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Vertical Liouville Foliations on the Big-Tangent Manifold of a Finsler Space

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Abstract. The present paper unifies some aspects concerning the vertical Liouville distributions on the tangent (cotangent) bundle of a Finsler (Cartan) space in the context of generalized geometry. More exactly, we consider the big-tangent manifold $\mathcal{T}M$ associated to a Finsler space (M, F) and of its $\mathcal{L}$-dual which is a Cartan space (M, K) and we define three Liouville distributions on $\mathcal{T}M$ which are integrable. We also find geometric properties of both leaves of Liouville distribution and the vertical distribution in our context.

1. Introduction and Preliminary Notions

1.1. Introduction

The vertical Liouville distribution on the tangent bundle of a (pseudo) Finsler space was defined for the first time in [4] where some aspects of the geometry of the vertical bundle are derived via vertical Liouville distribution. A similar study on the cotangent bundle of a Cartan space can be found in [11]. Also, other significant studies concerning the interrelations between natural foliations defined by Liouville fields on the tangent bundle of a Finsler space and the geometry of the Finsler space itself, as well as similar problems on Cartan spaces are intensively studied in [6] and [2], respectively. See also [11, 14, 19, 20].

As it is well known, in the *generalized geometry* initiated in [10], the tangent bundle TM of a smooth n-dimensional manifold M is replaced by the *big-tangent bundle* (or Pontryagin bundle) $TM \oplus T^*M$. On its total space the velocities and moments are considered as independent variables. This idea was proposed and developed in [21, 22] and later was used in the study of Hamiltonian-Jacobi theory for singular Lagrangian systems [13]. The geometry of the total space of the big-tangent bundle, called *big-tangent manifold*, is intensively studied in [25] and some its applications to mechanical systems can be found in [9].

Using the framework of the geometry on the big-tangent manifold, our aim in this paper is to extend some results concerning the vertical Liouville foliation in the context of generalized geometry. Thus, some aspects concerning the geometry of vertical bundle of the big-tangent bundle of a Finsler space can be obtained via our generalized Liouville distribution. This extension yields new subfoliations of the vertical foliation on the big-tangent bundle and some properties of these subfoliations are studied in the end of the paper. In this sense, we consider the big-tangent manifold $\mathcal{T}M$ associated to a Finsler space (M, F) and of

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its $\mathcal{L}$-dual which is a Cartan space (M, K). As usual, we reconsider the vertical Liouville distributions $V_{\mathcal{E}_1}$ and $V_{\mathcal{E}_2}$ from the case of vertical tangent (cotangent) bundle of a Finsler (Cartan) space, see [4, 11], for the case of vertical subbundles V_1 and V_2, respectively, with respect to Liouville vector fields $\mathcal{E}_1$ and $\mathcal{E}_2$. Next we define the Liouville distribution $V_{\mathcal{E}}$ with respect to the Liouville vector field $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, we prove that it is integrable (Theorem 2.2) and we study some of its properties (Theorem 2.5 and Proposition 2.6). Also, some links between the vertical Liouville foliations $V_{\mathcal{E}_1}$, $V_{\mathcal{E}_2}$ and $V_{\mathcal{E}}$, respectively, are established.

On the other hand, it is well known that the vertical Liouville distribution on the tangent (cotangent) bundle of a Finsler (Cartan) space is strongly related with the indicatrix of Finsler (Cartan) space, see [6, 11], which inherits a natural almost contact structure, see [1, 8]. Thus, our generalized vertical Liouville distribution, which is an odd dimensional integrable distribution over the big-tangent manifold, can serve as an example of a contact Lie algebroid. This example is presented in the recent paper [12] when the Finsler metric is the norm of a Riemannian metric. Also, some basic adapted connections with respect to vertical subfoliations of the big-tangent manifold can be studied as in [11, 14].

1.2. Preliminaries and notations

Let M be a n-dimensional smooth manifold, and we consider $\pi : TM \to M$ its tangent bundle, $\pi^* : T^*M \to M$ its cotangent bundle and $\tau \equiv \pi \oplus \pi^* : TM \oplus T^*M \to M$ its big-tangent bundle defined as Whitney sum of the tangent and cotangent bundles of M. The total space of the big-tangent bundle, called *big-tangent manifold*, is a $3n$-dimensional smooth manifold denoted here by $\mathcal{T}M$. Let us briefly recall some elementary notions about the big-tangent manifold $\mathcal{T}M$. For a detailed discussion about its geometry we refer [25].

Let $(U, (x^i))$ be a local chart on M. If $\{\frac{\partial}{\partial x^i}|_x\}$, $x \in U$ is a local frame of sections in the tangent bundle over U and $\{dx^i|_x\}$, $x \in U$ is a local frame of sections in the cotangent bundle over U, then by definition of the Whitney sum, $\{\frac{\partial}{\partial x^i}|_x, dx^i|_x\}$, $x \in U$ is a local frame of sections in the big-tangent bundle $TM \oplus T^*M$ over U. Every section (y, p) of τ over U takes the form $(y, p) = y^i \frac{\partial}{\partial x^i} + p_i dx^i$ and the local coordinates on $\tau^{-1}(U)$ will be defined as the triples (x^i, y^i, p_i), where $i = 1, \ldots, n = \dim M$, (x^i) are local coordinates on M, (y^i) are vector coordinates and (p_i) are covector coordinates.

