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Lacunary Statistical Convergence and Strongly Lacunary Summable Functions of Order α

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Abstract. The main purpose of this paper is to introduce and investigate the concepts of lacunary strong summability of order α and lacunary statistical convergence of order α of real-valued functions which are measurable (in the Lebesgue sense) in the interval $(1, \infty)$. Some relations between lacunary statistical convergence of order α and lacunary strong summability of order β are also given.

1. Introduction, Definitions and Preliminaries

The idea of statistical convergence was given by Zygmund [30] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [29] and Fast [12] and then reintroduced independently by Schoenberg [27]. Over the years and under different names, statistical convergence has been discussed in the Theory of Fourier Analysis, Ergodic Theory, Number Theory, Measure Theory, Trigonometric Series, Turnpike Theory and Banach Spaces. Later on it was further investigated from the sequence space viewpoint and linked with summability theory by Alotaibi and Alroqi [1], Çolak [5], Connor [6], Et *et al.* ([9], [10] and [11]), Fridy [14], Gadjiev and Orhan [16], Güngör *et al.* ([17] and [18]), Mohiuddine *et al.* [19], Mursaleen and Alotaibi [20], Nuray [22], Rath and Tripathy [25], Šalát [26], and Srivastava *et al.* ([4], [8], [21], [24] and [28]).

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \tag{1}$$

provided that the limit exists. Here, and in what follows, $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and that

$$\delta\left(\mathbb{E}^c\right) = 1 - \delta\left(\mathbb{E}\right).$$

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A sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if, for every $\varepsilon > 0$, we have

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geqq \varepsilon\}) = 0.$$

In this case, we write

$$x_k \xrightarrow{\text{stat}} L \quad \text{as} \quad k \to \infty \qquad \text{or} \qquad \text{S-}\lim_{k\to\infty} x_k = L.$$

The set of all statistically convergent sequences will be denoted by S.

By a lacunary sequence, we mean an increasing integer sequence $\theta = (k_r)_{r=0}^{\infty}$ such that

$$k_0 = 0 \qquad \text{and} \qquad h_r = (k_r - k_{r-1}) \to \infty \quad \text{as} \quad r \to \infty.$$

Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. In recent years, lacunary sequences were studied in [2], [7], [13], [15] and [23].

Borwein [3] has studied strongly summable functions. A real-valued function $x = x(t)$, measurable (in the Lebesgue sense) in the interval $(1, \infty)$, is said to be strongly summable to $L = L_x$ if

$$\lim_{n\to\infty} \frac{1}{n} \int_1^n |x(t) - L|^p \, dt = 0, \qquad (1 \leqq p < \infty). \tag{2}$$

We denote the space of all real-valued functions x by $\left[W_p\right]$ which are measurable (in the Lebesgue sense) in the interval $(1, \infty)$. The space $\left[W_p\right]$ is a normed space with the norm given by

$$\|x\| = \sup_{n\geqq 1} \left(\frac{1}{n} \int_1^n |x(t)|^p \, dt \right)^{\frac{1}{p}}. \tag{3}$$

In this paper, we introduce the concepts of lacunary strong summability of order α and lacunary statistical convergence of order α of real-valued functions which are measurable (in the Lebesgue sense) in the interval $(1, \infty)$.

2. First Set of Main Results

In this section, we give the first set of main results of this paper. In particular, one of our main results (see Theorem 2 below) gives the inclusion relations between lacunary statistically convergent functions of order α for different values of α involving real-valued functions which are measurable (in the Lebesgue sense) in the interval $(1, \infty)$. In Theorem 3, we present the relationship between the strong $\left[W_{\theta p}^{\alpha}\right]$-summability for different values of α for real-valued functions which are measurable (in the Lebesgue sense) in the interval $(1, \infty)$. Theorem 5 provides the relationship between the strong $\left[W_{\theta p}^{\alpha}\right]$-summability and the $\left[S_{\theta}^{\beta}\right]$-statistical convergence of real-valued functions which are measurable (in the Lebesgue sense) in the interval $(1, \infty)$.

