. Then we get $a' - a'x = a' - (a')^2a^2 = (a')^2a - a'a = -a'(a - aa')$ is nilpotent, and $x^2a' = a^4(a')^3 = a^2a' = x$. Thus, we prove that a' is s-Drazin invertible and $(a')' = x$. Suppose that $\mathrm{ind}(a) = n$. Then $(a' - a'x)^n = (-a')^n(a - aa')^n = 0$. This shows that $\mathrm{ind}(a') \le n$, as required. □

Remark 2.8. *Let $a \in R$ be s-Drazin invertible. Then a^n $(n > 1)$ and a' are both s-Drazin invertible. However, $\mathrm{ind}(a^n)$ or $\mathrm{ind}(a')$ need not be same as $\mathrm{ind}(a)$ in general. Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{C})$. Note that $A^2 = O$. Then A is s-Drazin invertible with $A' = O$ and $\mathrm{ind}(A) = 2$. But $\mathrm{ind}(A^2) = \mathrm{ind}(A') = 1$.*

Proposition 2.9. *Let R be a ring with involution. Then a is s-Drazin invertible if and only if a^* is s-Drazin invertible. In this case, $(a^*)' = (a')^*$ and $\mathrm{ind}(a) = \mathrm{ind}(a^*)$.*

Proof. Suppose that a is s-Drazin invertible with $\mathrm{ind}(a) = n$. Then we have $a^* - a^*(a')^* \in \mathrm{N}(R)$, $a^*(a')^* = (a')^*a^*$ and $[(a')^*]^2a^* = (a')^*$ by applying the properties of the involution. Here, $\mathrm{ind}(a) = \mathrm{ind}(a^*)$ is clear since $(x^*)^* = x$ for any $x \in R$. □

3. Cline's Formula and Jacobson's Lemma for the s-Drazin inverse

Cline [9] proved that if ab is Drazin invertible, then so is ba, and $(ba)^D = b((ab)^D)^2a$. This equality is called **Cline's formula**. It provides a technique to present the Drazin inverse of the sum of two elements (see [14, 23, 26, 29]). In this section, we also show Cline's formula for the s-Drazin inverse.

Theorem 3.1. *Let $a, b \in R$. If ab is s-Drazin invertible in R with $\mathrm{ind}(ab) = n$, then ba is also s-Drazin invertible. In this case, $(ba)' = b[(ab)']^2a$ and $\mathrm{ind}(ba) \leq n + 1$.*

Proof. Set $\alpha = ab$ and $\beta = ba$. Then we have $a\beta = \alpha a$ and $\beta b = b\alpha$. Suppose that there exists $\alpha' \in R$ such that $\alpha(1 - \alpha') \in \mathrm{N}(R)$, $\alpha\alpha' = \alpha'\alpha$ and $(\alpha')^2\alpha = \alpha'$. Clearly, α' is also a Drazin inverse of α. Take $x = b(\alpha')^2a$. By [9], we know that x is a Drazin inverse of β. This implies that $\beta x = x\beta$ and $x^2\beta = x$. So it is sufficient to show $\beta - \beta x$ is nilpotent.

Observing that
$$\beta - \beta x = \beta - (\beta b)(\alpha')^2a = \beta - b\alpha(\alpha')^2a = \beta - b\alpha'a = b(1 - \alpha')a,$$
we derive
$$[b(1 - \alpha')a]^{n+1} = b(1 - \alpha')(\alpha - \alpha\alpha')^n a = 0.$$
Thus, $\beta - \beta x$ is nilpotent. This shows that β is s-Drazin invertible and $\beta' = b(\alpha')^2a$ with $\mathrm{ind}(\beta) \leq n + 1$. □

Using the technique of block matrices, we obtain the following result.

Proposition 3.2. *Let $A \in M_{m\times n}(R)$ and $B \in M_{n\times m}(R)$. If AB has a s-Drazin inverse in $M_m(R)$, then so does BA in $M_n(R)$ and $(BA)' = B[(AB)']^2A$.*

In 2009, Patrício and Costa [22] asked if the Drazin invertibility of $1 - ab$ implies that of $1 - ba$. Patrício-Hartwig [24] and Cvetković-Ilić-Harte [8] answered independently this question affirmatively. Lam and Nielsen [19] investigated further Jacobson's Lemma for Drazin inverses, π-regular elements and unit π-regular elements in arbitrary rings. Now, we can prove that Jacobson's Lemma also holds for the s-Drazin inverse.