The change rules of these coordinates are:

$$\widetilde{x}^i = \widetilde{x}^i(x^j)\,,\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j\,,\ \widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i} p_j \tag{1}$$

and the local expressions of a vector field X and of a 1-form φ on $\mathcal{T}M$ are

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^i \frac{\partial}{\partial y^i} + \zeta_i \frac{\partial}{\partial p_i} \text{ and } \varphi = \alpha_i dx^i + \beta_i dy^i + \gamma^i dp_i. \tag{2}$$

For the big-tangent manifold $\mathcal{T}M$ we have the following projections

$$\tau : \mathcal{T}M \to M\,,\ \tau_1 : \mathcal{T}M \to TM\,,\ \tau_2 : \mathcal{T}M \to T^*M$$

on M and on the total spaces of tangent and cotangent bundle, respectively.

As usual, we denote by $V = V(\mathcal{T}M)$ the vertical bundle on the big-tangent manifold $\mathcal{T}M$ and it has the decomposition

$$V = V_1 \oplus V_2, \tag{3}$$

where $V_1 = \tau_1^{-1}(V(TM))$, $V_2 = \tau_2^{-1}(V(T^*M))$ and have the local frames $\{\frac{\partial}{\partial y^i}\}$, $\{\frac{\partial}{\partial p_i}\}$, respectively. The subbundles V_1, V_2 are the vertical foliations of $\mathcal{T}M$ by fibers of τ_1, τ_2, respectively, and $\mathcal{T}M$ has a multi-foliate structure [23, 24]. The *Liouville vector fields* (or Euler vector fields) are given by

$$\mathcal{E}_1 = y^i \frac{\partial}{\partial y^i} \in \Gamma(V_1)\,,\ \mathcal{E}_2 = p_i \frac{\partial}{\partial p_i} \in \Gamma(V_2)\,,\ \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \in \Gamma(V). \tag{4}$$

In the following we consider that the manifold M is endowed with a Finsler structure F, and we present a metric structure on V induced by F. According to [3, 5, 17], a function $F : TM \to [0, \infty)$ which satisfies the following conditions:

i) F is C^∞ on $TM^0 = TM - \{\text{zero section}\}$;
ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda \in \mathbb{R}_+$;
iii) the $n \times n$ matrix (g_{ij}), where $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$, is positive definite at all points of TM^0,

is called a *Finsler structure* on M and the pair (M, F) is called a *Finsler space*. We notice that in fact $F(x, y) > 0$, whenever $y \neq 0$.

There are some useful facts which follow from the above homogeneity condition ii) of the fundamental function of the Finsler space (M, F). By the Euler theorem on positively homogeneous functions we have, see [3, 5, 17]:

$$y_i = g_{ij}y^j \,,\; y^i = g^{ij}y_j \,,\; F^2 = g_{ij}y^iy^j = y_iy^i \,,\; C_{ijk}y^k = C_{ikj}y^k = C_{kij}y^k = 0, \tag{5}$$

where (g^{ij}) is the inverse matrix of (g_{ji}) and we have put $y_i = \frac{1}{2}\frac{\partial F^2}{\partial y^i}$, $C_{ijk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}$.

Also, for a given Finsler structure F on TM^0 there is a Cartan structure $K = F^*$ on $T^*M^0 := T^*M - \{\text{zero section}\}$ obtained by Legendre transformation of F (the $\mathcal{L}$-duality process, see [15, 16, 18]), that is a function $K : T^*M \to [0, \infty)$ which has the following properties:

i) K is C^∞ on T^*M^0;
ii) $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$;
iii) the $n \times n$ matrix (g^{*ij}), where $g^{*ij} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}$, is positive definite at all points of T^*M_0.

Also $K(x, p) > 0$, whenever $p \neq 0$. The properties of K imply that

$$p^i = g^{*ij}p_j \,,\; p_i = g^*_{ij}p^j \,,\; K^2 = g^{*ij}p_ip_j = p_ip^i \,,\; C^{ijk}p_k = C^{ikj}p_k = C^{kij}p_k = 0, \tag{6}$$

where (g^*_{ij}) is the inverse matrix of (g^{*ji}) and we have put $p^i = \frac{1}{2}\frac{\partial K^2}{\partial p_i}$, $C^{ijk} = -\frac{1}{4}\frac{\partial^3 K^2}{\partial p_i \partial p_j \partial p_k}$.

It is well-known that g_{ij} determines a metric structure on $V(TM)$ and g^{*ij} determines a metric structure on $V(T^*M)$. Similarly, every Finsler structure F on M determines a metric structure G on V by setting

$$G(X, Y) = g_{ij}(x, y)X^i_1(x, y, p)Y^j_1(x, y, p) + g^{*ij}(x, p)X^2_i(x, y, p)Y^2_j(x, y, p), \tag{7}$$

for every $X = X^i_1(x, y, p)\frac{\partial}{\partial y^i} + X^2_i(x, y, p)\frac{\partial}{\partial p_i}$, $Y = Y^j_1(x, y, p)\frac{\partial}{\partial y^j} + Y^2_j(x, y, p)\frac{\partial}{\partial p_j} \in \Gamma(V)$.

2. Vertical Liouville Foliations on $\mathcal{T}M$

In this section we reconsider the vertical Liouville distributions $V_{\mathcal{E}_1}$ and $V_{\mathcal{E}_2}$ from the case of vertical tangent (cotangent) bundle of a Finsler (Cartan) space, see [4, 11], for the case of vertical subbundles V_1 and V_2, respectively, with respect to Liouville vector fields $\mathcal{E}_1$ and $\mathcal{E}_2$. Next we define the Liouville distribution $V_{\mathcal{E}}$ with respect to the Liouville vector field $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, we prove that it is integrable and we study some of its properties. Also, some links between the vertical Liouville foliations $V_{\mathcal{E}_1}$, $V_{\mathcal{E}_2}$ and $V_{\mathcal{E}}$, respectively, are established.

