



## On Regularly Weighted Generated Sequences

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**Abstract.** We introduce a new concept which is called a regularly weighted generated sequence for sequences of real numbers. Moreover, we obtain some Tauberian conditions in terms of regularly weighted generated sequences for the power series method of summability, and generalize some classical Tauberian theorems given by Tietz [Acta Sci. Math. 54 (3-4) 355–365 (1990)].

### 1. Introduction

Let  $u = (u_n)$  be a real sequence. Let  $c_0$  and  $\ell_\infty$  denote the space of sequences converging to 0 and bounded sequences, respectively. Let  $\mathcal{B}_>$  denote the set of all sequences  $\alpha = (\alpha_n)$  such that for every  $(\alpha_n) \in \mathcal{B}_>$  there exists  $C > 0$  such that  $\alpha_n \geq -C$ .

Assume that  $p = (p_n)$  be a sequence of nonnegative numbers with  $p_0 > 0$ , that

$$P_n := \sum_{k=0}^n p_k \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and that

$$p(x) = \sum_{k=0}^{\infty} p_k x^k < \infty \text{ for } 0 \leq x < 1.$$

The  $n$ -th weighted mean of  $(u_n)$  is defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n} \sum_{k=0}^n p_k u_k.$$

A sequence  $(u_n)$  is said to be summable by the weighted mean method determined by the sequence  $p$ , in short,  $(\overline{N}, p)$  summable to a finite number  $s$  if

$$\lim_{n \rightarrow \infty} \sigma_{n,p}^{(1)}(u) = s. \tag{1}$$

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If  $p_n = 1$  for all nonnegative  $n$ , then  $(\overline{N}, p)$  summability method reduces to Cesàro summability method. If  $\sum_{k=0}^{\infty} p_k u_k x^k$  is convergent for  $0 \leq x < 1$ , and

$$\lim_{x \rightarrow 1^-} \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k u_k x^k = s, \tag{2}$$

we say that  $(u_n)$  is summable to  $s$  by the power series method  $(J, p)$ , and we write  $u_n \rightarrow s (J, p)$ .

If  $p_n = 1$  for all nonnegative  $n$ , then the  $(J, p)$  summability method reduces to Abel summability method.

The sequence  $\Delta u = (\Delta u_n)$ , which is the backward difference of  $(u_n)$ , is defined by  $\Delta u_n = u_n - u_{n-1}$  for  $n \geq 1$  and  $\Delta u_0 = u_0$ .

For any nonnegative integer  $m$ , we define  $\Delta_m u_n = \Delta(\Delta_{m-1} u_n) = \Delta_{m-1}(\Delta u_n)$  with  $\Delta_0 u_n = u_n$ .

A sequence  $(\mu_n)$  of real numbers is called totally monotone if  $\Delta_m \mu_n \geq 0$  for all nonnegative integers  $m$  and  $n$ .

Baron and Tietz [1] proved that if  $(u_n)$  is  $(J, p)$  summable to  $s$  and  $(\frac{p_n}{P_n})$  is totally monotone, then  $(\sigma_{n,p}^{(1)}(u))$  is  $(J, p)$  summable to  $s$ .

The difference between  $u_n$  and  $\sigma_{n,p}^{(1)}(u)$ , which is called the weighted Kronecker identity [2], is given by

$$u_n - \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u) \tag{3}$$

where  $V_{n,p}^{(0)}(\Delta u) = \frac{1}{P_n} \sum_{k=1}^n P_{k-1} \Delta u_k$ .

For each integer  $m \geq 0$ , we define  $\sigma_{n,p}^{(m)}(u)$  and  $V_{n,p}^{(m)}(\Delta u)$  by

$$\sigma_{n,p}^{(m)}(u) = \begin{cases} \frac{1}{P_n} \sum_{k=0}^n p_k \sigma_{k,p}^{(m-1)}(u) & , m \geq 1 \\ u_n & , m = 0 \end{cases}$$

and

$$V_{n,p}^{(m)}(\Delta u) = \begin{cases} \frac{1}{P_n} \sum_{k=0}^n p_k V_{k,p}^{(m-1)}(\Delta u) & , m \geq 1 \\ V_{n,p}^{(0)}(\Delta u) & , m = 0 \end{cases}$$

respectively.