Definition 1. Let $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0, 1]$ and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$. A real-valued function x is said to be lacunary strongly summable of order α (or, equivalently, $\left[W_{\theta p}^{\alpha}\right]$-summable) if there is a number $L = L_x$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \int_{k_{r-1}}^{k_r} |x(t) - L|^p \, dt = 0 \qquad (1 \leqq p < \infty), \tag{4}$$

where h_r^{α} denotes the αth power $(h_r)^{\alpha}$ of h_r. In this case, we write

$$\left[W_{\theta p}^{\alpha}\right]\text{-}\lim x(t) = L.$$

The set of all lacunary strongly summable sequences of order α will be denoted by $\left[W_{\theta p}^{\alpha}\right]$. For $\theta = (2^r)$, we shall simply write $\left[W_{p}^{\alpha}\right]$ instead of $\left[W_{\theta p}^{\alpha}\right]$. Moreover, in the special case when $\alpha = 1$ and $\theta = (2^r)$, we shall write $\left[W_{p}\right]$ instead of $\left[W_{\theta p}^{\alpha}\right]$. We denote the class of functions by $\left[W_{p}^{\alpha}\right]_0$ in $\left[W_{p}^{\alpha}\right]$ for which $L = 0$.

Definition 2. Let $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0,1]$ and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1,\infty)$. A real-valued function x is said to be lacunary statistically convergent of order α $\left(\text{or, equivalently, } \left[S_{\theta}^{\alpha}\right]\text{-statistically convergent}\right)$ to a number $L = L_x$ if, for every $\varepsilon > 0$, we have

$$\lim_{r} \frac{1}{h_r^{\alpha}} \left|\left\{t \in I_r : \left|x(t) - L\right| \geqq \varepsilon\right\}\right| = 0. \tag{5}$$

The set of all lacunary statistically convergent sequences of order α will be denoted by $\left[S_{\theta}^{\alpha}\right]$. In this case, we write

$$\left[S_{\theta}^{\alpha}\right]\text{-}\lim x(t) = L.$$

For $\theta = (2^r)$, we shall simply write $\left[S^{\alpha}\right]$ instead of $\left[S_{\theta}^{\alpha}\right]$. Furthermore, in the special case when $\alpha = 1$ and $\theta = (2^r)$, we use $\left[S\right]$ instead of $\left[S_{\theta}^{\alpha}\right]$.

The following example show that the $\left[S_{\theta}^{\alpha}\right]$-statistical convergence and the $\left[W_{\theta p}^{\alpha}\right]$-summability are well defined for $0 < \alpha \leqq 1$, but (in general) they are not well defined for $\alpha > 1$.

Example. Let $\theta = (k_r)$ be a lacunary sequence and α be fixed real number such that $0 < \alpha \leqq 1$. Define $x = x(t)$ by

$$x(t) = \begin{cases} 1 & (t = 2r) \\ 0 & (t \neq 2r) \end{cases} \qquad (r \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

We then have

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \left|\left\{t \in I_r : |x(t) - 1| \geqq \varepsilon\right\}\right| \leqq \lim_{r\to\infty} \frac{k_r - k_{r-1}}{2h_r^{\alpha}} = \lim_{r\to\infty} \frac{h_r}{2h_r^{\alpha}} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \left|\left\{t \in I_r : |x(t) - 0| \geqq \varepsilon\right\}\right| \leqq \lim_{r\to\infty} \frac{k_r - k_{r-1}}{2h_r^{\alpha}} = \lim_{r\to\infty} \frac{h_r}{2h_r^{\alpha}} = 0,$$

for $\alpha > 1$, such that $x = x(t)$ lacunary statistically convergence of order α. However, this is impossible.

The proof of each of the following results is straightforward. Therefore, we omitted the proofs.