2.1. *Vertical Liouville distributions $V_{\mathcal{E}_1}$ and $V_{\mathcal{E}_2}$*

Following [4], [11] we define two vertical Liouville distributions on $\mathcal{T}M$ as the complementary orthogonal distributions in V_1 and V_2 to the line distributions spanned by the Liouville vector fields $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

By (4) and (5) we have

$$G(\mathcal{E}_1, \mathcal{E}_1) = F^2. \tag{8}$$

Using G and $\mathcal{E}_1$, we define the V_1-vertical one form ζ_1 by

$$\zeta_1(X_1) = \frac{1}{F}G(X_1, \mathcal{E}_1) \,,\; \forall\, X_1 = X^i_1(x, y, p)\frac{\partial}{\partial y^i} \in \Gamma(V_1). \tag{9}$$

Let us denote by $\{\mathcal{E}_1\}$ the line vector bundle over $\mathcal{T}M$ spanned by $\mathcal{E}_1$ and we define the *first vertical Liouville distribution* as the complementary orthogonal distribution $V_{\mathcal{E}_1}$ to $\{\mathcal{E}_1\}$ in V_1 with respect to G. Thus, $V_{\mathcal{E}_1}$ is defined by ζ_1, that is

$$\Gamma\left(V_{\mathcal{E}_1}\right) = \{X_1 \in \Gamma(V_1) \,:\, \zeta_1(X_1) = 0\}. \tag{10}$$

We get that every V_1-vertical vector field $X_1 = X_1^i(x,y,p)\frac{\partial}{\partial y^i}$ can be expressed in the form:

$$X_1 = P_1X_1 + \frac{1}{F}\zeta_1(X_1)\mathcal{E}_1, \tag{11}$$

where P_1 is the projection morphism of V_1 on $V_{\mathcal{E}_1}$.

Also, by direct calculus, we get

$$G(X_1, P_1Y_1) = G(P_1X_1, P_1Y_1) = G(X_1, Y_1) - \zeta_1(X_1)\zeta_1(Y_1),\ \forall\, X_1, Y_1 \in \Gamma(V_1). \tag{12}$$

Let us consider $\{\theta^i\}$ the dual basis of $\{\frac{\partial}{\partial y^i}\}$. Then, with respect to the basis $\{\theta^i\}$ and $\left\{\theta^j \otimes \frac{\partial}{\partial y^i}\right\}$, respectively, ζ_1 and P_1 are locally given by

$$\zeta_1 = \overset{1}{\zeta_i}\,\theta^i\,,\ P_1 = \overset{1}{P^i_j}\,\theta^j \otimes \frac{\partial}{\partial y^i}\,,\ \overset{1}{\zeta_i} = \frac{y_i}{F}\,,\ \overset{1}{P^i_j} = \delta^i_j - \frac{y_jy^i}{F^2}, \tag{13}$$

where δ^i_j are the components of the Kronecker delta.

As usual for tangent bundle of a Finsler space (see Theorem 3.1 from [4]), the first vertical Liouville distribution $V_{\mathcal{E}_1}$ is integrable and it defines a foliation on $\mathcal{T}M$, called the *first vertical Liouville foliation* on the big-tangent manifold $\mathcal{T}M$. Also, some geometric properties of the leaves of vertical foliation V_1 can be derived via the first vertical Liouville foliation $V_{\mathcal{E}_1}$.

Similarly, by (4) and (6) we have

$$G(\mathcal{E}_2, \mathcal{E}_2) = K^2, \tag{14}$$

and using G and $\mathcal{E}_2$, we define the V_2-vertical one form ζ_2 by

$$\zeta_2(X_2) = \frac{1}{K}G(X_2, \mathcal{E}_2)\,,\ \forall\, X_2 = X_i^2(x,y,p)\frac{\partial}{\partial p_i} \in \Gamma(V_2). \tag{15}$$

Let us denote by $\{\mathcal{E}_2\}$ the line vector bundle over $\mathcal{T}M$ spanned by $\mathcal{E}_2$ and we define the *second vertical Liouville distribution* as the complementary orthogonal distribution $V_{\mathcal{E}_2}$ to $\{\mathcal{E}_2\}$ in V_2 with respect to G. Thus, $V_{\mathcal{E}_2}$ is defined by ζ_2, that is

$$\Gamma\left(V_{\mathcal{E}_2}\right) = \{X_2 \in \Gamma(V_2) \,:\, \zeta_2(X_2) = 0\}. \tag{16}$$

We get that every V_2-vertical vector field $X_2 = X_i^2(x,y,p)\frac{\partial}{\partial p_i}$ can be expressed in the form:

$$X_2 = P_2X_2 + \frac{1}{K}\zeta_2(X_2)\mathcal{E}_2, \tag{17}$$

where P_2 is the projection morphism of V_2 on $V_{\mathcal{E}_2}$.