Lemma 3.3. *If a is s-Drazin invertible in R with $\mathrm{ind}(a) = n$, then $1 - a$ is also s-Drazin invertible. In this case, $(1 - a)' = \Sigma_{i=0}^{n-1} a^i(1 - aa')$ and $\mathrm{ind}(1 - a) = n$.*

Proof. Suppose that a is s-Drazin invertible in R with the s-Drazin inverse a'. Since $a - aa'$ is nilpotent, we have $1 - a + aa' = 1 - (a - aa')$ is a unit in R. Set
$$x = (1 - a + aa')^{-1}(1 - aa').$$
Note that aa' is an idempotent. Then
$$(1 - a)(1 - aa') = (1 - a + aa')(1 - aa').$$
Thus, we have
$$\begin{aligned}(1 - a) - (1 - a)x &= (1 - a) - (1 - a + aa')(1 - a + aa')^{-1}(1 - aa')\\ &= aa' - a \in \mathrm{N}(R).\end{aligned}$$

We can show by a routine way that $x^2(1-a) = x$ and $(1-a)x = x(1-a)$. This shows that x is a s-Drazin inverse of the element $1-a$ and $\mathrm{ind}(1-a) \le \mathrm{ind}(a)$. Similarly, $\mathrm{ind}(a) \le \mathrm{ind}(1-a)$ since $a = 1-(1-a)$. Thus, we get $\mathrm{ind}(1-a) = n$. Next, we will show that $x = \Sigma_{i=0}^{n-1} a^i(1-aa')$. From $(a-aa')^n = 0$, we know

$$(1-a+aa')^{-1} = \Sigma_{i=0}^{n-1}(a-aa')^i.$$

Note that $aa' = a'a$ and $1-aa'$ is an idempotent. Then

$$\begin{aligned} x &= (1-a+aa')^{-1}(1-aa') \\ &= \Sigma_{i=0}^{n-1}(a-aa')^i(1-aa') \\ &= \Sigma_{i=0}^{n-1}a^i(1-aa'), \end{aligned}$$

as required. □

Theorem 3.4. *Let $a, b \in R$. If $1-ab$ is s-Drazin invertible with $\mathrm{ind}(1-ab) = n$, then so is $1-ba$ and the s-Drazin inverse*

$$(1-ba)' = \Sigma_{i=0}^{n}(ba)^i - b[\Sigma_{i=0}^{n}\Sigma_{j=0}^{n-1}(ab)^i(1-ab)^j][1-(1-ab)(1-ab)']a.$$

Proof. Note that $ab = 1-(1-ab)$. Then one can show that, by Lemma 3.3, ab is s-Drazin invertible in R and

$$(ab)' = [\Sigma_{j=0}^{n-1}(1-ab)^j][1-(1-ab)(1-ab)'].$$

So ba is also s-Drazin invertible and $(ba)' = b[(ab)']^2a$ with $\mathrm{ind}(ba) \le n+1$ by Theorem 3.1. Again by Lemma 3.3, we can prove that $1-ba$ is s-Drazin invertible and

$$\begin{aligned} (1-ba)' &= [\Sigma_{i=0}^{n}(ba)^i][1-(ba)(ba)'] \\ &= [\Sigma_{i=0}^{n}(ba)^i][1-(ba)b[(ab)']^2a] \\ &= [\Sigma_{i=0}^{n}(ba)^i][1-b(ab)[(ab)']^2a] \\ &= [\Sigma_{i=0}^{n}(ba)^i][1-b(ab)'a] \\ &= [\Sigma_{i=0}^{n}(ba)^i][1-b[\Sigma_{j=0}^{n-1}(1-ab)^j][1-(1-ab)(1-ab)']a] \\ &= \Sigma_{i=0}^{n}(ba)^i - b[\Sigma_{i=0}^{n}\Sigma_{j=0}^{n-1}(ab)^i(1-ab)^j][1-(1-ab)(1-ab)']a. \end{aligned}$$

The proof is completed. □

4. Additive Results for s-Drazin Inverse

The problem of Drazin inverse of the sum of two Drazin invertible elements was first considered by Drazin in [11], it was proved that $(a+b)^D = a^D + b^D$ under the condition that $ab = ba = 0$ in associative rings. Recently, the representations of the Drazin inverse have been studied by many authors. Hartwig, Wang and Wei [14] presented the formula for $(a+b)^D$ when $ab = 0$. Zhuang, Chen et al [30] showed that, under the condition $ab = ba$, $a+b$ is Drazin invertible if and only if $1+a^Db$ is Drazin invertible. More additive results for Drazin inverse can be found in [23, 26, 27, 29]. We will present some additive results for s-Drazin inverse under the conditions that $ab = 0$ or $ab = ba$.

Lemma 4.1. *Let R be a ring. If a and b are both s-Drazin invertible with $\mathrm{ind}(a) = r$ and $\mathrm{ind}(b) = s$, then $A = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}$ is s-Drazin invertible in $M_2(R)$ and the s-Drazin inverse*

$$A' = \begin{pmatrix} b' & z \\ 0 & a' \end{pmatrix},$$

where $z = (b')^2[\Sigma_{i=0}^{r-1}(b')^ica^i](1-aa') + (1-bb')[\Sigma_{i=0}^{s-1}b^ic(a')^i](a')^2 - b'ca'$.