Theorem 1. *Let $0 < \alpha \leqq 1$. Suppose also that x and y are real-valued functions which are measurable* (*in the Lebesgue sense*) *in the interval* $(1,\infty)$. *Then each of the following assertions holds true*:

(i) *If* $\left[S_{\theta}^{\alpha}\right]$-$\lim x(t) = L$ *and* $c \in \mathbb{R}$, *then*

$$\left[S_{\theta}^{\alpha}\right]\text{-}\lim cx(t) = cL.$$

(ii) *If* $\left[S_{\theta}^{\alpha}\right]$-$\lim x(t) = L_1$ *and* $\left[S_{\theta}^{\alpha}\right]$-$\lim y(t) = L_2$, *then*

$$\left[S_\theta^\alpha\right]\text{-}\lim\left[x(t)+y(t)\right]=L_1+L_2.$$

(iii) *If* $\left[W_{\theta p}^\alpha\right]$-$\lim x(t)=L$ *and* $c\in\mathbb{R}$, *then*

$$\left[W_{\theta p}^\alpha\right]\text{-}\lim cx(t)=cL.$$

(iv) *If* $\left[W_{\theta p}^\alpha\right]$-$\lim x(t)=L_1$ *and* $\left[W_{\theta p}^\alpha\right]$-$\lim y(t)=L_2$, *then*

$$\left[W_{\theta p}^\alpha\right]\text{-}\lim\left[x(t)+y(t)\right]=L_1+L_2.$$

Theorem 2. *Let* $\theta=(k_r)$ *be a lacunary sequence and the parameters* α *and* β *be fixed real numbers such that* $0<\alpha\leqq\beta\leqq 1$. *Suppose also that* x *is a real-valued function which is measurable* (*in the Lebesgue sense*) *in the interval* $(1,\infty)$. *Then*

$$\left[S_\theta^\alpha\right]\subseteq\left[S_\theta^\beta\right]$$

and the inclusion is strict for some α *and* β *sucht that* $\alpha<\beta$.

Proof. The inclusion part of the proof is easy. Taking $\theta=(2^r)$, we show the strictness of the inclusion $\left[S_\theta^\alpha\right]\subseteq\left[S_\theta^\beta\right]$ for a special case. Define $x=x(t)$ by

$$x(t)=\begin{cases}2 & (t=m^2)\\ 0 & (t\neq m^2)\end{cases}\qquad (m\in\mathbb{N}=\{1,2,3,\cdots\}).$$

Then $x\in\left[S_\theta^\beta\right]$ for $\frac{1}{2}<\beta\leqq 1$, but $x\notin\left[S_\theta^\alpha\right]$ for $0<\alpha\leqq\frac{1}{2}$. This implies that the inclusion $\left[S_\theta^\alpha\right]\subseteq\left[S_\theta^\beta\right]$ is strict for $\alpha,\beta\in(0,1]$, such that $\alpha\in(0,\frac{1}{2}]$ and $\beta\in(\frac{1}{2},1]$. □

Theorem 2 yields the following corollary.

Corollary 1. *If* x *is lacunary statistically convergent of order* α *to* L, *then it is lacunary statistically convergent to* L *for each* $\alpha\in(0,1]$.

Theorem 3. *Let* $\theta=(k_r)$ *be a lacunary sequence and* α *and* β *be fixed real numbers such that* $0<\alpha\leqq\beta\leqq 1$. *Also let* p *be a positive real number and* x *be a real-valued function which is measurable* (*in the Lebesgue sense*) *in the interval* $(1,\infty)$. *Then*

$$\left[W_{\theta p}^\alpha\right]\subseteq\left[W_{\theta p}^\beta\right].$$

The inclusion is strict for some α and β sucht that $\alpha<\beta$.

Proof. The inclusion part of the proof is easy. Taking $\theta=(2^r)$, and $p=1$ we show the strictness of the inclusion $\left[W_{\theta p}^\alpha\right]\subseteq\left[W_{\theta p}^\beta\right]$ for a special case. Define $x=x(t)$ by

$$x(t)=\begin{cases}1 & (t=m^2)\\ 0 & (t\neq m^2)\end{cases}\qquad (m\in\mathbb{N}=\{1,2,3,\cdots\}).$$

Then $x\in\left[W_{\theta p}^\beta\right]$ for $\frac{1}{2}<\beta\leqq 1$, but $x\notin\left[W_{\theta p}^\alpha\right]$ for $0<\alpha\leqq\frac{1}{2}$. This implies that the inclusion $\left[W_{\theta p}^\alpha\right]\subseteq\left[W_{\theta p}^\beta\right]$ is strict for $\alpha,\beta\in(0,1]$, such that $\alpha\in(0,\frac{1}{2}]$ and $\beta\in(\frac{1}{2},1]$. □