The weighted classical control modulo of  $(u_n)$  is defined by  $\omega_{n,p}^{(0)}(u) = \frac{P_{n-1}}{p_n} \Delta u_n$  and the weighted general control modulo of integer order  $m \geq 1$  of  $(u_n)$  is defined by  $\omega_{n,p}^{(m)}(u) = \omega_{n,p}^{(m-1)}(u) - \sigma_{n,p}^{(1)}(\omega_{n,p}^{(m-1)}(u))$  (see [2]).

For a sequence  $(u_n)$  and any integer  $m \geq 1$ , the identities

$$\frac{P_{n-1}}{p_n} \Delta \sigma_{n,p}^{(m)}(u) = V_{n,p}^{(m-1)}(\Delta u),$$

and

$$\sigma_{n,p}^{(1)} \left( \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(m-1)}(\Delta u) \right) = \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(m)}(\Delta u),$$

and

$$V_{n,p}^{(m-1)}(\Delta u) - V_{n,p}^{(m)}(\Delta u) = \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(m)}(\Delta u)$$

are given in [8].

A sequence  $(u_n)$  is called slowly oscillating [6] if

$$(u_m - u_n) \in c_0, \text{ for } \frac{P_m}{P_n} \rightarrow 1, (m > n \rightarrow \infty).$$

Denote by  $\mathcal{SO}$  the class of all slowly oscillating sequences.

If  $(\omega_{n,p}^{(0)}(u)) \in \ell_\infty$  with the condition

$$1 \leq \frac{P_m}{P_n} \rightarrow 1 \text{ when } 1 < \frac{m}{n} \rightarrow 1 (n \rightarrow \infty). \tag{4}$$

holds, then  $(u_n) \in \mathcal{SO}$ . Indeed,  $(\omega_{n,p}^{(0)}(u)) \in \ell_\infty$  implies that  $\left| \frac{P_{n-1}}{P_n} \Delta u_n \right| \leq C$ , for some  $C > 0$ . Therefore,

$$|u_m - u_n| \leq C \sum_{j=n+1}^m \frac{p_j}{P_{j-1}} \leq C \left( \frac{P_m}{P_n} - 1 \right), \text{ for } m > n.$$

A sequence  $(u_n)$  is said to be slowly decreasing if

$$\liminf_{n \rightarrow \infty} (u_m - u_n) \geq 0, \text{ for } \frac{P_m}{P_n} \rightarrow 1.$$

Denote by  $\mathcal{SD}$  the class of all slowly decreasing sequences.

If  $(\omega_{n,p}^{(0)}(u)) \in \mathcal{B}_>$  with the condition (4) holds, then  $(u_n) \in \mathcal{SD}$ . Indeed,  $(\omega_{n,p}^{(0)}(u)) \in \mathcal{B}_>$  implies that  $\left( \frac{P_{n-1}}{P_n} \Delta u_n \right) \geq -C$ , for some  $C > 0$ . Therefore,

$$u_m - u_n \geq -C \sum_{j=n+1}^m \frac{p_j}{P_{j-1}} \geq C \left( \frac{P_m}{P_n} - 1 \right), \text{ for } m > n.$$

In the classical Tauberian theory, the convergence retrieval problem of the sequence  $(u_n)$  out of the existence of a generalized limit, such as (1) and (2) with some conditions, so-called Tauberian conditions, on  $(u_n)$  reduces to proving that

$$\lim_{n \rightarrow \infty} u_n = s. \tag{5}$$

Hardy's theorem [5] asserts that if the limit (1) exists and  $(\omega_{n,p}^{(0)}(u)) \in \ell_\infty$ , then (5) holds. Çanak and Totur [2] obtained a one-sided Tauberian theorem that if the limit (1) exists and  $(\omega_{n,p}^{(1)}(u)) \in \mathcal{B}_>$  with certain conditions on  $(p_n)$ , then (5) holds. Çanak and Totur [3] introduced a Tauberian condition of slowly oscillating type for the power series method  $(J, p)$ .

Tietz [7] gave some significant generalizations of Hardy's theorem and obtained some classical Tauberian theorems which are analogous to Hardy-Littlewood's theorem [4] and Schmidt's theorem [6] for the power series method of summability.