Similarly, by direct calculus, we get

$$G(X_2, P_2Y_2) = G(P_2X_2, P_2Y_2) = G(X_2, Y_2) - \zeta_2(X_2)\zeta_2(Y_2),\ \forall\, X_2, Y_2 \in \Gamma(V_2). \tag{18}$$

Let us consider $\{k_i\}$ the dual basis of $\{\frac{\partial}{\partial p_i}\}$. Then, with respect to the basis $\{k_i\}$ and $\left\{k_j \otimes \frac{\partial}{\partial p_i}\right\}$, respectively, ζ_2 and P_2 are locally given by

$$\zeta_2 = \overset{2}{\zeta^i}\,k_i\,,\ P_2 = \overset{2}{P^j_i}\,k_j \otimes \frac{\partial}{\partial p_i}\,,\ \overset{2}{\zeta^i} = \frac{p^i}{K}\,,\ \overset{2}{P^i_j} = \delta^i_j - \frac{p_jp^i}{K^2}. \tag{19}$$

As usual for cotangent bundle of a Cartan space (see Theorem 2.1 from [11]), the second vertical Liouville distribution $V_{\mathcal{E}_2}$ is integrable and it defines a foliation on $\mathcal{T}M$, called the *second vertical Liouville foliation* on the big-tangent manifold $\mathcal{T}M$. Also, some geometric properties of the leaves of vertical foliation V_2 can be derived via the second vertical Liouville foliation $V_{\mathcal{E}_2}$.

2.2. Vertical Liouville distribution $V_{\mathcal{E}}$

In this subsection we unify the concepts presented in the previous subsection and we define a vertical Liouville distribution on $\mathcal{T}M$ as the complementary orthogonal distribution in V to the line distribution spanned by the Liouville vector field $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$. We prove that this distribution is an integrable one, and also, we find some geometric properties of both leaves of Liouville distribution and the vertical distribution on the big-tangent manifold $\mathcal{T}M$. Finally, some links between the vertical Liouville foliations $V_{\mathcal{E}_1}$, $V_{\mathcal{E}_2}$ and $V_{\mathcal{E}}$, respectively, are established.

By (4), (5) and (6) we have

$$G(\mathcal{E},\mathcal{E}) = F^2 + K^2. \tag{20}$$

Now, by means of G and $\mathcal{E}$, we define the vertical one form ζ by

$$\zeta(X) = \frac{1}{\sqrt{F^2+K^2}}G(X,\mathcal{E}),\ \forall\, X = X_1^i(x,y,p)\frac{\partial}{\partial y^i} + X_i^2(x,y,p)\frac{\partial}{\partial p_i} \in \Gamma(V). \tag{21}$$

Let us denote by $\{\mathcal{E}\}$ the line vector bundle over $\mathcal{T}M$ spanned by $\mathcal{E}$ and we define the *vertical Liouville distribution* as the complementary orthogonal distribution $V_{\mathcal{E}}$ to $\{\mathcal{E}\}$ in V with respect to G. Thus, $V_{\mathcal{E}}$ is defined by ζ, that is

$$\Gamma(V_{\mathcal{E}}) = \{X \in \Gamma(V) \,:\, \zeta(X) = 0\}. \tag{22}$$

We get that every vertical vector field $X = X_1^i(x,y,p)\frac{\partial}{\partial y^i} + X_i^2(x,y,p)\frac{\partial}{\partial p_i}$ can be expressed in the form:

$$X = PX + \frac{1}{\sqrt{F^2+K^2}}\zeta(X)\mathcal{E}, \tag{23}$$

where P is the projection morphism of V on $V_{\mathcal{E}}$.

Also, by direct calculus, we get

$$G(X,PY) = G(PX,PY) = G(X,Y) - \zeta(X)\zeta(Y),\ \ \forall\, X,Y \in \Gamma(V). \tag{24}$$

With respect to the basis $\{\theta^i, k_i\}$ and $\left\{\theta^j \otimes \frac{\partial}{\partial y^i}, \theta^j \otimes \frac{\partial}{\partial p_i}, k_j \otimes \frac{\partial}{\partial y^i}, k_j \otimes \frac{\partial}{\partial p_i}\right\}$, respectively, ζ and P are locally given by

$$\zeta = \zeta_i\theta^i + \zeta^i k_i\,,\ P = \overset{1}{P}{}^i_j\,\theta^j \otimes \frac{\partial}{\partial y^i} + \overset{2}{P}{}^j_i\,k_j \otimes \frac{\partial}{\partial p_i} + \overset{3}{P}_{ij}\,\theta^j \otimes \frac{\partial}{\partial p_i} + \overset{4}{P}{}^{ij}\,k_j \otimes \frac{\partial}{\partial y^i}, \tag{25}$$

where their local components are expressed by

$$\zeta_i = \frac{y_i}{\sqrt{F^2+K^2}}\,,\ \zeta^i = \frac{p^i}{\sqrt{F^2+K^2}}, \tag{26}$$

$$\overset{1}{P}{}^i_j = \delta^i_j - \frac{y_j y^i}{F^2+K^2}\,,\ \overset{2}{P}{}^i_j = \delta^i_j - \frac{p^i p_j}{F^2+K^2}\,,\ \overset{3}{P}_{ij} = -\frac{y_j p_i}{F^2+K^2}\,,\ \overset{4}{P}{}^{ij} = -\frac{p^j y^i}{F^2+K^2}. \tag{27}$$

Remark 2.1. *We have the following relations between ζ, P, ζ_1, ζ_2, P_1 and P_2:*

$$\zeta(X) = \frac{F}{\sqrt{F^2+K^2}}\zeta_1(X_1) + \frac{K}{\sqrt{F^2+K^2}}\zeta_2(X_2), \tag{28}$$

$$P(X) = P_1(X_1) + P_2(X_2) + \frac{1}{F^2+K^2}\left(\frac{\zeta_1(X_1)}{F} - \frac{\zeta_2(X_2)}{K}\right)(K^2\mathcal{E}_1 - F^2\mathcal{E}_2), \tag{29}$$

for every vertical vector field $X = X_1 + X_2 = X_1^i(x,y,p)\frac{\partial}{\partial y^i} + X_i^2(x,y,p)\frac{\partial}{\partial p_i}$.