Corollary 2. *If x is lacunary strongly summable of order α to L, then it is lacunary strongly summable to L for each $\alpha \in (0,1]$.*

Theorem 4. *Let $0 < \alpha \leqq 1$ and $\theta = (k_r)$ be a lacunary sequence. Also let p be a positive real number and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1,\infty)$. If $\liminf_r q_r > 1$, then*

$$\left[W_p^\alpha\right] \subset \left[W_{\theta p}^\alpha\right].$$

Proof. Suppose that $\liminf_r q_r > 1$ and that there exists $\delta > 0$ such that $1 + \delta \leqq q_r$ for all $r \geqq 1$. Then, for $x(t) \in \left[W^\alpha\right]_0$, we get

$$\begin{aligned}\frac{1}{h_r^\alpha}\int_{k_{r-1}}^{k_r} |x(t)|^p\,dt &= \frac{1}{h_r^\alpha}\int_1^{k_r} |x(t)|^p\,dt - \frac{1}{h_r^\alpha}\int_1^{k_{r-1}} |x(t)|^p\,dt \\ &= \frac{k_r^\alpha}{h_r^\alpha}\left(\frac{1}{k_r^\alpha}\int_1^{k_r} |x(t)|^p\,dt\right) - \frac{k_{r-1}^\alpha}{h_r^\alpha}\left(\frac{1}{k_{r-1}^\alpha}\int_1^{k_{r-1}} |x(t)|^p\,dt\right).\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r^\alpha}{h_r^\alpha} \leqq \frac{(1+\delta)^\alpha}{\delta^\alpha} \qquad \text{and} \qquad \frac{k_{r-1}^\alpha}{h_r^\alpha} \leqq \frac{1}{\delta^\alpha},$$

which shows that $x(t) \in \left[W_{\theta p}^\alpha\right]_0$. The general inclusion

$$\left[W_p^\alpha\right] \subset \left[W_{\theta p}^\alpha\right],$$

asserted by Theorem 4, would follow by linearity. □

Theorem 5. *Let $\theta = (k_r)$ be a lacunary sequence. Also let p be a positive real number and $\alpha, \beta \in (0,1]$ such that $\alpha \leqq \beta$. Suppose that x is a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1,\infty)$. If the function x is $\left[W_{\theta p}^\alpha\right]$-summable to L, then it is $\left[S_\theta^\beta\right]$-statically convergent to L.*

Proof. For any function x which is $\left[W_{\theta p}^\alpha\right]$-summable and for a given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{h_r^\alpha}\int_{k_{r-1}}^{k_r} |x(t)-L|^p\,dt &= \frac{1}{h_r^\alpha}\int\limits_{\substack{k_{r-1} \\ (|x(t)-L| \geqq \varepsilon)}}^{k_r} |x(t)-L|^p\,dt + \frac{1}{h_r^\alpha}\int\limits_{\substack{k_{r-1} \\ (|x(t)-L| < \varepsilon)}}^{k_r} |x(t)-L|^p\,dt \\ &\geqq \frac{1}{h_r^\alpha}\int\limits_{\substack{k_{r-1} \\ (|x(t)-L| \geqq \varepsilon)}}^{k_r} |x(t)-L|^p\,dt \\ &\geqq \frac{1}{h_r^\beta}\left|\{t \in I_r : |x(t)-L| \geqq \varepsilon\}\right| \varepsilon^p,\end{aligned}$$

which proves the assertion of Theorem 5 that the function x is $\left[S_\theta^\beta\right]$-statically convergent to L. □

The following results are easily derivable from Theorem 5.