**Theorem 1.1.** [7, Theorem 4.1] *Let the condition (4) be satisfied. If  $(u_n)$  is  $(J, p)$  summable to  $s$  and  $(\omega_{n,p}^{(0)}(u)) \in \mathcal{B}_>$ , then  $(u_n)$  converges to  $s$ .*

**Theorem 1.2.** [7, Theorem 3.9] *Let the condition (4) be satisfied. If  $(u_n)$  is  $(J, p)$  summable to  $s$  and  $(u_n) \in \mathcal{SD}$ , then  $(u_n)$  converges to  $s$ .*

Notice that Theorem 1.1 generalizes Theorem 1.2.

## 2. Regularly Weighted Generated Sequences

Now, we introduce a new concept for sequences of real numbers.

**Definition 2.1.** Let  $\mathcal{L}$  be any linear space of sequences. If there exists a subclass  $\mathcal{A}$  of  $\mathcal{L}$  such that

$$u_n = \alpha_n + \sum_{k=1}^n \frac{p_k}{P_{k-1}} \alpha_k + u_0$$

for some  $(\alpha_n) \in \mathcal{A}$  and for all nonnegative integers  $n$ , then we say that the sequence  $(u_n)$  is regularly weighted generated by the sequence  $(\alpha_n)$ , and  $(\alpha_n)$  is called a weighted generator of  $(u_n)$ .

The class of all sequences which are regularly weighted generated by  $(\alpha_n)$  in  $\mathcal{A}$  is denoted by  $U(\mathcal{A})$ .

Notice that since  $\sigma_{n,p}^{(1)}(u) = u_0 + \sum_{k=1}^n \frac{p_k}{P_{k-1}} V_{k,p}^{(0)}(\Delta u)$ , the identity (3) can be rewritten as

$$u_n = V_{n,p}^{(0)}(\Delta u) + \sum_{k=1}^n \frac{p_k}{P_{k-1}} V_{k,p}^{(0)}(\Delta u) + u_0. \tag{6}$$

We see by (6) that the sequence  $(V_{n,p}^{(0)}(\Delta u))$  is a weighted generator of  $(u_n)$ .

**Example 2.2.**

- (a) If  $\mathcal{A}$  is the class of all bounded and slowly oscillating sequences, then  $U(\mathcal{A})$  is the class of all slowly oscillating sequences with the condition (4).
- (b) If  $\mathcal{A}$  is the class of one-sided bounded sequences, then  $U(\mathcal{A})$  is the class of all slowly decreasing sequences with the condition (4).

The representation (6) of the sequence  $(u_n)$  suggests that we set conditions on the weighted generator sequence  $(V_{n,p}^{(0)}(\Delta u))$  of  $(u_n)$  rather than the sequence  $(u_n)$  itself to obtain some classical Tauberian conditions. Furthermore, the representation (6) of  $(u_n)$  reveals some information about the structure of the sequences on which we should impose Tauberian conditions to get convergence of the sequence  $(u_n)$  out of the power series method of summability of  $(u_n)$ .

**3. Main Results**

Our first result is the following theorem similar to classical Tauberian theorems.

**Theorem 3.1.** Let  $(p_n)$  satisfy the condition (4), and let  $(\frac{p_n}{P_n})$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$  and  $(u_n) \in U(\mathcal{SD})$ , then  $(u_n)$  converges to  $s$ .

*Proof.* Since  $(u_n) \in U(\mathcal{SD})$ ,  $u_n = \alpha_n + \sum_{k=1}^n \frac{p_k}{P_{k-1}} \alpha_k + u_0$ , for some  $(\alpha_n) \in \mathcal{SD}$ . Hence we have

$$\frac{P_{n-1}}{p_n} \Delta u_n - \sigma_{n,p}^{(1)} \left( \frac{P_{n-1}}{p_n} \Delta u_n \right) = \frac{P_{n-1}}{p_n} \Delta \alpha_n.$$

It follows from the identity

$$\frac{P_{n-1}}{p_n} \Delta u_n - \frac{P_{n-1}}{p_n} \Delta \sigma_{n,p}^{(1)}(u) = \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u), \tag{7}$$

that we get  $\frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) = \frac{P_{n-1}}{p_n} \Delta \alpha_n$ . Taking the weighted means of (7), we have

$$\frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) = V_{n,p}^{(0)}(\Delta \alpha).$$

Since  $(\alpha_n) \in \mathcal{SD}$ , by Example 2.2, it follows  $(V_{n,p}^{(0)}(\Delta \alpha)) \in \mathcal{B}_>$ .