Proof. Suppose that a and b have s-Drazin inverses a' and b' in R, respectively. By Lemma 2.2, a' and b' are also Drazin inverses for a and b, respectively. This implies that $X = \begin{pmatrix} b' & z \\ 0 & a' \end{pmatrix}$ is a Drazin inverse of A by [13], where $z = (b')^2[\Sigma_{i=0}^{r-1}(b')^i ca^i](1-aa') + (1-bb')[\Sigma_{i=0}^{s-1} b^i c(a')^i](a')^2 - b'ca'$. Thus, we have $AX = XA$ and $X^2A = X$. It is sufficient to show that $A - AX$ is nilpotent. Note that $a - aa'$ and $b - bb'$ are both nilpotent in R. Then $A - AX = \begin{pmatrix} b-bb' & c-bz-ca' \\ 0 & a-aa' \end{pmatrix}$ is also nilpotent, as required. □

Theorem 4.2. *Let R be a ring. If $a,\ b$ are both s-Drazin invertible and $ab = 0$, then $a + b$ is s-Drazin invertible and the s-Drazin inverse*

$$(a+b)' = (1-bb')[\Sigma_{i=0}^{s-1} b^i (a')^{i+1}] + [\Sigma_{i=0}^{r-1}(b')^{i+1} a^i](1-aa'),$$

where $\mathrm{ind}(a) = r$ *and* $\mathrm{ind}(b) = s$.

Proof. Let $A = \begin{pmatrix} 1 \\ a \end{pmatrix}$ and $B = (b,\ 1)$. Then $AB = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ since $ab = 0$, and $BA = a + b$. Note that a and b are both s-Drazin invertible in R. Thus, by Lemma 4.1, AB is s-Drazin invertible in $M_2(R)$, and $(AB)' = \begin{pmatrix} b' & z \\ 0 & a' \end{pmatrix}$ where $z = (b')^2[\Sigma_{i=0}^{r-1}(b')^i a^i](1-aa') + (1-bb')[\Sigma_{i=0}^{s-1} b^i (a')^i](a')^2 - b'a'$. By Proposition 3.2, $BA = a + b$ has a s-Drazin inverse in R and

$$\begin{aligned}
(a+b)' &= (BA)' = B((AB)')^2 A \\
&= (b,\ 1)\begin{pmatrix} b' & z \\ 0 & a' \end{pmatrix}^2 \begin{pmatrix} 1 \\ a \end{pmatrix} \\
&= b' + bb'za + bza'a + a' \\
&= (1-bb')[\Sigma_{i=0}^{s} b^i (a')^{i+1}] + [\Sigma_{i=0}^{r}(b')^{i+1} a^i](1-aa') \\
&= (1-bb')[\Sigma_{i=0}^{s-1} b^i (a')^{i+1}] + [\Sigma_{i=0}^{r-1}(b')^{i+1} a^i](1-aa').
\end{aligned}$$

The proof is completed. □

The following lemma is well known (also see [30, Lemma 1]).

Lemma 4.3. *Let $a,\ b \in R$ be nilpotent and $ab = ba$. Then $a + b$ is also nilpotent.*

Lemma 4.4. *Let $a,\ b \in R$ be s-Drazin invertible and $ab = ba$. Then*
(1) $a,\ b,\ a'$ and b' commute each other.
(2) ab is s-Drazin invertible and $(ab)' = b'a'$.

Proof. (1). Note that a' and b' are also Drazin inverses for a and b, respectively. Then it follows from [30, Lemma 2(1)].

(2). By [30, Lemma 2(2)], the element ab is Drazin invertible with the Drazin inverse $b'a'$. This implies that $(ab)(b'a') = (b'a')(ab)$ and $(b'a')^2 ab = b'a'$. Next, it is sufficient to show that $ab - abb'a'$ is nilpotent in R. Since $a - aa'$ and $b - bb'$ are both nilpotent, it follows that $ab - abb'a' = b(a - aa') + aa'(b - bb')$ is nilpotent by Lemma 4.3. □

Theorem 4.5. *Let $a,\ b \in R$ be s-Drazin invertible and $ab = ba$. Then $a + b$ is s-Drazin invertible if and only if $1 + a'b$ is s-Drazin invertible. Moreover, we have*

$$(a+b)' = (1+a'b)'a' + b'[1 + (a-aa')b']^{-1}(1-aa'),$$

and

$$(1+a'b)' = 1 - aa' + a^2a'(a+b)'.$$

Proof. Suppose that $1+a'b$ is s-Drazin invertible. Then, by Lemma 4.4(1), a, a', b, b' and $(1+a'b)'$ commute each other. Note that $a-aa'$ is nilpotent. Then $1+(a-aa')b'$ is a unit in R and commutes with a, a', b, b'. Since $(a')^2a = a'$ and $aa' = a'a$, we get $1-aa'$ is an idempotent and $aa'(1-aa') = 0$. Set $x = (1+a'b)'a'+b'[1+(a-aa')b']^{-1}(1-aa')$. Next, we will show that

$$\text{(i) } (a+b)x = x(a+b), \text{ (ii) } x^2(a+b) = x, \text{ (iii) } (a+b)-(a+b)x \text{ is nilpotent.}$$

(i). It follows from Lemma 4.4(1).