Theorem 2.2. *The vertical Liouville distribution $V_{\mathcal{E}}$ is integrable and it defines a foliation on $\mathcal{T}M$, called vertical Liouville foliation on the big-tangent manifold $\mathcal{T}M$.*

Proof. Follows using an argument similar to that used in [4]. Let $X, Y \in \Gamma(V_{\mathcal{E}})$. As V is an integrable distribution on $\mathcal{T}M$, it is sufficient to prove that $[X, Y]$ has no component with respect to $\mathcal{E}$.

It is easy to see that a vertical vector field $X = X_1^i(x, y, p)\frac{\partial}{\partial y^i} + X_i^2(x, y, p)\frac{\partial}{\partial p_i}$ is in $\Gamma(V_{\mathcal{E}})$ if and only if

$$g_{ij}(x, y)X_1^i y^j + g^{*ij}(x, p)X_i^2 p_j = 0. \tag{30}$$

Differentiating (30) with respect to y^k we get

$$\frac{\partial g_{ij}}{\partial y^k}X_1^i y^j + g_{ik}X_1^i + g_{ij}\frac{\partial X_1^i}{\partial y^k}y^j + g^{*ij}p_j\frac{\partial X_i^2}{\partial y^k} = 0\,,\ \forall\, k = 1, \ldots, n \tag{31}$$

and taking into account the relation $\frac{\partial g_{ij}}{\partial y^k}y^j = 0$ (see (5)), one gets

$$g_{ik}X_1^i + g_{ij}y^j\frac{\partial X_1^i}{\partial y^k} + g^{*ij}p_j\frac{\partial X_i^2}{\partial y^k} = 0\,,\ \forall\, k = 1, \ldots, n. \tag{32}$$

Similarly, differentiating (30) with respect to p_k we get

$$g_{ij}y^j\frac{\partial X_1^i}{\partial p_k} + g^{*ik}X_i^2 + \frac{\partial g^{*ij}}{\partial p_k}X_i^2 p_j + g^{*ij}p_j\frac{\partial X_i^2}{\partial p_k} = 0\,,\ \forall\, k = 1, \ldots, n \tag{33}$$

and taking into account the relation $\frac{\partial g^{*ij}}{\partial p_k}p_j = 0$ (see (6)), one gets

$$g^{*ik}X_i^2 + g_{ij}y^j\frac{\partial X_1^i}{\partial p_k} + g^{*ij}p_j\frac{\partial X_i^2}{\partial p_k} = 0\,,\ \forall\, k = 1, \ldots, n. \tag{34}$$

Let $X = X_1^i(x, y, p)\frac{\partial}{\partial y^i} + X_i^2(x, y, p)\frac{\partial}{\partial p_i}$, $Y = Y_1^j(x, y, p)\frac{\partial}{\partial y^j} + Y_j^2(x, y, p)\frac{\partial}{\partial p_j} \in \Gamma(V)$. Then, by direct calculations using (32) and (34), we have

$$\begin{aligned} G([X, Y], \mathcal{E}) &= g_{jk}y^k\left(X_1^i\frac{\partial Y_1^j}{\partial y^i} - Y_1^i\frac{\partial X_1^j}{\partial y^i}\right) + g^{*ik}p_k X_1^j\frac{\partial Y_i^2}{\partial y^j} - g_{ik}y^k Y_j^2\frac{\partial X_1^i}{\partial p_j} \\ &\quad + g_{ik}y^k X_j^2\frac{\partial Y_1^i}{\partial p_j} - g^{*ik}p_k Y_1^j\frac{\partial X_i^2}{\partial y^j} + g^{*ik}p_k\left(X_j^2\frac{\partial Y_i^2}{\partial p_j} - Y_j^2\frac{\partial X_i^2}{\partial p_j}\right) \\ &= -g_{ij}Y_1^i X_1^j + g_{ij}X_1^i Y_1^j - g^{*ij}Y_i^2 X_j^2 + g^{*ij}X_i^2 Y_j^2 \\ &= 0 \end{aligned}$$

which completes the proof. □

Remark 2.3. *The proof of Theorem 2.2 can be also obtained using an argument similar to [7]. More exactly, if we consider $P(\frac{\partial}{\partial y^j}) = \overset{1}{P}{}^i_j\frac{\partial}{\partial y^i} + \overset{3}{P}_{ij}\frac{\partial}{\partial p_i}$ and $P(\frac{\partial}{\partial p_j}) = \overset{4}{P}{}^{ij}\frac{\partial}{\partial y^i} + \overset{2}{P}{}^j_i\frac{\partial}{\partial p_i}$, by direct calculus we obtain*

$$P(\frac{\partial}{\partial y^j})(\sqrt{F^2 + K^2}) = P(\frac{\partial}{\partial p_j})(\sqrt{F^2 + K^2}) = 0. \tag{35}$$