Corollary 3. *Let $\theta = (k_r)$ be a lacunary sequence. Also let p be a positive real number and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$. Then each of the following assertions holds true:*

(i) *If the function x is $\left[W^{\alpha}_{\theta p}\right]$-summable to L, then it is $\left[S^{\alpha}_{\theta}\right]$-statically convergent to L.*

(ii) *If the function x is $\left[W^{\alpha}_{\theta p}\right]$-summable to L, then it is $\left[S_{\theta}\right]$ -statistically convergent to L.*

Theorem 6. *Let $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0, 1]$ and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$. If $\liminf_r q_r > 1$, then*

$$\left[S^{\alpha}\right] \subseteq \left[S^{\alpha}_{\theta}\right].$$

Proof. Suppose that $\liminf_r q_r > 1$. Then there exists a $\delta > 0$ such that $q_r \geqq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geqq \frac{\delta}{1+\delta} \Longrightarrow \left(\frac{h_r}{k_r}\right)^{\alpha} \geqq \left(\frac{\delta}{1+\delta}\right)^{\alpha} \Longrightarrow \frac{1}{k_r^{\alpha}} \geqq \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}} \frac{1}{h_r^{\alpha}}. \tag{6}$$

Now let x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$. Then, for every $\varepsilon > 0$ and for sufficiently large r, we get

$$\begin{aligned} \frac{1}{k_r^{\alpha}} \left|\{k \leqq k_r : |x(t) - L| \geqq \varepsilon\}\right| &\geqq \frac{1}{k_r^{\alpha}} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right| \\ &\geqq \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}} \frac{1}{h_r^{\alpha}} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right|. \end{aligned} \tag{7}$$

The proof of Theorem 6 is completed by (6) and (7). □

Theorem 7. *Let $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0, 1]$ and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$. If $\limsup_r q_r < \infty$, then*

$$\left[S^{\alpha}_{\theta}\right] \subseteq \left[S\right].$$

Proof. Under the hypothesis that $\limsup_r q_r < \infty$, there is an $H > 0$ such that $q_r < H$ for all r. Suppose that x is a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$ and put $N_r = \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right|$. By the definition of the $\left[S^{\alpha}_{\theta}\right]$-statistical convergence, given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that, for $0 < \alpha \leqq 1$,

$$\frac{N_r}{h_r^{\alpha}} < \varepsilon \Longrightarrow \frac{N_r}{h_r} < \varepsilon \qquad (r > r_0).$$

The rest of the proof of Theorem 7 follows readily from Lemma 3 in [15]. □

3. Further Results and their Applications

Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$. Suppose also that the parameters α and β are fixed real numbers such that $0 < \alpha \leqq \beta \leqq 1$. We shall now give some inclusion relations between the sets of $\left[S^{\alpha}_{\theta}\right]$-statistically convergent sequences and $\left[W^{\alpha}_{\theta p}\right]$-summable sequences for different choices for α and θ.

Theorem 8. *Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. Also let $0 < \alpha \leqq \beta \leqq 1$ and x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$.*

Then each of the following assertions holds true:

(i) *If*

$$\lim_{r\to\infty}\inf\frac{h_r^{\alpha}}{\ell_r^{\beta}}>0, \tag{8}$$

then

$$\left[S_{\theta'}^{\beta}\right]\subseteq\left[S_{\theta}^{\alpha}\right].$$

(ii) *If*

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^{\beta}}=1 \tag{9}$$

then

$$\left[S_{\theta}^{\alpha}\right]\subseteq\left[S_{\theta'}^{\beta}\right].$$

Proof. We divide our proof of Theorem 8 in the following two parts.

(i) Let x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1,\infty)$ and suppose that $I_r\subset J_r$ for all $r\in\mathbb{N}$. If the condition (8) is satisfied, then (for a given $\varepsilon>0$), we have

$$\{t\in J_r:|x(t)-L|\geqq\varepsilon\}\supseteq\{t\in I_r:|x(t)-L|\geqq\varepsilon\},$$

and so

$$\frac{1}{\ell_r^{\beta}}\left|\{t\in J_r:|x(t)-L|\geqq\varepsilon\}\right|\geqq\frac{h_r^{\alpha}}{\ell_r^{\beta}}\frac{1}{h_r^{\alpha}}\left|\{t\in I_r:|x(t)-L|\geqq\varepsilon\}\right| \tag{10}$$

for all $r\in\mathbb{N}$, where

$$I_r=(k_{r-1},k_r],\quad J_r=(s_{r-1},s_r],\quad h_r=k_r-k_{r-1}\quad\text{and}\quad \ell_r=s_r-s_{r-1}.$$

Now, proceeding to the limit as $r\to\infty$ in the last inequality (10) and using (8), we get

$$\left[S_{\theta'}^{\beta}\right]\subseteq\left[S_{\theta}^{\alpha}\right],$$

just as asserted by Theorem 8(i).