Therefore,

$$\left(\frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u)\right) \in \mathcal{B}_{>}. \tag{8}$$

Since  $(u_n)$  is  $(J, p)$  summable to  $s$  and  $\left(\frac{p_n}{P_n}\right)$  is totally monotone,  $(V_{n,p}^{(1)}(\Delta u))$  is  $(J, p)$  summable to 0 by the weighted Kronecker identity. Hence, by (8), Theorem 1.1 yields

$$(V_{n,p}^{(1)}(\Delta u)) \in c_0. \tag{9}$$

Since  $(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{B}_{>}$ , it follows from the identity

$$\frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) = V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)$$

that we obtain  $(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{B}_{>}$ , by (8) and (9). Thus,

$$(\sigma_{n,p}^{(1)}(u)) \in \mathcal{SD}. \tag{10}$$

On the other hand, since  $(u_n) \in U(\mathcal{SD})$ , we have

$$(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{SD} \tag{11}$$

from the identity (6). It follows by (10) and (11) that  $(u_n) \in \mathcal{SD}$ . Consequently, the proof is completed by Theorem 1.2.  $\square$

**Corollary 3.2.** *Let  $(p_n)$  satisfy the condition (4), and let  $\left(\frac{p_n}{P_n}\right)$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$ , and  $(\omega_{n,p}^{(0)}(u)) \in U(\mathcal{B}_{>})$ , then  $(u_n)$  converges to  $s$ .*

*Proof.* Since  $(\omega_{n,p}^{(0)}(u)) \in U(\mathcal{B}_{>})$ ,  $\omega_{n,p}^{(0)}(u) = \alpha_n + \sum_{k=1}^n \frac{p_n}{P_{n-1}} \alpha_k + u_0$ , for some  $(\alpha_n) \in \mathcal{B}_{>}$ . Hence we have

$$\frac{P_{n-1}}{p_n} \Delta \omega_{n,p}^{(0)}(u) = \frac{P_{n-1}}{p_n} \Delta \alpha_n + \alpha_n.$$

It follows from the identity  $\frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) = \alpha_n$  that

$$\left(\frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u)\right) \in \mathcal{B}_{>}. \tag{12}$$

The condition (12) implies that  $(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{SD}$ . Hence, it follows from the identity (6) that  $(u_n) \in U(\mathcal{SD})$ .  $\square$

**Corollary 3.3.** *Let  $(p_n)$  satisfy the condition (4), and let  $\left(\frac{p_n}{P_n}\right)$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$ , and  $(u_n) \in U(\mathcal{SO})$ , then  $(u_n)$  converges to  $s$ .*

**Corollary 3.4.** *Let  $(p_n)$  satisfy the condition (4), and let  $\left(\frac{p_n}{P_n}\right)$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$ , and  $(u_n) \in \mathcal{SO}$ , then  $(u_n)$  converges to  $s$ .*

*Proof.* Since  $(u_n) \in \mathcal{SO}$ , then  $(V_n^{(0)}(\Delta u)) \in \mathcal{SO}$ . Hence, it follows from the identity (6) that  $(u_n) \in U(\mathcal{SO})$ .  $\square$

**Theorem 3.5.** *Let  $(p_n)$  satisfy the condition (4), and let  $\left(\frac{p_n}{P_n}\right)$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$  and  $(V_{n,p}^{(0)}(\Delta u)) \in U(\mathcal{SD})$ , then  $(u_n)$  converges to  $s$ .*

*Proof.* Since  $(V_{n,p}^{(0)}(\Delta u)) \in U(\mathcal{SD})$ ,  $V_{n,p}^{(0)}(\Delta u) = \alpha_n + \sum_{k=1}^n \frac{p_n}{p_{n-1}} \alpha_k + V_{n,p}^{(0)}(\Delta u)$ , for some  $(\alpha_n) \in \mathcal{SD}$ . Hence, we have  $\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) - \sigma_{n,p}^{(1)}(\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u)) = \frac{p_{n-1}}{p_n} \Delta \alpha_n$ . That is,

$$\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) - \frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) = \frac{p_{n-1}}{p_n} \Delta(V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)) = \frac{p_{n-1}}{p_n} \Delta \alpha_n. \tag{13}$$