Now, since $V = V_{\mathcal{E}} \oplus \{\mathcal{E}\}$ is integrable, the Lie brackets of vector fields from $V_{\mathcal{E}}$ are given by

$$\left[P(\frac{\partial}{\partial y^i}), P(\frac{\partial}{\partial y^j})\right] = A_{ij}^k P(\frac{\partial}{\partial y^k}) + B_{ijk}P(\frac{\partial}{\partial p_k}) + C_{ij}\mathcal{E}, \tag{36}$$

$$\left[P(\frac{\partial}{\partial y^i}),P(\frac{\partial}{\partial p_j})\right]=D_i^{jk}P(\frac{\partial}{\partial y^k})+E_{ik}^jP(\frac{\partial}{\partial p_k})+F_i^j\mathcal{E},\tag{37}$$

$$\left[P(\frac{\partial}{\partial p_i}),P(\frac{\partial}{\partial p_j})\right]=G^{ijk}P(\frac{\partial}{\partial y^k})+H_k^{ij}P(\frac{\partial}{\partial p_k})+L^{ij}\mathcal{E},\tag{38}$$

for some locally defined functions A_{ij}^k, B_{ijk}, C_{ij}, D_i^{jk}, E_{ik}^j, F_i^j, G^{ijk}, H_k^{ij} and L^{ij}, respectively. We notice that by the homogeneity condition of F and K we have $\mathcal{E}(\sqrt{F^2+K^2})=\sqrt{F^2+K^2}$. Now, if we apply the vector fields in both sides of formulas (36), (37) *and* (38) *to the function $\sqrt{F^2+K^2}$ and using* (35), *we obtain $C_{ij}\sqrt{F^2+K^2}=F_i^j\sqrt{F^2+K^2}=L^{ij}\sqrt{F^2+K^2}=0$. This implies that $C_{ij}=F_i^j=L^{ij}=0$, and then the vertical Liouville distribution $V_{\mathcal{E}}$ is integrable.*

As usual, the Theorem 2.2, we may say that the geometry of the leaves of vertical foliation V should be derived from the geometry of the leaves of vertical Liouville foliation $V_{\mathcal{E}}$ and of integral curves of $\mathcal{E}$. In order to obtain this interplay, we consider a leaf L_V of V given locally by $x^i=a^i$, $i=1,\ldots,n$, where the a^i's are constants. Then, $g_{ij}(a,y)$ and $g^{*ij}(a,p)$ are the components of a Riemannian metric $G_{L_V}=G|_{L_V}$ on L_V. If we denote by ∇ the Levi-Civita connection on L_V with respect to G_{L_V} then its local expression is

$$\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=C_{ij}^k(a,y)\frac{\partial}{\partial y^k}\,,\ \nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j}=0\,,\ \nabla_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial y^j}=0\,,\ \nabla_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j}=C_k^{ij}(a,p)\frac{\partial}{\partial p_k},\tag{39}$$

where $C_{ij}^k(a,y)=\frac{1}{2}g^{lk}(a,y)\frac{\partial g_{jl}(a,y)}{\partial y^i}$ and $C_k^{ij}(a,p)=-\frac{1}{2}g^*_{lk}(a,p)\frac{\partial g^{*jl}(a,p)}{\partial p_i}$.

Contracting $C_{ij}^k(a,y)$ by y^j and $C_k^{ij}(a,p)$ by p_j, respectively, we deduce

$$C_{ij}^k(a,y)y^j=0\ ,\ C_k^{ij}(a,p)p_j=0.\tag{40}$$

In the following lemma we obtain the covariant derivatives with respect to ∇ of $\mathcal{E}$, ζ and P, respectively.

Lemma 2.4. *On any leaf L_V of V, we have*

$$\nabla_X\left(\frac{\mathcal{E}}{\sqrt{F^2+K^2}}\right)=\frac{PX}{\sqrt{F^2+K^2}},\tag{41}$$

$$(\nabla_X\zeta)\,Y=\frac{1}{\sqrt{F^2+K^2}}G_{L_V}(PX,PY),\tag{42}$$

and

$$(\nabla_XP)\,Y=-\frac{1}{F^2+K^2}\left[G_{L_V}(PX,PY)\mathcal{E}+\sqrt{F^2+K^2}\zeta(Y)PX\right]\tag{43}$$

for any $X,Y\in\Gamma(TL_V)$.

Proof. We take $X=X_1^i(a,y,p)\frac{\partial}{\partial y^i}+X_i^2(a,y,p)\frac{\partial}{\partial p_i}$, $Y=Y_1^j(a,y,p)\frac{\partial}{\partial y^j}+Y_j^2(a,y,p)\frac{\partial}{\partial p_j}\in\Gamma(TL_V)$ and the relation (41) follows by:

$$\begin{aligned}\nabla_X\left(\frac{\mathcal{E}}{\sqrt{F^2+K^2}}\right) &= \frac{X_1^i}{\sqrt{F^2+K^2}}\left(\delta_i^j-\frac{y^jy_i}{F^2+K^2}\frac{\partial}{\partial y^j}-\frac{p_jy_i}{F^2+K^2}\frac{\partial}{\partial p_j}\right)\\ &\quad+\frac{X_i^2}{\sqrt{F^2+K^2}}\left(\delta_j^i-\frac{p_jp^i}{F^2+K^2}\frac{\partial}{\partial p_j}-\frac{y^jp^i}{F^2+K^2}\frac{\partial}{\partial y^j}\right)\\ &= \frac{1}{\sqrt{F^2+K^2}}\left(X_1^i\overset{1}{P}{}_i^j\frac{\partial}{\partial y^j}+X_1^i\overset{3}{P}{}_{ji}\frac{\partial}{\partial p_j}+X_i^2\overset{4}{P}{}^{ji}\frac{\partial}{\partial y^j}+X_i^2\overset{2}{P}{}_j^i\frac{\partial}{\partial p_j}\right)\\ &= \frac{PX}{\sqrt{F^2+K^2}}.\end{aligned}$$