(ii) Let x be a real-valued function which is measurable (in the Lebesgue sense) in the interval $(1,\infty)$ and suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and the condition (9) is satisfied. Since $I_r \subset J_r$ $(r \in \mathbb{N})$ for $\varepsilon > 0$, we obtain

$$\begin{aligned}
\frac{1}{\ell_r^\beta} \left|\{t \in J_r : |x(t) - L| \geqq \varepsilon\}\right| &= \frac{1}{\ell_r^\beta} \left|\{s_{r-1} < t \leqq k_{r-1} : |x(t) - L| \geqq \varepsilon\}\right| \\
&\quad + \frac{1}{\ell_r^\beta} \left|\{k_r < t \leqq s_r : |x_k - L| \geqq \varepsilon\}\right| \\
&\quad + \frac{1}{\ell_r^\beta} \left|\{k_{r-1} < t \leqq k_r : |x(t) - L| \geqq \varepsilon\}\right| \\
&\leqq \frac{k_{r-1} - s_{r-1}}{\ell_r^\beta} + \frac{s_r - k_r}{\ell_r^\beta} + \frac{1}{\ell_r^\beta} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right| \\
&= \frac{\ell_r - h_r}{\ell_r^\beta} + \frac{1}{\ell_r^\beta} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right| \\
&\leqq \frac{\ell_r - h_r^\beta}{h_r^\beta} + \frac{1}{h_r^\beta} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right| \\
&\leqq \left(\frac{\ell_r}{h_r^\beta} - 1\right) + \frac{1}{h_r^\alpha} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right| \qquad (11)
\end{aligned}$$

for all $r \in \mathbb{N}$. By using the condition (9), we have

$$\frac{\ell_r}{h_r^\beta} - 1 \geqq 0.$$

Moreover, since x is a real-valued function which is lacunary statistical convergence of order α to L, the second term of the last member of the inequality (11) tends to 0 as $r \to \infty$. This implies that

$$\left[S_\theta^\alpha\right] \subseteq \left[S_{\theta'}^\beta\right],$$

which is asserted by Theorem 8(ii). □

The following results are derivable easily from Theorem 8.

Corollary 4. *Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$ and x be a real-valued function which is measurable* (*in the Lebesgue sense*) *in the interval* $(1, \infty)$. *If the condition* (8) *is satisfied, then*

(i) $\left[S_{\theta'}^\alpha\right] \subseteq \left[S_\theta^\alpha\right]$ *for each* $\alpha \in (0, 1]$;

(ii) $\left[S_{\theta'}\right] \subseteq \left[S_\theta^\alpha\right]$ *for each* $\alpha \in (0, 1]$;

(iii) $\left[S_{\theta'}\right] \subseteq \left[S_\theta\right]$.

Furthermore, if the condition (9) *is satisfied, then*

(i) $\left[S_\theta^\alpha\right] \subseteq \left[S_{\theta'}^\alpha\right]$ *for each* $\alpha \in (0, 1]$;

(ii) $\left[S_\theta^\alpha\right] \subseteq \left[S_{\theta'}\right]$ *for each* $\alpha \in (0, 1]$;

(iii) $\left[S_\theta\right] \subseteq \left[S_{\theta'}\right]$.

Theorem 9. *Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. Also let the parameters α and β be such that $0 < \alpha \leqq \beta \leqq 1$. If the condition* (8) *holds true and if x is a real-valued function which is measurable* (*in the Lebesgue sense*) *in the interval* $(1, \infty)$, *then*

$$\left[W^{\beta}_{\theta' p}\right] \subset \left[W^{\alpha}_{\theta p}\right].$$

Proof. The proof of the Theorem 9 is akin to the proof of the Theorem 8. Therefore, we omitted the proof. □

By applying Theorem 9, we can deduce the following corollary.