Taking the weighted means of (13), we have  $\frac{p_{n-1}}{p_n} \Delta(V_{n,p}^{(1)}(\Delta u) - V_{n,p}^{(2)}(\Delta u)) = V_{n,p}^{(0)}(\Delta \alpha)$ . Since  $(\alpha_n) \in \mathcal{SD}$ , by Example 2.2,  $(V_{n,p}^{(0)}(\Delta \alpha)) \in \mathcal{B}_>$ . Therefore,

$$\left(\frac{p_{n-1}}{p_n} \Delta(V_{n,p}^{(1)}(\Delta u) - V_{n,p}^{(2)}(\Delta u))\right) \in \mathcal{B}_>. \tag{14}$$

Since  $(u_n)$  is  $(J, p)$  summable to  $s$ , then it follows from the identity

$$\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(2)}(\Delta u) = V_{n,p}^{(1)}(\Delta u) - V_{n,p}^{(2)}(\Delta u)$$

that  $(\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(2)}(\Delta u))$  is  $(J, p)$  summable to 0.

Hence from (14), by Theorem 1.1, we obtain

$$\left(\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(2)}(\Delta u)\right) \in c_0. \tag{15}$$

From (14), we have  $(\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u)) \in \mathcal{B}_>$ . It from the  $(J, p)$  summability of  $(u_n)$  follows that  $(V_{n,p}^{(1)}(\Delta u))$  is  $(J, p)$  summable to 0, then we have

$$(V_{n,p}^{(1)}(\Delta u)) \in c_0, \tag{16}$$

by Theorem 1.1. It follows from the identity

$$\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) = V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)$$

that

$$(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{B}_> \quad \text{and} \quad (\sigma_{n,p}^{(1)}(u)) \in \mathcal{SD}. \tag{17}$$

On the other hand, since  $(V_{n,p}^{(0)}(\Delta u)) \in U(\mathcal{SD})$ , we have

$$\left(\frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u)\right) \in \mathcal{SD} \tag{18}$$

by the identity (6). Thus, from (16) and (18), we obtain  $(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{SD}$ . From (17), we have  $(u_n) \in \mathcal{SD}$  by the weighted Kronecker identity. Consequently, the proof is completed by Theorem 1.2.

□

**Corollary 3.6.** Let  $(p_n)$  satisfy the condition (4), and let  $(\frac{p_n}{P_n})$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$ , and  $(\omega_{n,p}^{(1)}(u)) \in U(\mathcal{B}_>)$ , then  $(u_n)$  converges to  $s$ .

*Proof.* Since  $(\omega_{n,p}^{(1)}(u)) \in U(\mathcal{B}_>)$ ,  $\omega_{n,p}^{(1)}(u) = \alpha_n + \sum_{k=1}^n \frac{p_n}{p_{n-1}} \alpha_k + u_0$ , for some  $(\alpha_n) \in \mathcal{B}_>$ . Hence we have

$$\frac{p_{n-1}}{p_n} \Delta \omega_{n,p}^{(1)}(u) = \frac{p_{n-1}}{p_n} \Delta \alpha_n + \alpha_n.$$

It follows from the identity  $\frac{p_{n-1}}{p_n} \Delta \left( \frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) \right) = \alpha_n$ , that we get

$$\frac{p_{n-1}}{p_n} \Delta \left( \frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) \right) \in \mathcal{B}_>. \tag{19}$$

The condition (19) implies that  $\left( \frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) \right) \in \mathcal{SD}$ . Hence, it follows from the identity (6) that  $\left( V_{n,p}^{(0)}(\Delta u) \right) \in U(\mathcal{SD})$ .  $\square$

**Corollary 3.7.** *Let  $(p_n)$  satisfy the condition (4), and let  $\left( \frac{p_n}{P_n} \right)$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$ , and  $(V_{n,p}^{(0)}(\Delta u)) \in U(\mathcal{SO})$ , then  $(u_n)$  converges to  $s$ .*

**Corollary 3.8.** *Let  $(p_n)$  satisfy the condition (4), and let  $\left( \frac{p_n}{P_n} \right)$  be totally monotone. If  $(u_n)$  is  $(J, p)$  summable to  $s$ , and  $(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{SO}$ , then  $(u_n)$  converges to  $s$ .*

*Proof.* Since  $(V_{n,p}^{(0)}(\Delta u)) \in \mathcal{SO}$ , then  $\left( \frac{p_{n-1}}{p_n} \Delta V_{n,p}^{(1)}(\Delta u) \right) \in \mathcal{SO}$ . Hence, it follows from the identity (6) that  $\left( V_{n,p}^{(0)}(\Delta u) \right) \in U(\mathcal{SO})$ .  $\square$

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