For the relation (42) we have

$$\begin{aligned}(\nabla_X\zeta)Y &= X(\zeta(Y))-\zeta(\nabla_XY)\\ &= X_1^iY_1^j\frac{\partial\zeta_j}{\partial y^i}+X_1^iY_j^2\frac{\partial\zeta^j}{\partial y^i}+X_i^2Y_1^j\frac{\partial\zeta_j}{\partial p_i}+X_i^2Y_j^2\frac{\partial\zeta^j}{\partial p_i}\\ &= \frac{X_1^iY_1^j}{\sqrt{F^2+K^2}}\left(g_{ij}-\frac{y_iy_j}{F^2+K^2}\right)-\frac{X_1^iY_j^2p^jy_i}{(F^2+K^2)\sqrt{F^2+K^2}}\\ &\quad-\frac{X_i^2Y_1^jy_jp^i}{(F^2+K^2)\sqrt{F^2+K^2}}+\frac{X_i^2Y_j^2}{\sqrt{F^2+K^2}}\left(g^{*ij}-\frac{p^jp^i}{F^2+K^2}\right).\end{aligned}$$

On the other hand we have

$$\begin{aligned}G_{L_V}(PX,PY) &= G_{L_V}(X,Y)-\zeta(X)\zeta(Y)\\ &= X_1^iY_1^jg_{ij}+X_i^2Y_j^2g^{*ij}-\frac{(X_1^iy_i+X_i^2p^i)(Y_1^jy_j+Y_j^2p^j)}{F^2+K^2}\end{aligned}$$

and the relation (42) follows easily.

The relation (43) follows using (23), (41) and (42). □

Theorem 2.5. *Let (M,F) be a n-dimensional Finsler space and L_V, $L_{V_\mathcal{E}}$ and γ be a leaf of V, a leaf of $V_\mathcal{E}$ that lies in L_V, and an integral curve of $\frac{\mathcal{E}}{\sqrt{F^2+K^2}}$, respectively. Then the following assertions are valid:*

i) γ is a geodesic of L_V with respect to ∇.
ii) $L_{V_\mathcal{E}}$ is totally umbilical immersed in L_V.
iii) $L_{V_\mathcal{E}}$ lies in the generalized indicatrix $I_a=\{(y,p)\in T_aM^0\oplus T_a^*M^0\ :\ F^2(a,y)+K^2(a,p)=1\}$ and has constant mean curvature equal to -1.

Proof. Replace X by $\frac{\mathcal{E}}{\sqrt{F^2+K^2}}$ in (41) and we obtain i). Taking into account that $\frac{\mathcal{E}}{\sqrt{F^2+K^2}}$ is the unit normal vector field of $L_{V_\mathcal{E}}$, the second fundamental form B of $L_{V_\mathcal{E}}$ as a hypersurface of L_V is given by

$$B(X,Y)=\frac{1}{\sqrt{F^2+K^2}}G_{L_V}(\nabla_XY,\mathcal{E})\ ,\ \forall\,X,Y\in\Gamma(TL_{V_\mathcal{E}}). \tag{44}$$

On the other hand, by using (41) and taking into account that G_{L_V} is parallel with respect to ∇, we deduce that

$$G_{L_V}(\nabla_XY,\mathcal{E})=-G_{L_V}(X,Y)\,,\ \forall\,X,Y\in\Gamma(TL_{V_\mathcal{E}}). \tag{45}$$

Hence,

$$B(X,Y)=-\frac{1}{\sqrt{F^2+K^2}}G_{L_V}(X,Y),\ \forall\,X,Y\in\Gamma(TL_{V_\mathcal{E}}), \tag{46}$$

that is, $L_{V_\mathcal{E}}$ is totally umbilical immersed in L_V. Now, we have

$$\frac{g_{ij}y^i}{\sqrt{F^2+K^2}}+\frac{g^{*ij}p_i}{\sqrt{F^2+K^2}}=\frac{\partial\sqrt{F^2+K^2}}{\partial y^j}+\frac{\partial\sqrt{F^2+K^2}}{\partial p_j} \tag{47}$$

which says that $\frac{\mathcal{E}}{\sqrt{F^2+K^2}}$ is a unit normal vector field for both $L_{V_\mathcal{E}}$ and the component I_a. Thus, $L_{V_\mathcal{E}}$ lies in I_a and $F^2(a,y)+K^2(a,p)=1$ at any point $(y,p)\in L_{V_\mathcal{E}}$. Then (46) becomes

$$B(X,Y)=-G_{L_V}(X,Y),\ \forall\,X,Y\in\Gamma(TL_{V_\mathcal{E}}) \tag{48}$$

which implies that

$$\frac{1}{2n-1}\sum_{i=1}^{2n-1}\varepsilon_i B(E_i,E_i)=-1, \tag{49}$$

where $\{E_i\}$ ia an orthonormal frame field on $L_{V_{\mathcal{E}}}$ of signature $\{\varepsilon_i\}$. Hence, the mean curvature of $L_{V_{\mathcal{E}}}$ is -1 which completes the proof. □

Proposition 2.6. *Let (M,F) be a n-dimensional Finsler space and L_V be a leaf of the vertical foliation V. Then the sectional curvature of any nondegenerate plane section on L_V which contain the vertical Liouville vector field $\mathcal{E}$ is equal to zero.*

Proof. Denote by R_{L_V} the curvature tensor field of ∇ on L_V. Then, by using (41) and (43), we obtain

$$R_{L_V}(X,\mathcal{E})\mathcal{E}=-\left(1-\frac{\mathcal{E}(\sqrt{F^2+K^2})}{\sqrt{F^2+K^2}}\right)PX \tag{50}$$

for every vector field X on L_V. Now, taking into account $\mathcal{E}(\sqrt{F^2+K^2})=\sqrt{F^2+K^2}$, the sectional curvature of a plane section $\{X,\mathcal{E}\}$ vanishes on L_V. □

Remark 2.7. *Let (M,F) be a n-dimensional Finsler space. Then there exist no leaves of V which are positively or negatively curved.*

Finally, let us study certain relations between the vertical Liouville foliations $V_{\mathcal{E}_1}$, $V_{\mathcal{E}_2}$ and $V_{\mathcal{E}}$, respectively.