Corollary 5. *Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If the condition* (8) *holds true and if x is a real-valued function which is measurable* (*in the Lebesgue sense*) *in the interval* $(1, \infty)$, *then*

(i) $\left[W^{\alpha}_{\theta' p}\right] \subset \left[W^{\alpha}_{\theta p}\right]$ *for each* $\alpha \in (0, 1]$;

(ii) $\left[W_{\theta' p}\right] \subset \left[W^{\alpha}_{\theta p}\right]$ *for each* $\alpha \in (0, 1]$;

(iii) $\left[W_{\theta' p}\right] \subset \left[W_{\theta p}\right]$.

Theorem 10. *Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. Also let the parameters α and β be such that $0 < \alpha \leqq \beta \leqq 1$. Suppose that the condition* (8) *holds true and that x is a real-valued function which is measurable* (*in the Lebesgue sense*) *in the interval* $(1, \infty)$. *If x is $\left[W^{\beta}_{\theta' p}\right]$-summable to L, then it is $\left[S^{\alpha}_{\theta}\right]$-statistically convergent to L.*

Proof. Let the real-valued function x be measurable (in the Lebesgue sense) in the interval $(1, \infty)$ and let x be $\left[W^{\beta}_{\theta' p}\right]$-summable to L. Then, for a given $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{\ell_r^{\beta}} \int_{s_{r-1}}^{s_r} |x(t) - L|^p \, dt &= \frac{1}{\ell_r^{\beta}} \int\limits_{\substack{s_{r-1} \\ (|x(t)-L| \geqq \varepsilon)}}^{s_r} |x(t) - L|^p \, dt + \frac{1}{\ell_r^{\beta}} \int\limits_{\substack{s_{r-1} \\ (|x(t)-L| < \varepsilon)}}^{s_r} |x(t) - L|^p \, dt \\
&\geqq \frac{1}{\ell_r^{\beta}} \int\limits_{\substack{k_{r-1} \\ (|x(t)-L| \geqq \varepsilon)}}^{k_r} |x(t) - L|^p \, dt \\
&\geqq \frac{1}{\ell_r^{\beta}} \left|\{t \in I_r : |x(t) - L| \geqq \varepsilon\}\right| \varepsilon^p \\
&\geqq \frac{h_r^{\alpha}}{\ell_r^{\beta}} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : |x_k - L| \geqq \varepsilon\}\right| \varepsilon^p. \qquad (12)
\end{aligned}$$

Thus, in view of (12), we conclude that x is $\left[S^{\alpha}_{\theta}\right]$-statistically convergent to L. This completes the proof of Theorem 10. □

Corollary 6 can easily be proven by applying Theorem 10.

Corollary 6. *Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. Also let α be a real number such that $0 < \alpha \leqq 1$. If the condition* (8) *holds true and if x be a real-valued function which is* *measurable* (*in the Lebesgue sense*) *in the interval* $(1, \infty)$, *then each of the following assertions holds true*:

(i) *If* x *is* $\left[W^{\alpha}_{\theta' p}\right]$*-summable to* L*, then it is* $\left[S^{\alpha}_{\theta}\right]$*-statistically convergent to* L*;*

(ii) *If* x *is* $\left[W_{\theta' p}\right]$*-summable to* L*, then it is* $\left[S^{\alpha}_{\theta}\right]$*-statistically convergent to* L*;*

(iii) *If* x *is* $\left[W_{\theta' p}\right]$*-summable to* L*, then it is* $\left[S_{\theta}\right]$*-statistically convergent to* L*.*

Several other interesting applications and consequences of our results (Theorems 1 to 10 and Corollaries 1 to 6) can be deduced. We left such details for the targeted readers.