(ii). Note that $a'(1-aa') = 0$. Then

$$x^2 = [(1+a'b)']^2(a')^2 + (b')^2[1+(a-aa')b']^{-2}(1-aa').$$

From $(a')^2a = a'$, we have

$$[(1+a'b)']^2(a')^2(a+b) = [(1+a'b)']^2(1+a'b)a' = (1+a'b)'a'.$$

Since $(b')^2b = b'$ and $1-aa'$ is an idempotent, we can calculate directly

$$(*) \qquad (1-aa')(b')^2(a+b) = (1-aa')(1+ab')b' = [1+(a-aa')b'](1-aa')b'.$$

Thus, we can simplify

$$\begin{aligned} x^2(a+b) &= [(1+a'b)']^2(a')^2(a+b) + [1+(a-aa')b']^{-2}(1-aa')(b')^2(a+b) \\ &= (1+a'b)'a' + [1+(a-aa')b']^{-1}(1-aa')b' \\ &= x. \end{aligned}$$

(iii). Note that

$$(1+a'b)'a'(a+b) = (1+a'b)'(a')^2a(a+b) = (1+a'b)'(1+a'b)aa'.$$

And from $(*)$ and $(b')^2b = b'$, we have

$$\begin{aligned} b'[1+(a-aa')b']^{-1}(1-aa')(a+b) &= (b')^2b[1+(a-aa')b']^{-1}(1-aa')(a+b) \\ &= b[1+(a-aa')b']^{-1}[1+(a-aa')b'](1-aa')b' \\ &= (1-aa')bb'. \end{aligned}$$

Hence,

$$\begin{aligned} (a+b)-(a+b)x &= (a+b) - (1+a'b)'(1+a'b)aa' - (1-aa')bb' \\ &= [(1+a'b)-(1+a'b)'(1+a'b)]aa' + (a-aa')(1+a'b) + (b-bb')(1-aa'). \end{aligned}$$

Note that $a-aa'$, $b-bb'$ and $(1+a'b)-(1+a'b)'(1+a'b)$ are all nilpotent. According to Lemma 4.3, it is enough to prove that $(a+b)-(a+b)x$ is nilpotent. This shows that $a+b$ is s-Drazin invertible with the s-Drazin inverse x.

Conversely, suppose that $a+b$ is s-Drazin invertible. Set $a_1 = 1-aa'$ and $b_1 = a'(a+b)$. Then we may rewrite $1+a'b = a_1+b_1$. Note that a_1 is an idempotent and a' is s-Drazin invertible with $(a')' = a^2a'$ by Proposition 2.7. Then by Lemma 4.4(2), we know that b_1 is s-Drazin invertible and

$$(b_1)' = [a'(a+b)]' = (a+b)'a^2a'.$$

Observing

$$1+a_1'b_1 = 1+a_1b_1 = 1.$$

This is enough to show that a_1+b_1 is s-Drazin invertible and

$$(1+a'b)' = (a_1+b_1)' = 1-aa'+a^2a'(a+b)'.$$

□

Corollary 4.6. *Let $a, b \in R$ be s-Drazin invertible with* $\mathrm{ind}(a) = k$*, and let $ab = ba$. If $1 + a'b$ is s-Drazin invertible, then $a + b$ is s-Drazin invertible and*

$$(a + b)' = (1 + a'b)'a' + \Sigma_{i=0}^{k-1}(-a)^i(b')^{i+1}(1 - aa').$$

Proof. It follows that $[(a - aa')b']^k = 0$ from $\mathrm{ind}(a) = k$. This implies that

$$[1 + (a - aa')b']^{-1} = \Sigma_{i=0}^{k-1}(aa' - a)^i(b')^i.$$

Note that $1 - aa'$ is an idempotent. Then $(aa' - a)^i(1 - aa') = (-a)^i(1 - aa')$. Thus, we get

$$\begin{aligned}(a + b)' &= (1 + a'b)'a' + b'[1 + (a - aa')b']^{-1}(1 - aa')\\ &= (1 + a'b)'a' + \Sigma_{i=0}^{k-1}(-a)^i(b')^{i+1}(1 - aa').\end{aligned}$$

The proof is completed. □

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