We notice that we have the following decompositions of the vertical distribution:

$$V=V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}\oplus\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\} \text{ and } V=V_{\mathcal{E}}\oplus\{\mathcal{E}\}. \tag{51}$$

Taking into account that $[\overset{1}{P}{}^i_j\frac{\partial}{\partial y^i},\overset{2}{P}{}^k_l\frac{\partial}{\partial p_l}]=0$ and $[\mathcal{E}_1,\mathcal{E}_2]=0$ we get that both distributions $V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}$ and $\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\}$ are integrable. Evidently, $\{\mathcal{E}\}\subset\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\}$ and by (28) we have also $V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}\subset V_{\mathcal{E}}$. Thus, we have the following vertical subfoliations on $\mathcal{T}M$:

$$\{\mathcal{E}\}\subset\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\}\subset V\,,\; V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}\subset V_{\mathcal{E}}\subset V. \tag{52}$$

The relations (51) says that $\{\mathcal{E}\}$ and $V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}$ have the same orthogonal complement in $\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\}$ and in $V_{\mathcal{E}}$, respectively. It is a line distribution $\{\mathcal{E}'\}$, where $\mathcal{E}'=K^2\mathcal{E}_1-F^2\mathcal{E}_2$, see (29) (or by direct calculations in $G(\alpha_1\mathcal{E}_1+\alpha_2\mathcal{E}_2,\mathcal{E})=0$ it results $\alpha_1=K^2$ and $\alpha_2=-F^2$). Thus

$$\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\}=\{\mathcal{E}\}\oplus\{\mathcal{E}'\}\,,\; V_{\mathcal{E}}=V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}\oplus\{\mathcal{E}'\}. \tag{53}$$

Proposition 2.8. *The leaves of the foliation $\{\mathcal{E}_1\}\oplus\{\mathcal{E}_2\}$ are totally geodesic submanifolds of the leaves of vertical foliation V.*

Proof. Follows easily taking into account that $\nabla_{\mathcal{E}_1}\mathcal{E}_1=\mathcal{E}_1$, $\nabla_{\mathcal{E}_1}\mathcal{E}_2=\nabla_{\mathcal{E}_2}\mathcal{E}_1=0$, $\nabla_{\mathcal{E}_2}\mathcal{E}_2=\mathcal{E}_2$. □

Also by direct calculus we obtain $\nabla_{\mathcal{E}'}\mathcal{E}'=-K^2F^2\mathcal{E}+(K^2-F^2)\mathcal{E}'\notin\Gamma(\{\mathcal{E}'\})$, which leads to

Proposition 2.9. *If γ is an integral curve of $\mathcal{E}'$ then it is not a geodesic of a leaf of vertical foliation V.*

A natural question is if between the foliations $V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}$ and $V_{\mathcal{E}}$ exists certain relations. Although the leaves of $V_{\mathcal{E}_1}$ are totally umbilical submanifolds of the leaves of V_1, the leaves of $V_{\mathcal{E}_2}$ are totally umbilical submanifolds of the leaves of V_2 and the leaves of $V_{\mathcal{E}}$ are totally umbilical submanifolds of the leaves of V, we have

Theorem 2.10. *The leaves of $V_{\mathcal{E}_1} \oplus V_{\mathcal{E}_2}$ are not totally umbilical submanifolds of the leaves of $V_{\mathcal{E}}$.*

Proof. Taking into account that $\frac{\mathcal{E}'}{FK\sqrt{F^2+K^2}}$ is the unit normal vector field of $L_{V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}}$, the second fundamental form B' of $L_{V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}}$ as hypersurface of $L_{V_{\mathcal{E}}}$ is given by

$$B'(X',Y') = \frac{1}{FK\sqrt{F^2+K^2}} G_{L_V}\left(\nabla_{X'}Y', \mathcal{E}'\right),\ \forall\, X',Y' \in \Gamma\left(TL_{V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}}\right). \tag{54}$$

Taking into account that G_{L_V} is parallel with respect to ∇, we deduce that

$$G_{L_V}\left(\nabla_{X'}Y', \mathcal{E}'\right) = -G_{L_V}(Y', \nabla_{X'}\mathcal{E}'),\ \forall\, X',Y' \in \Gamma\left(TL_{V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}}\right). \tag{55}$$

Now, let us take $X' = P_1(X_1) + P_2(X_2)$ and $Y' = P_1(Y_1) + P_2(Y_2)$ for every $X_1, Y_1 \in \Gamma(V_1)$ and $X_2, Y_2 \in \Gamma(V_2)$. Then by direct calculus we get

$$\nabla_{X'}\mathcal{E}' = K^2 P_1(X_1) - F^2 P_2(X_2). \tag{56}$$

Thus the relation (54) becomes

$$B'(X',Y') = \frac{-1}{FK\sqrt{F^2+K^2}} G_{L_V}\left(K^2 P_1(X_1) - F^2 P_2(X_2), Y'\right) \neq \lambda G_{L_V}(X',Y'), \tag{57}$$

that is, $L_{V_{\mathcal{E}_1}\oplus V_{\mathcal{E}_2}}$ is not totally umbilical immersed in $L_{V_{\mathcal{E}}}$. □

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