References

[1] A. Alotaibi and A. M. Alroqi, Statistical convergence in a paranormed space, *J. Inequal. Appl.* **2012** (2012), Article ID 39, 1–6.
[2] V. K. Bhardwaj and M. Singh, Some sequences defined by Orlicz functions, *Demonstratio Math.* **33** (2000), 571–582.
[3] D. Borwein, Linear functionals connected with strong Cesàro summability, *J. London Math. Soc.* **40** (1965), 628–634.
[4] N. L. Braha, H. M. Srivastava and S.A. Mohiuddine, A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean, *Appl. Math. Comput.* **228** (2014), 162–169.
[5] R. Çolak, Statistical convergence of order α, in *Modern Methods in Analysis and Its Applications* (M. Mursaleen, Editor), pp. 121–129, Anamaya Publications, New Delhi, 2010.
[6] J. S. Connor, The statistical and strong p-Cesàro convergence of sequences, *Analysis* **8** (1988), 47–63.
[7] G. Das and S. K. Mishra, Banach limits and lacunary strong almost convegence, *J. Orissa Math. Soc.* **2** (1983), 61–70.
[8] E. Erkuş, O. Duman and H. M. Srivastava, Statistical approximation of certain positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials, *Appl. Math. Comput.* **182** (2006), 213–222.
[9] M. Et, Strongly almost summable difference sequences of order m defined by a modulus, *Studia Sci. Math. Hungar.* **40** (2003), 463–476.
[10] M. Et, M. Çınar and M. Karakaş, On λ-statistical convergence of order α of sequences of function, *J. Inequal. Appl.* **2013** (2013), Article ID 204, 1–8.
[11] M. Et, S. A. Mohiuddine and A. Alotaibi, On λ-statistical convergence and strongly λ-summable functions of order α, *J. Inequal. Appl.* **2013** (2013), Article ID 469, 1–8.
[12] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.
[13] A. R. Freedman, J. J. Sember and M. Raphael, Some Cesàro-type summability spaces, *Proc. London Math. Soc.* (*Ser.* 2) **37** (1978), 508–520.
[14] J. A. Fridy, On statistical convergence, *Analysis* **5** (1985), 301–313.
[15] J. A. Fridy and C. Orhan, Lacunary statistical convergence, *Pacific J. Math.* **160** (1993), 43–51.
[16] A. D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* **32** (2002), 129–138.
[17] M. Güngör, M. Et and Y. Altın, Strongly (V_σ, λ, q)-summable sequences defined by Orlicz functions, *Appl. Math. Comput.* **157** (2004), 561–571.
[18] M. Güngör and M. Et, Δ^m-strongly almost summable sequences defined by Orlicz functions, *Indian J. Pure Appl. Math.* **34** (2003), 1141–1151.
[19] S. A. Mohiuddine, A. Alotaibi and M. Mursaleen, Statistical convergence of double sequences in locally solid Riesz spaces, *Abstr. Appl. Anal.* **2002** (2012), Article ID 719729, 1–9.
[20] M. Mursaleen and A. Alotaibi, Statistical summability and approximation by de la Vallée Poussin mean, *Appl. Math. Lett.* **24** (2011), 320–324.
[21] M. Mursaleen, A. Khan, H. M. Srivastava and K.S. Nisar, Operators constructed by means of q-Lagrange polynomials and A-statistical approximation, *Appl. Math. Comput.* **219** (2013), 6911–6918.
[22] F. Nuray, λ-strongly summable and λ-statistically convergent functions, *Iranian J. Sci. Technol. Trans. A Sci.* **34** (2010), 335–338.
[23] F. Nuray and B. Aydin, Strongly summable and statistically convergent functions, *Inform. Technol. Valdymas* **1** (**30**) (2004), 74–76.
[24] M. A. Özarslan, O. Duman and H. M. Srivastava, Statistical approximation results for Kantorovich-type operators involving some special polynomials, *Math. Comput. Modelling* **48** (2008), 388–401.
[25] D. Rath and B. C. Tripathy, On statistically convergent and statistically Cauchy sequences, *Indian J. Pure. Appl. Math.* **25** (1994), 381–386.
[26] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca* **30** (1980), 139–150.
[27] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* **66** (1959), 361–375.
[28] H. M. Srivastava, M. Mursaleen and A. Khan, Generalized equi-statistical convergence of positive linear operators and associated approximation theorems, *Math. Comput. Modelling* **55** (2012), 2040–2051.
[29] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* **2** (1951), 73–74.
[30] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, London and New York, 1979.